



ESTIMATION OF NAKAGAMI DISTRIBUTION PARAMETERS IN DESCRIBING A FADING RADIO-COMMUNICATION CHANNEL

ABSTRACT

This article presents a review of issues related to the estimation of Nakagami distribution parameters. This distribution is often used for modeling transmission in a fading radio-communication channel, and in addition it well approximates other distributions.

Key words:

radio-communication fading channel, envelope probability distribution, estimation of distribution parameters.

INTRODUCTION

During transmission a radio signal experiences random variations which are caused by interference occurring in a transmission channel. In order to describe this interference various probabilistic models are used. The main model of a received fading radio signal is a two-parameter distribution, e.g. the Rice distribution, and the Nakagami distribution. In order to describe a fading absolute diffuse signal a one-parameter distribution is used, e.g. the Rayleigh distribution. The Hoyt, Weibull one sided normal distribution, the Beckman three-parameter and four parameter distribution [14, 15] are also often used. A transmitted signal $u(t)$ when transmitted through a radio-communication channel experiences random fading

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$k(t)$, i.e. multiplicative and additive interference $n(t)$. A signal $y(t)$ received by a receiver is a sum of a useful signal $s(t)$ and additive interference $n(t)$, that is [14]

$$y(t) = s(t) + n(t) = k(t) \cdot u(t) + n(t) = r(t) \cos[\omega_o t + \phi(t)] + n(t), \quad (1)$$

where:

$r(t) \geq 0$ — useful signal envelope;

$\phi(t)$ — useful signal instantaneous phase;

ω_o — mean pulsation (angular frequency).

It should be remembered that a useful signal envelope depends exclusively on fading only in the case of signals with angular modulation. In further considerations we assume that all signals and interference occurring in an analogue model of a radio-communication channel are stationary. In addition we assume slow fading variations in comparison with the time of one elementary signal existence. Then in mathematical transformations we can use random variations instead of stochastic processes.

THE NAKAGAMI DISTRIBUTION

It follows from the analysis of the literature that the Nakagami distribution and the Rice distribution are among the most often used for modeling fading [14, 15]. The Rice distribution is often used for modeling diffuse, a multipath transmission of a harmonic signal, when a dominant signal without fading occurs on one of the paths. This distribution is often used for modeling transmission in a satellite channel. The Nakagami distribution describes an even wider class of fading. Let the useful signal $s(t)$ be described with the dependence

$$s(t) = r(t) \cos[\omega_o t + \phi(t)], \quad (2)$$

In order to describe the signal envelope $s(t)$ the Nakagami distribution having the density

$$p(r) = \frac{2}{\Gamma(m)} \left(\frac{m}{\Omega} \right)^m r^{2m-1} \exp\left(-\frac{m}{\Omega} r^2 \right) \quad (3)$$

is used,

where:

Ω — mean signal power defined as

$$\Omega = E[r^2]; \quad (4)$$

$\Gamma(a) = \int_0^{\infty} t^{a-1} \exp(-t) dt$ — function gamma [2, 3];

m — depths of fading, is the inverse of the standardized variance of the useful signal envelope square, i.e. the inverse of the standardized mean signal power.

Parameter m is calculated with the dependence

$$m = \frac{\Omega^2}{E[r^2 - \Omega]^2} \geq 0.5. \quad (5)$$

The Nakagami distribution is a chi distribution, in which parameter m can also take non-integer values. The Nakagami distribution is often referred to as distribution m.

A signal having the Nakagami envelope distribution has the following moment value of the k -order [13, 15]

$$E[r^k] = \int_0^{\infty} r^k p(r) = \frac{\Gamma\left(m + \frac{k}{2}\right)}{\Gamma(m)} \sqrt{\left(\frac{\Omega}{m}\right)^k}. \quad (6)$$

The Nakagami distribution approximates other distributions well. We obtain especially:

- one-sided standard distribution, when $m = 0.5$;
- the Rayleigh distribution, when $m = 1$;
- c) the Rice distribution (Nakagami - n), when $\Omega = 2\sigma^2 + a^2$ and $m = \frac{\Omega^2}{\Omega^2 - a^4}$,

where a is the dominant amplitude of a determined harmonic signal, $2\sigma^2$ is a variance of a narrow path signal (interference) having normal distribution of instantaneous value and mean value equal to zero, both of the signals being components of the Rice signal. Parameters m and Ω of the Rice distribution can be calculated when second order moments of the envelope are known. The dependence (7) makes it possible to determine parameters of the Rice distribution using the Nakagami distribution

$$a^2 = \frac{\Omega}{m} \sqrt{m^2 - m}, \quad \sigma^2 = \frac{\Omega}{2m} \left(m - \sqrt{m^2 - m} \right) = 0.5 (\Omega - a^2). \quad (7)$$

