



ON VARIABLE EXPONENT AMALGAM SPACES

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Abstract

We derive some of the basic properties of weighted variable exponent Lebesgue spaces $L_w^{p(\cdot)}(\mathbb{R}^n)$ and investigate embeddings of these spaces under some conditions. Also a new family of Wiener amalgam spaces $W(L_w^{p(\cdot)}, L_v^q)$ is defined, where the local component is a weighted variable exponent Lebesgue space $L_w^{p(\cdot)}(\mathbb{R}^n)$ and the global component is a weighted Lebesgue space $L_v^q(\mathbb{R}^n)$. We investigate the properties of the spaces $W(L_w^{p(\cdot)}, L_v^q)$. We also present new Hölder-type inequalities and embeddings for these spaces.

1 Introduction

A number of authors worked on amalgam spaces or some special cases of these spaces. The first appearance of amalgam spaces can be traced to N. Wiener [26]. But the first systematic study of these spaces was undertaken by F. Holland [18], [19]. The *amalgam* of L^p and l^q on the real line is the space $(L^p, l^q)(\mathbb{R})$ (or shortly (L^p, l^q)) consisting of functions f which are locally in L^p and have l^q behavior at infinity in the sense that the norms over $[n, n+1]$ form an l^q -sequence. For $1 \leq p, q \leq \infty$ the norm

$$\|f\|_{p,q} = \left[\sum_{n=-\infty}^{\infty} \left[\int_n^{n+1} |f(x)|^p dx \right]^{\frac{q}{p}} \right]^{\frac{1}{q}} < \infty$$

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makes (L^p, l^q) into a Banach space. If $p = q$ then (L^p, l^q) reduces to L^p . A generalization of Wiener's definition was given by H.G. Feichtinger in [10], describing certain Banach spaces of functions (or measures, distributions) on locally compact groups by global behaviour of certain local properties of their elements. C. Heil [17] gave a good summary of results concerning amalgam spaces with global components being weighted $L^q(\mathbb{R})$ spaces. For a historical background of amalgams see [16]. The variable exponent Lebesgue spaces (or generalized Lebesgue spaces) $L^{p(\cdot)}$ appeared in literature for the first time already in a 1931 article by W. Orlicz [22]. The major study of this spaces was initiated by O. Kovacik and J. Rakosnik [20], where basic properties such as Banach space, reflexivity, separability, uniform convexity, Hölder inequalities and embeddings of type $L^{p(\cdot)} \hookrightarrow L^{q(\cdot)}$ were obtained in higher dimension Euclidean spaces. Also there are recent many interesting and important papers appeared in variable exponent Lebesgue spaces (see, [4], [5], [6], [8], [9]). The spaces $L^{p(\cdot)}$ and classical Lebesgue spaces L^p have many common properties, but a crucial difference between this spaces is that $L^{p(\cdot)}$ is not invariant under translation in general (Ex. 2.9 in [20] and Lemma 2.3 in [6]). Moreover, the Young theorem $\|f * g\|_{p(\cdot)} \leq \|f\|_{p(\cdot)} \|g\|_1$ is not valid for $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$. But the Young theorem was proved in a special form and derived more general statement in [25]. Aydın and Gürkanlı [3] defined the weighted variable Wiener amalgam spaces $W(L^{p(\cdot)}, L_w^q)$ where the local component is a variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ and the global component is a weighted Lebesgue space $L_w^q(\mathbb{R}^n)$. They proved new Hölder-type inequalities and embeddings for these spaces. They also showed that under some conditions the Hardy-Littlewood maximal function does not map the space $W(L^{p(\cdot)}, L_w^q)$ into itself.

Let $0 < \mu(\Omega) < \infty$. It is known that $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ if and only if $p(x) \leq q(x)$ for a.e. $x \in \Omega$ by Theorem 2.8 in [20]. This paper is concerned with embeddings properties of $L_w^{p(\cdot)}(\mathbb{R}^n)$ with respect to variable exponents and weight functions. We will discuss the continuous embedding $L_{w_2}^{p_2(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{w_1}^{p_1(\cdot)}(\mathbb{R}^n)$ under different conditions. We investigate the properties of the spaces $W(L_w^{p(\cdot)}, L_v^q)$. We also present new Hölder-type inequalities and embeddings for these spaces.