ESTIMATION OF PARAMETERS IN NAKAGAMI DISTRIBUTION BASED ON MEASURED DATA

In a radio-communication channel deep fading occurs. Therefore, in order to calculate parameters in the Nakagami distribution on the basis of measured data magnitudes proportional to the envelope logarithm (expressed in decibels) are used. It is assumed that the logarithmic envelope $v(t)$ is determined by the dependence

$$v(t) = 20 \log \frac{r}{w}, \quad (8)$$

where:

- r — random variable having the Nakagami distribution;
- w — reference value (base for standardization).

The variable v has the Nakagami exponential distribution of probability density [4, 15]

$$p(v) = \frac{2 m^m}{K \Gamma(m)} \exp \left\{ m \left[\frac{2(v - v_0)}{K} - \exp \left(2 \frac{v - v_0}{K} \right) \right] \right\}, \quad (9)$$

where:

$$K = 20 \log e = 8.686 ;$$

$$v_0 = 10 \log \left(\frac{\Omega}{w^2} \right).$$

We assume that the reference value $w = \sqrt{\Omega}$. Then $v_0 = 0$ and logarithmically standardized variable v_n defined as

$$v_n = 20 \log \left(\frac{r}{\sqrt{\Omega}} \right) \quad (10)$$

has the standardized Nakagami exponential distribution of density probability [4]

$$p(v_n) = \frac{2 m^m}{K \Gamma(m)} \exp \left\{ m \left[\frac{2 v_n}{K} - \exp \left(\frac{2 v_n}{K} \right) \right] \right\}. \quad (11)$$

The mean value, root mean square value and variable variance v_n can be calculated using the dependence [4]

$$E(v_n) = \frac{K}{2} [\Psi(m) - \ln(m)]; \quad (12)$$

$$E(v_n^2) = \frac{K^2}{2} \{ [\Psi(m) - \ln(m)]^2 - \Psi'(m) \}; \quad (13)$$

$$Var(v_n) = Var(v) = \frac{K^2}{4} \Psi'(m), \quad (14)$$

where:

$\Psi(x)$ — Euler's psi function (digamma function), i.e. derivative logarithmic gamma function;

$\Psi'(x)$ — derivative of Euler's psi function;

$\Gamma'(x)$ — derivative of gamma function [2, 3].

The parameters m and Ω in the Nakagami distribution can be calculated using standardized moments of the Nakagami exponential distribution. The variance determined in (14), which is not dependent on the reference value, is the most useful for estimating the parameter m . In the expression (14) there occurs the derivative of Euler's psi function. Therefore, an analytical determination of the parameter m as a solution to the equation (14) is impossible. Function $\Psi'(m)$ can be approximated using the equation

$$\Psi'(m) = C m^B. \quad (15)$$

Constants C and B of the approximating function can be determined using the regression method. The accurate values of the function $\Psi'(m)$ for some selected values of m can be calculated on the basis of the dependence [3]

$$\Psi'(1) = \frac{\pi^2}{6}; \quad \Psi'(0.5) = \frac{\pi^2}{2}; \quad \Psi'(L) = \Psi'(1) - \sum_{l=1}^{L-1} \frac{1}{l^2}; \quad L = 2, 3, \dots \quad (16)$$

$$\Psi'(L + 0.5) = \Psi'(0.5) - 4 \sum_{l=1}^L \frac{1}{(2l-1)^2} \quad L = 1, 2, 3, \dots$$

After using the regression method for the values $\Psi'(m)$ calculated on the basis of the dependence (16) and for $m \in <0.5, \dots, 10>$ the following values of the approximating function parameters were obtained $B = -1.2343$ and $C = 1.6645$, that is

$$\Psi'(m) = 1.6645 m^{-1.2343}. \quad (17)$$

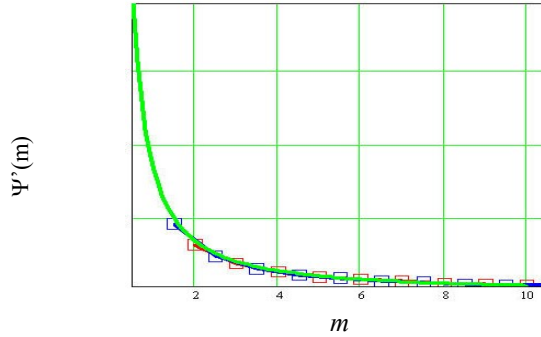


Fig. 1. The accurate values of the function $\Psi'(m)$ (line with squares) and the values calculated on the basis of the approximating function (continuous line)

The selection of the parameter m range was based on the analysis of dynamic probability of element error, for $m \in < 0.5, \dots, 10 >$ significant influence of the parameter m on the transmission quality is recorded [15]. The curvilinear correlation coefficient, being the estimation of the quality of the obtained approximation is 0.9941. Figure 1 presents a diagram of the function $\Psi'(m)$ as the result of using the regression analysis (continuous line). The figure also shows the accurate values of the function (line with squares) calculated on the basis of the dependence (16). The dependence (17) makes it possible to obtain the estimator $\hat{m}$ for the parameter m , which assumes the form

$$\hat{m} = 16.3201 (\hat{\sigma}_v^2)^{-0.8102}, \quad (18)$$

where:

$$\hat{\sigma}_v^2 = \frac{1}{L-1} \sum_{l=1}^L [v_l - \hat{E}(v)]^2 \quad \text{— estimator of logarithmic variance of envelope } v;$$

$$\hat{E}(v) = \frac{1}{L} \sum_{l=1}^L v_l \quad \text{— estimator of mean logarithmic value of envelope } v;$$

$$L \quad \text{— number of measurements.}$$

The second parameter in the Nakagami distribution can be calculated by comparing the standardized Nakagami exponential distribution with the non-standardized Nakagami distribution. The value v_0 can be presented as

$$v_0 = E(v) - E(v_n) = E(v) - \frac{K}{2} [\Psi(m) - \ln(m)]. \quad (19)$$

We also use the regression method. We assume the approximation

$$\ln(m) - \Psi(m) = D m^E; \quad D > 0. \quad (20)$$

As a result of using the regression method, for the values of the function $\Psi(m)$ determined on the basis of the table [3], the following values of the approximating function parameters $E = 1.0787$ and $D = 0.5904$ were obtained. The curvilinear correlation coefficient is 0.9992. Taking into account the obtained result and the equation (12) the mean value of variable v_n can be written as

$$E(v_n) = -2.5641 m^{-1.0787}. \quad (21)$$

Thus the value of the parameter Ω can be calculated on the basis of the dependence

$$v_0 = 10 \log \left(\frac{\Omega}{w^2} \right) = E(v) + 2.5641 m^{-1.0787}. \quad (22)$$

Using the measured data we estimate the mean value of logarithmic envelope v . The estimate $\hat{\Omega}$ of the parameter Ω can be obtained using the following estimator

$$\hat{\Omega} = w^2 10^{0.1 \hat{v}_0}, \quad (23)$$

where

$$\hat{v}_0 = \hat{E}(v) + 2.5641 \hat{m}^{-1.0787} - \text{estimator } v_0.$$

The presented estimators of the Nakagami distribution parameters require estimating the mean value and variance of envelope. Examples of somewhat different ways, based, among others, on maximizing probability functions, were presented in publications [1, 6–8, 11, 16, 19, 20]. For first-order approximation of function $\Psi(m)$, i.e. $\Psi(m) = \ln(m) - (1/2 m)$, in [7] obtained was a parameter estimator m of the Nakagami distribution determined with the dependence

$$\hat{m}_1 = \frac{1}{2 \Delta}, \quad (24)$$

where

$$\Delta = \ln \left[\frac{1}{N} \sum_{i=1}^N r_i^2 \right] - \frac{1}{N} \sum_{i=1}^N \ln(r_i^2).$$