2 Definition and Preliminary Results

In this paper all sets and functions are Lebesgue measurable. The Lebesgue measure and the characteristic function of a set $A \subset \mathbb{R}^n$ will be denoted by $\mu(A)$ and χ_A , respectively. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces and $X \subset Y$. $X \hookrightarrow Y$ means that X is a subspace of Y and the iden-

tity operator I from X into Y is continuous. This implies that there exists a constant $C > 0$ such that

$$\|u\|_Y \leq C \|u\|_X$$

for all $u \in X$.

The space $L^1_{loc}(\mathbb{R}^n)$ consists of all (classes of) measurable functions f on $\mathbb{R}^n$ such that $f\chi_K \in L^1(\mathbb{R}^n)$ for any compact subset $K \subset \mathbb{R}^n$. It is a topological vector space with the family of seminorms $f \rightarrow \|f\chi_K\|_{L^1}$. A Banach function space (shortly BF-space) on $\mathbb{R}^n$ is a Banach space $(B, \|\cdot\|_B)$ of measurable functions which is continuously embedded into $L^1_{loc}(\mathbb{R}^n)$, i.e. for any compact subset $K \subset \mathbb{R}^n$ there exists some constant $C_K > 0$ such that $\|f\chi_K\|_{L^1} \leq C_K \|f\|_B$ for all $f \in B$. A BF-space $(B, \|\cdot\|_B)$ is called solid if $g \in L^1_{loc}(\mathbb{R}^n)$, $f \in B$ and $|g(x)| \leq |f(x)|$ almost everywhere (shortly a.e.) implies that $g \in B$ and $\|g\|_B \leq \|f\|_B$. A BF-space $(B, \|\cdot\|_B)$ is solid iff it is a $L^\infty(\mathbb{R}^n)$ -module. We denote by $C_c(\mathbb{R}^n)$ and $C^\infty_c(\mathbb{R}^n)$ the space of all continuous, complex-valued functions with compact support and the space of infinitely differentiable functions with compact support in $\mathbb{R}^n$ respectively. The character operator M_t is defined by $M_t f(y) = \langle y, t \rangle f(y)$, $y \in \mathbb{R}^n$, $t \in \mathbb{R}^n$. $(B, \|\cdot\|_B)$ is strongly character invariant if $M_t B \subseteq B$ and $\|M_t f\|_B = \|f\|_B$ for all $f \in B$ and $t \in \mathbb{R}^n$.

We denote the family of all measurable functions $p : \mathbb{R}^n \rightarrow [1, \infty)$ (called the variable exponent on $\mathbb{R}^n$) by the symbol $\mathcal{P}(\mathbb{R}^n)$. For $p \in \mathcal{P}(\mathbb{R}^n)$ put

$$p_* = \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x), \quad p^* = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x).$$

For every measurable functions f on $\mathbb{R}^n$ we define the function

$$\varrho_p(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx.$$

The function ϱ_p is a convex modular; that is, $\varrho_p(f) \geq 0$, $\varrho_p(f) = 0$ if and only if $f = 0$, $\varrho_p(-f) = \varrho_p(f)$ and ϱ_p is convex. The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is defined as the set of all μ -measurable functions f on $\mathbb{R}^n$ such that $\varrho_p(\lambda f) < \infty$ for some $\lambda > 0$, equipped with the Luxemburg norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \varrho_p\left(\frac{f}{\lambda}\right) \leq 1 \right\}.$$

If $p^* < \infty$, then $f \in L^{p(\cdot)}(\mathbb{R}^n)$ iff $\varrho_p(f) < \infty$. If $p(x) = p$ is a constant function, then the norm $\|\cdot\|_{p(\cdot)}$ coincides with the usual Lebesgue norm $\|\cdot\|_p$. The space $L^{p(\cdot)}(\mathbb{R}^n)$ is a particular case of the so-called Orlicz-Musielak space [20]. The function p always denotes a variable exponent and we assume that $p^* < \infty$.