And for the second-order approximation of the function $\Psi(m)$, i.e. $\Psi(m) = \ln(m) - (1/2m) - (1/12m^2)$ obtained was

$$\hat{m}_2 = \frac{6 + \sqrt{36 + 48\Delta}}{24\Delta}. \quad (25)$$

In [1] a comparative analysis was presented for 3 different estimators of the parameter m , which can be written as

a) for standardized estimator

$$\hat{m}_{zn} = \frac{\hat{\mu}_2^2}{\hat{\mu}_4 - \hat{\mu}_2^2}; \text{ where } \hat{\mu}_k = \frac{1}{N} \sum_{i=1}^N r_i^k, \text{ therefore } \hat{\Omega} = \hat{\mu}_2; \quad (26)$$

b) for Tolparev-Polyakov estimator

$$\hat{m}_{TP} = \frac{1 + \sqrt{1 + \frac{4}{3} \ln\left(\frac{\mu_2}{M}\right)}}{4 \ln\left(\frac{\mu_2}{M}\right)}, \text{ where } M = \sqrt[N]{\prod_{i=1}^N r_i^2}; \quad (27)$$

c) for Lorenz estimator

$$\hat{m}_L = \frac{4.4}{\sqrt{\hat{\mu}_2^{dB} - (\hat{\mu}_1^{dB})^2}} + \frac{17.4}{\left[\hat{\mu}_2^{dB} - (\hat{\mu}_1^{dB})^2\right]^{1.29}}, \text{ where } \hat{\mu}_k^{dB} = \frac{1}{N} \sum_{i=1}^N [20 \log(r_i)]^k. \quad (28)$$

It follows from the analysis presented in [1] that the standardized Lorenz estimators generate similar results. The Tolparev-Polyakov estimator is convenient for calculating where the number of measurements is small.

Another form of estimator for the parameter m in the Nakagami distribution is presented in publications [6, 17, 18]. In order to calculate it a third-moment and first-moment quotient was used. This quotient, after transformations and after taking into account the dependence $\Gamma(a+1) = a \cdot \Gamma(a)$ is shown as

$$\frac{\hat{\mu}_3}{\hat{\mu}_1} = \frac{\left(m + \frac{1}{2}\right)\Omega}{m}. \quad (29)$$

From the dependence (29) we obtain the following form of estimator of the parameter m

$$\hat{m}_t = \frac{\hat{\mu}_1 \hat{\mu}_2}{2(\hat{\mu}_3 - \hat{\mu}_1 \hat{\mu}_2)}. \quad (30)$$

In publication [6] it was shown, using numerical calculations, that the estimator $\hat{m}_t$ approximates the parameter m better than the standardized estimator $\hat{m}_{zn}$.

In publications [6, 8, 9] further generalizations were made and another estimator was presented. To calculate it k order moments were used for a random variable calculated with the dependence

$$x_i = \sqrt[p]{r_i}, \text{ for } i = 1, 2, \dots, N; p > 0, \quad (31)$$

k -order moment of the variable x assumes the form determined with the equation [6, 8, 9]

$$E[x^k] = \mu_{k/p} = \frac{\Gamma\left(m + \frac{k}{2p}\right)}{\Gamma(m)} \cdot \sqrt[p]{\left(\frac{\Omega}{m}\right)^k}. \quad (32)$$

Obviously for $p = 1$ we obtain a dependence for k -order of random variable representing the Nakagami envelope, which is determined by means of the equation (6). In order to calculate the values of estimator of the parameter m the order $2p + 1$ and first order moment quotient was taken into account, that is [6]

$$\frac{E[x^{2p+1}]}{E[x]} = \left(1 + \frac{1}{2mp}\right) \cdot \Omega. \quad (33)$$

After transformations we obtain [6]

$$m = \frac{E[x] \cdot \Omega}{2p \{E[x^{2p+1}] - E[x] \cdot \Omega\}} \quad (34)$$

and a new estimator [6]

$$\hat{m}_{\frac{1}{p}} = \frac{\hat{\mu}_{\frac{1}{p}} \hat{\mu}_2}{2p \left(\hat{\mu}_{2+\frac{1}{p}} - \hat{\mu}_{\frac{1}{p}} \hat{\mu}_2 \right)}. \quad (35)$$

It follows from the analysis presented in publication [6] that the estimation quality of the parameter m in the Nakagami distribution increases together with the increase in the parameter p . In addition, in a special case when $p = 0.5$ we obtain a standardized estimator, i.e. $\hat{m}_2 = \hat{m}_{zn}$, and for $p = 1$ we obtain $\hat{m}_1 = \hat{m}_t$. For $p = 2$ we obtain another form of estimator determined with the dependence [6]

$$\hat{m}_{\frac{1}{2}} = \frac{\hat{\mu}_{\frac{1}{2}} \hat{\mu}_2}{4 \left(\hat{\mu}_{\frac{5}{2}} - \hat{\mu}_{\frac{1}{2}} \hat{\mu}_2 \right)}. \quad (36)$$

It follows from the analysis of publications [6, 8, 9] that the variance of estimator $\hat{m}_{\frac{1}{2}}$ needs to be verified. It must be compared with the variance of estimators $\hat{m}_1$ or $\hat{m}_l$.