Definition 2.1. Let w be a measurable, positive a.e. and locally μ -integrable function on $\mathbb{R}^n$. Such functions are called weight functions. By a Beurling weight on $\mathbb{R}^n$ we mean a measurable and locally bounded function w on $\mathbb{R}^n$ satisfying $1 \leq w(x)$ and $w(x+y) \leq w(x)w(y)$ for all $x, y \in \mathbb{R}^n$. Let $1 \leq p < \infty$ be given. By the classical weighted Lebesgue space $L_w^p(\mathbb{R}^n)$ we denote the set of all μ -measurable functions f for which the norm

$$\|f\|_{p,w} = \|fw\|_p = \left(\int_{\mathbb{R}^n} |f(x)w(x)|^p dx \right)^{1/p} < \infty.$$

We say that $w_1 \prec w_2$ if and only if there exists a $C > 0$ such that $w_1(x) \leq Cw_2(x)$ for all $x \in \mathbb{R}^n$. Two weight functions are called equivalent and written $w_1 \approx w_2$, if $w_1 \prec w_2$ and $w_2 \prec w_1$ [13], [15].

Lemma 2.2. (a) A Beurling weight function w is also weight function in general.

(b) For each $p \in \mathcal{P}(\mathbb{R}^n)$, both $w^{p(\cdot)}$ and $w^{-p(\cdot)}$ are locally integrable.

Proof. (a) Let any compact subset $K \subset \mathbb{R}^n$ be given. Since w is locally bounded function, then we write

$$\sup_{x \in K} w(x) < \infty.$$

Hence

$$\int_K w(x) dx \leq \left(\sup_{x \in K} w(x) \right) \mu(K) < \infty.$$

(b) Since $w(x) \geq 1$, then

$$\int_K w(x)^{p(x)} dx \leq \int_K w(x)^{p^*} dx \leq \left(\sup_{x \in K} w(x)^{p^*} \right) \mu(K) < \infty.$$

Also $w(x) \neq 0$ and $w(x)^{-1} \leq 1$

$$\int_K w(x)^{-p(x)} dx \leq \int_K w(x)^{-p^*} dx \leq \left(\sup_{x \in K} w(x)^{-p^*} \right) \mu(K) < \infty.$$

□

Let w be a Beurling weight function on $\mathbb{R}^n$ and $p \in \mathcal{P}(\mathbb{R}^n)$. The weighted variable exponent Lebesgue space $L_w^{p(\cdot)}(\mathbb{R}^n)$ is defined as the set of all measurable functions f , for which

$$\|f\|_{p(\cdot),w} = \|fw\|_{p(\cdot)} < \infty.$$

The space $(L_w^{p(\cdot)}(\mathbb{R}^n), \|\cdot\|_{p(\cdot),w})$ is a Banach space. Throughout this paper we assume that w is a Beurling weight.

Proposition 2.3. (i) The embeddings $L_w^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n)$ is continuous and the inequality

$$\|f\|_{p(\cdot)} \leq \|f\|_{p(\cdot),w}$$

is satisfied for all $f \in L_w^{p(\cdot)}(\mathbb{R}^n)$.

(ii) $C_c(\mathbb{R}^n) \subset L_w^{p(\cdot)}(\mathbb{R}^n)$.

(iii) $C_c(\mathbb{R}^n)$ is dense in $L_w^{p(\cdot)}(\mathbb{R}^n)$.

(iv) $L_w^{p(\cdot)}(\mathbb{R}^n)$ is a BF-space.

(v) $L_w^{p(\cdot)}$ is a Banach module over L^∞ with respect to pointwise multiplication.

Proof. (i) Assume $f \in L_w^{p(\cdot)}(\mathbb{R}^n)$. Since $w(x)^{p(x)} \geq 1$, then

$$\begin{aligned} |f(x)|^{p(x)} &\leq |f(x)w(x)|^{p(x)}, \\ \varrho_p(f) &\leq \varrho_{p,w}(f) < \infty. \end{aligned}$$

This implies that $L_w^{p(\cdot)}(\mathbb{R}^n) \subset L^{p(\cdot)}(\mathbb{R}^n)$. Also by using the inequality $|f(x)| \leq |f(x)w(x)|$ and definition of $\|\cdot\|_{p(\cdot)}$, then

$$\|f\|_{p(\cdot)} \leq \|fw\|_{p(\cdot)} = \|f\|_{p(\cdot),w}.$$

(ii) Let $f \in C_c(\mathbb{R}^n)$ be any function such that $\text{supp } f = K$ compact. For $p^* < \infty$ it is known that $C_c(\mathbb{R}^n) \subset L^{p(\cdot)}(\mathbb{R}^n)$ by Lemma 4 in [1] and $\varrho_p(f) < \infty$. Hence we have

$$\begin{aligned} \varrho_{p,w}(f) &= \varrho_p(fw) = \int_K |f(x)|^{p(x)} w(x)^{p(x)} dx \\ &\leq \left(\sup_{x \in K} w(x)^{p^*} \right) \varrho_p(f) < \infty \end{aligned}$$

and $C_c(\mathbb{R}^n) \subset L_w^{p(\cdot)}(\mathbb{R}^n)$.

(iii) It is known that $C_c^\infty(\mathbb{R}^n)$ is dense in $L_w^{p(\cdot)}(\mathbb{R}^n)$ by Corollary 2.5 in [2]. Hence $C_c(\mathbb{R}^n)$ is dense in $L_w^{p(\cdot)}(\mathbb{R}^n)$.

(iv) Let $K \subset \mathbb{R}^n$ be a compact subset and $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$. By Hölder inequality for generalized Lebesgue spaces [20], we write

$$\begin{aligned} \int_K |f(x)| dx &\leq C \|\chi_K\|_{q(\cdot)} \|f\|_{p(\cdot)} \\ &\leq C \|\chi_K\|_{q(\cdot),w} \|f\|_{p(\cdot),w} \end{aligned}$$

for all $f \in L_w^{p(\cdot)}(\mathbb{R}^n)$, where χ_K is the characteristic function of K . It is known that $\|\chi_K\|_{q(\cdot),w} < \infty$ if and only if $\varrho_{q,w}(\chi_K) < \infty$ for $q^* < \infty$. Then we have

$$\varrho_{q,w}(\chi_K) = \int_K w(x)^{q(x)} dx = \left(\sup_{x \in K} w(x)^{q^*} \right) \mu(K) < \infty.$$

That means $L_w^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{loc}^1(\mathbb{R}^n)$.

(v) We know that $L_w^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space. Also it is known that $L^\infty(\mathbb{R}^n)$ is a Banach algebra with respect to pointwise multiplication. Let $(f, g) \in L^\infty(\mathbb{R}^n) \times L_w^{p(\cdot)}(\mathbb{R}^n)$. Then

$$\begin{aligned} \varrho_{p,w}(fg) &= \int_{\mathbb{R}} |f(x)g(x)|^{p(x)} w(x)^{p(x)} dx \\ &\leq \max \left\{ 1, \|f\|_\infty^{p^*} \right\} \int_{\mathbb{R}} |g(x)w(x)|^{p(x)} dx < \infty. \end{aligned}$$

We also have

$$\begin{aligned} \varrho_{p,w}\left(\frac{fg}{\|f\|_\infty \|g\|_{p(\cdot),w}}\right) &\leq \int_{\mathbb{R}} \frac{|f(x)g(x)|^{p(x)}}{\|f\|_\infty^{p(x)} \|g\|_{p(\cdot),w}^{p(x)}} dx \leq \int_{\mathbb{R}} \frac{\|f\|_{L^\infty}^{p(x)} |g(x)|^{p(x)}}{\|f\|_\infty^{p(x)} \|g\|_{p(\cdot),w}^{p(x)}} dx \\ &= \varrho_{p,w}\left(\frac{g}{\|g\|_{p(\cdot),w}}\right) \leq 1. \end{aligned}$$

Hence by the definition of the norm $\|\cdot\|_{p(\cdot),w}$ of the weighted variable exponent Lebesgue space, we obtain $\|fg\|_{p(\cdot),w} \leq \|f\|_{L^\infty} \|g\|_{p(\cdot),w}$. The remaining part of the proof is easy. $\square$

Proposition 2.4. (i) The space $L_w^{p(\cdot)}(\mathbb{R}^n)$ is strongly character invariant.
(ii) The function $t \rightarrow M_t f$ is continuous from $\mathbb{R}^n$ into $L_w^{p(\cdot)}(\mathbb{R}^n)$.

Proof. (i) Let take any $f \in L_w^{p(\cdot)}(\mathbb{R}^n)$. We define a function g such that $g(x) = M_t f(x)$ for all $t \in \mathbb{R}^n$. Hence we have

$$|g(x)| = |M_t f(x)| = |\langle x, t \rangle f(x)| = |f(x)|$$

and

$$\|M_t f\|_{p(\cdot),w} = \|g\|_{p(\cdot),w} = \|f\|_{p(\cdot),w}.$$

(ii) Since $C_c(\mathbb{R}^n)$ is dense in $L_w^{p(\cdot)}(\mathbb{R}^n)$ by Proposition 2.3, then given any $f \in L_w^{p(\cdot)}(\mathbb{R}^n)$ and $\varepsilon > 0$, there exists $g \in C_c(\mathbb{R}^n)$ such that