Another estimator in the Nakagami distribution was calculated using the known quotient of moments of variable $x_i = \sqrt[p]{r_i}$ [9]

$$\Delta_{a,b,p} = \frac{\frac{\mu_a}{p}}{\frac{\mu_b}{p} (\mu_2)^{\frac{a-b}{2p}}} = \frac{\Gamma\left(m + \frac{a}{2p}\right)}{\Gamma\left(m + \frac{b}{2p}\right) m^{\frac{a-b}{2p}}}, \quad (37)$$

where

$a \neq b$ — parameters which are integers.

It follows from the analysis presented in publication [9] that the equation (37) can be approximated using the dependence

$$\begin{aligned} \Delta_{a,b,p} &\approx 1 + \frac{(a-b)(a+b-2p)}{8p^2 m} + \frac{(a-b)(a-b-2p)}{384p^4 m^2} [3(a+b-2p)^2 - 2p(a-b+2p)] = \\ &= c_0 + \frac{c_1}{m} + \frac{c_2}{m^2}, \end{aligned} \quad (38)$$

where coefficients c_0, c_1, c_2 depend on the values of parameters a, b, p .

For $a=1$ and $b=0$ obtained was another estimator m determined as [9]

$$\hat{m}_A = \frac{-c_1 - \sqrt{c_1^2 - 4(c_0 - \Delta_{1,0,p})c_2}}{2(c_0 - \Delta_{1,0,p})}, \quad (39)$$

where

$$\Delta_{1,0,p} = \frac{\frac{\mu_1}{p}}{(\mu_2)^{\frac{1}{2p}}},$$

and the coefficients c_0, c_1, c_2 , in order to facilitate the analysis, were presented in publication [9] in the tabular form.

One more estimator was obtained after taking into account the dependence [9]

$$\Delta_G = \frac{\left(\frac{\mu_1}{p}\right)^2}{\frac{\mu_2}{p}} = \frac{\left[\Gamma\left(m + \frac{1}{2p}\right)\right]^2}{\Gamma\left(m + \frac{1}{p}\right)\Gamma(m)} \approx 1 - \frac{1}{4p^2 m} - \frac{4p^2 - 4p - 1}{32p^4 m^2} + \dots \quad (40)$$

Taking into account only the first two components of the approximating polynomial an estimator, determined with the equation [9], was obtained

$$\hat{m}_G = \frac{c_1}{\Delta_G - c_0} . \quad (41)$$

The presented different dependences which can be used to determine the parameter m in the Nakagami distribution do not exhaust the broad analysis of this issue covered in world literature. Other considerations were, among others, discussed in publications [5, 10–12, 16].

CONCLUSIONS

This article constitutes a review of the issues relating to the estimation of the Nakagami distribution parameters. In world literature this issue has been attracting interest for many years, which is shown by numerous publications. To the article author's knowledge, there is a lack of such considerations in the Polish literature. The presented Nakagami distribution estimator parameters can be used to assess the transmission quality in a radio-communication fading channel. They can also be used to design optimal receivers. At present, at the Gdynia Maritime Academy experimental research is being carried out in the real propagation environment, which will make it possible to estimate parameters of a radio-communication channel and to model transmission in a radio-communication fading channel. The aim of further investigations will also be comparative analysis of particular estimators.