$$\|f - g\|_{p(\cdot),w} < \frac{\varepsilon}{3}.$$

Let assume that $\text{supp } g = K$. Thus for every $t \in \mathbb{R}^n$, we have $\text{supp}(M_t g - g) \subset K$. If one uses the inequality

$$\begin{aligned} |M_t g(x) - g(x)| &= |\langle x, t \rangle g(x) - g(x)| = |g(x)| |\langle x, t \rangle - 1| \\ &\leq |g(x)| \sup_{x \in K} |\langle x, t \rangle - 1| = |g(x)| \|\langle \cdot, t \rangle - 1\|_{\infty, K}, \end{aligned}$$

we have

$$\|M_t g - g\|_{p(\cdot), w} \leq \|\langle \cdot, t \rangle - 1\|_{\infty, K} \|g\|_{p(\cdot), w}.$$

It is known that $\|\langle \cdot, t \rangle - 1\|_{\infty, K} \rightarrow 0$ for $t \rightarrow 0$. Also, we have

$$\begin{aligned} \|M_t f - f\|_{p(\cdot), w} &\leq \|M_t f - M_t g\|_{p(\cdot), w} + \|M_t g - g\|_{p(\cdot), w} + \|f - g\|_{p(\cdot), w} \\ &= 2 \|f - g\|_{p(\cdot), w} + \|\langle \cdot, t \rangle - 1\|_{\infty, K} \|g\|_{p(\cdot), w}. \end{aligned}$$

Let us take the neighbourhood U of $0 \in \mathbb{R}^n$ such that

$$\|\langle \cdot, t \rangle - 1\|_{\infty, K} < \frac{\varepsilon}{3 \|g\|_{p(\cdot), w}}$$

for all $t \in U$. Then we have

$$\|M_t f - f\|_{p(\cdot), w} < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3 \|g\|_{p(\cdot), w}} \|g\|_{p(\cdot), w} = \varepsilon$$

for all $t \in U$. □

Definition 2.5. Let $p_1(\cdot)$ and $p_2(\cdot)$ be exponents on $\mathbb{R}^n$. We say that $p_2(\cdot)$ is non-weaker than $p_1(\cdot)$ if and only if $\Phi_{p_2}(x, t) = t^{p_2(x)}$ is non-weaker than $\Phi_{p_1}(x, t) = t^{p_1(x)}$ in the sense of Musielak [21], i.e. there exist constants $K_1, K_2 > 0$ and $h \in L^1(\mathbb{R}^n)$, $h \geq 0$, such that for a.e. $x \in \mathbb{R}^n$ and all $t \geq 0$

$$\Phi_{p_1}(x, t) \leq K_1 \Phi_{p_2}(x, K_2 t) + h(x).$$

We write $p_1(\cdot) \preceq p_2(\cdot)$.

Let $p_1(\cdot) \preceq p_2(\cdot)$. Then the embedding $L^{p_2(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p_1(\cdot)}(\mathbb{R}^n)$ was proved by Lemma 2.2 in [6].

Proposition 2.6. (i) If $w_1 \prec w_2$, then $L_{w_2}^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{w_1}^{p(\cdot)}(\mathbb{R}^n)$.

(ii) If $w_1 \approx w_2$, then $L_{w_1}^{p(\cdot)}(\mathbb{R}^n) = L_{w_2}^{p(\cdot)}(\mathbb{R}^n)$.

(iii) Let $0 < \mu(\Omega) < \infty$, $\Omega \subset \mathbb{R}^n$. If $w_1 \prec w_2$ and $p_1(\cdot) \leq p_2(\cdot)$, then $L_{w_2}^{p_2(\cdot)}(\Omega) \hookrightarrow L_{w_1}^{p_1(\cdot)}(\Omega)$.

Proof. (i) Let $f \in L_{w_2}^{p(\cdot)}(\mathbb{R}^n)$. Since $w_1 \prec w_2$, there exists a $C > 0$ such that $w_1(x) \leq C w_2(x)$ for all $x \in \mathbb{R}^n$. Hence we write

$$|f(x) w_1(x)| \leq C |f(x) w_2(x)|.$$

This implies that

$$\|f\|_{p(\cdot), w_1} \leq C \|f\|_{p(\cdot), w_2}.$$

for all $f \in L_{w_2}^{p(\cdot)}(\mathbb{R}^n)$.

(ii) Obvious.

(iii) Let $f \in L_{w_2}^{p_2(\cdot)}(\Omega)$ be given. By using (i), we have $f \in L_{w_1}^{p_2(\cdot)}(\Omega)$ and $fw_1 \in L^{p_2(\cdot)}(\Omega)$. Since $p_1(\cdot) \leq p_2(\cdot)$, then $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ by Theorem 2.8 in [20] and

$$\begin{aligned} \|fw_1\|_{p_1(\cdot)} &\leq C_1 \|fw_1\|_{p_2(\cdot)} \\ &\leq C_1 C_2 \|f\|_{p_2(\cdot), w_2}. \end{aligned}$$

Hence $L_{w_2}^{p_2(\cdot)}(\Omega) \hookrightarrow L_{w_1}^{p_1(\cdot)}(\Omega)$. $\square$

Proposition 2.7. If $p_1(\cdot) \preceq p_2(\cdot)$ and $w_1 \prec w_2$, then $L_{w_2}^{p_2(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{w_1}^{p_1(\cdot)}(\mathbb{R}^n)$.

Proof. Since $p_1(\cdot) \preceq p_2(\cdot)$, then $L_{w_2}^{p_2(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{w_2}^{p_1(\cdot)}(\mathbb{R}^n)$ by Theorem 8.5 of [21]. Also by using Proposition 2.6, we have $L_{w_2}^{p_1(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{w_1}^{p_1(\cdot)}(\mathbb{R}^n)$. $\square$

Remark 2.8. By the closed graph theorem in Banach space, to prove that there is a continuous embedding $L_{w_2}^{p_2(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{w_1}^{p_1(\cdot)}(\mathbb{R}^n)$, one need only prove $L_{w_2}^{p_2(\cdot)}(\mathbb{R}^n) \subset L_{w_1}^{p_1(\cdot)}(\mathbb{R}^n)$.

Let w_1, w_2 be weights on $\mathbb{R}^n$. The space $L_{w_1}^{p_1(\cdot)}(\mathbb{R}^n) \cap L_{w_2}^{p_2(\cdot)}(\mathbb{R}^n)$ is defined as the set of all measurable functions f , for which

$$\|f\|_{w_1, w_2}^{p_1(\cdot), p_2(\cdot)} = \|f\|_{p_1(\cdot), w_1} + \|f\|_{p_2(\cdot), w_2} < \infty.$$

Proposition 2.9. Let w_1, w_2, w_3 and w_4 be weights on $\mathbb{R}^n$. If $w_1 \prec w_3$ and $w_2 \prec w_4$, then $L_{w_3}^{p_1(\cdot)}(\mathbb{R}^n) \cap L_{w_4}^{p_2(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{w_1}^{p_1(\cdot)}(\mathbb{R}^n) \cap L_{w_2}^{p_2(\cdot)}(\mathbb{R}^n)$.

Proof. Obvious. $\square$

Corollary 2.10. If $w_1 \approx w_3$ and $w_2 \approx w_4$, then $L_{w_3}^{p_1(\cdot)}(\mathbb{R}^n) \cap L_{w_4}^{p_2(\cdot)}(\mathbb{R}^n) = L_{w_1}^{p_1(\cdot)}(\mathbb{R}^n) \cap L_{w_2}^{p_2(\cdot)}(\mathbb{R}^n)$.

Proposition 2.11. If $p_1(x) \leq p_2(x) \leq p_3(x)$ and $w_2 \prec w_1$, then

$$L_{w_1}^{p_1(\cdot)}(\mathbb{R}^n) \cap L_{w_1}^{p_3(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{w_2}^{p_2(\cdot)}(\mathbb{R}^n).$$

Proof. Since $p_1(x) \leq p_2(x) \leq p_3(x)$, then we write

$$\begin{aligned} |f(x)w_1(x)|^{p_2(x)} &\leq |f(x)w_1(x)|^{p_1(x)} \chi_{\{x: |f(x)w_1(x)| \leq 1\}} + \\ &\quad + |f(x)w_1(x)|^{p_3(x)} \chi_{\{x: |f(x)w_1(x)| \geq 1\}}. \end{aligned}$$