REFERENCES

- [1] Abdi A., Kaves M., *Performance comparison of three different estimator for the Nakagami m parameter using Monte Carlo simulation*, 'Communications Letters', IEEE, April 2000, Vol. 4, pp. 119–121.
- [2] Abramowitz M., Stegun I., *Handbook of Mathematical Functions*, New York — Dover 1972.
- [3] Antoniewicz J., *Tablice funkcji dla inżynierów*, PWN, Warszawa 1969 [Table of functions for engineers — available in Polish].
- [4] Buch T., *Messung der Verteilungs parameter der Empfangs feldstarke bei Funkwellen ausbreitungs untersuchungen*, 'Nachrichten Technik Elektronik', 1982, 32, H. 6.
- [5] Chen X., Liu S., Fan P., *Hardware implementation on m parameter ML estimation of Nakagami m fading channel*, IEEE, 2015, pp. 1395–1399.
- [6] Cheng J., Beaulieu N. C., *Generalized moment estimators for the Nakagami m fading parameter*, 'Communications Letters', IEEE, April 2002, Vol. 6, Issue 4, pp. 144–146.
- [7] Cheng J., Beaulieu N., *Maximum-likelihood based estimation of the Nakagami m parameter*, 'Communications Letters', IEEE, March 2001, Vol. 5, No. 3, pp. 101–103.
- [8] Gaeddeert J., Annamalai A., *Further results on Nakagami m parameter estimation*, 'Communications Letters', IEEE, January 2005, Vol. 9, No. 1, pp. 22–24.
- [9] Gaeddeert J., Annamalai A., *Further results on Nakagami m parameter estimation*, IEEE, Vehicular Technology Conference, 2004, Vol. 6, pp. 4255–4259.
- [10] Hadziahmetović N., Milis M., Dokić M. A., Hadžialić M., *Estimation of Nakagami distribution parameters based on signal samples corrupted with multiplicative and additive disturbances*, International Symposium ELMAR 2007, Zadar, Croatia, pp. 235–238.
- [11] Hadžialić M., Milis M., Hadziahmetović N., Sarajilić A., *Moment-based maximum likelihood-based quotiential estimation of the Nakagami m fading parameter*, IEEE, Vehicular Technology Conference, Spring 2007, pp. 549–553.
- [12] Liu S., Fan P., Xiong K., Wang G., *Novel ML estimation of m parameter of the noisy Nakagami m channel*, International Workshop on High Mobility Wireless Communications, 2014, pp. 183–193.
- [13] Noga K. M., *Statystyki drugiego rzędu kanału Rayleigha oraz Nakagamiego*, 'Kwartalnik Elektroniki i Telekomunikacji', 2006, Vol. 52, No. 4, pp. 617–628 [Second-order statistics of Rayleigh and Nakagami channel — available in Polish].
- [14] Noga K., *Zaawansowane modele probabilistyczne sygnałów w kanałach z zanikami*, 'Kwartalnik Elektroniki i Telekomunikacji', 2000, No. 1, pp. 47–62 [Advanced probabilistic models of signals in channel with fading — available in Polish].
- [15] Simon S. M., Alouini M. S., *Digital communication over fading channel*, J. Wiley & Sons, 2005.
- [16] Tepedelenlioglu C., Gao P., *Practical issues in the estimation of Nakagami m parameter*, IEEE GLOBECOM, 2003, pp. 972–976.
- [17] Wang N., Song X., Cheng J., *Estimating the Nakagami m fading parameter by the generalized method of moments*, IEEE, International Conference of Communications, 2011.
- [18] Wang N., Song X., Cheng J., *Generalized method of moments estimation of the Nakagami m fading parameter*, 'Transactions on Wireless Communications', IEEE, September 2012, Vol. 11, No. 9, pp. 3316–3325.
- [19] Young-Chai K., Alouini M. S., *Estimation of Nakagami fading channel parameters with application to optimized transmitter diversity systems*, IEEE, International Conference on Communications ICC 2001, June 2001, Vol. 6, pp. 1690–1695.
- [20] Zhang Q., *A note on the estimation of Nakagami m fading parameter*, 'Communications Letters', IEEE, January 2002, Vol. 6, No. 6, pp. 237–238.

ESTYMACJA PARAMETRÓW ROZKŁADU NAKAGAMIEGO OPISUJĄCEGO KANAŁ Z ZANIKAMI

STRESZCZENIE

W artykule przedstawiono estymatory rozkładu Nakagamiiego. Rozkład ten jest często stosowany do modelowania transmisji w kanale radiokomunikacyjnym z zanikami, ponadto dobrze aproksymuje inne rozkłady.

Słowa kluczowe:

kanał radiokomunikacyjny z zanikami, rozkład prawdopodobieństwa obwiedni, estymacja parametrów rozkładu.