Hence $L_{w_1}^{p_1(\cdot)}(\mathbb{R}^n) \cap L_{w_1}^{p_3(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{w_1}^{p_2(\cdot)}(\mathbb{R}^n)$. Also by using Proposition 2.6, we have $L_{w_1}^{p_2(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{w_2}^{p_2(\cdot)}(\mathbb{R}^n)$. $\square$

Corollary 2.12. Let $1 \leq p_* \leq p(x) \leq p^* < \infty$ for all $x \in \mathbb{R}^n$ and $w_2 \prec w_1$, then

$$L_{w_1}^{p_*}(\mathbb{R}^n) \cap L_{w_1}^{p^*}(\mathbb{R}^n) \hookrightarrow L_{w_2}^{p(\cdot)}(\mathbb{R}^n).$$

Proof. The proof is completed by Proposition 2.11. $\square$

For any $f \in L^1(\mathbb{R}^n)$, the Fourier transform of f is denoted by $\widehat{f}$ and defined by

$$\widehat{f}(x) = \int_{\mathbb{R}^n} e^{-it \cdot x} f(t) dt.$$

It is known that $\widehat{f}$ is a continuous function on $\mathbb{R}^n$, which vanishes at infinity and the inequality $\|\widehat{f}\|_\infty \leq \|f\|_1$ is satisfied. Let the Fourier algebra $\{\widehat{f} : f \in L^1(\mathbb{R}^n)\}$ with by $A(\mathbb{R}^n)$ and is given the norm $\|\widehat{f}\|_A = \|f\|_1$.

Let ω be an arbitrary Beurling's weight function on $\mathbb{R}^n$. We next introduce the homogeneous Banach space

$$A^\omega(\mathbb{R}^n) = \{\widehat{f} : f \in L_\omega^1(\mathbb{R}^n)\}$$

with the norm $\|\widehat{f}\|_\omega = \|f\|_{1,\omega}$. It is known that $A^\omega(\mathbb{R}^n)$ is a Banach algebra under pointwise multiplication [23]. We set $A_0^\omega(\mathbb{R}^n) = A^\omega(\mathbb{R}^n) \cap C_c(\mathbb{R}^n)$ and equip it with the inductive limit topology of the subspaces $A_K^\omega(\mathbb{R}^n) = A^\omega(\mathbb{R}^n) \cap C_K(\mathbb{R}^n)$, $K \subset \mathbb{R}^n$ compact, equipped with their $\|\cdot\|_\omega$ norms. For every $h \in A_0^\omega(\mathbb{R}^n)$ we define the semi-norm q_h on $A_0^\omega(\mathbb{R}^n)'$ by $q_h(h') = |\langle h, h' \rangle|$, where $A_0^\omega(\mathbb{R}^n)'$ is the topological dual of $A_0^\omega(\mathbb{R}^n)$. The locally convex topology on $A_0^\omega(\mathbb{R}^n)'$ defined by the family $(q_h)_{h \in A_0^\omega(\mathbb{R}^n)}$ of seminorms is called the topology $\sigma(A_0^\omega(\mathbb{R}^n)', A_0^\omega(\mathbb{R}^n))$ or the weak star topology.

Lemma 2.13. Let $r^* < \infty$. Then $A_K^\omega(\mathbb{R}^n)$ is continuously embedded into $L_w^{r(\cdot)}(\mathbb{R}^n)$ for every compact subsets $K \subset \mathbb{R}^n$, i.e $A_K^\omega(\mathbb{R}^n) \hookrightarrow L_w^{r(\cdot)}(\mathbb{R}^n)$.

Proof. Using the classical result $A_K^\omega(\mathbb{R}^n) \hookrightarrow L_w^{r^*}(\mathbb{R}^n) \cap L_w^{r^*}(\mathbb{R}^n)$ and $L_w^{r^*}(\mathbb{R}^n) \cap L_w^{r(\cdot)}(\mathbb{R}^n) \hookrightarrow L_w^{r(\cdot)}(\mathbb{R}^n)$ by Corollary 2.12, then $A_K^\omega(\mathbb{R}^n) \hookrightarrow L_w^{r(\cdot)}(\mathbb{R}^n)$. $\square$

Theorem 2.14. $L_w^{p(\cdot)}(\mathbb{R}^n)$ is continuously embedded into $A_0^\omega(\mathbb{R}^n)'$.

Proof. Let $f \in L_w^{p(\cdot)}(\mathbb{R}^n)$ and $h \in A_0^\omega(\mathbb{R}^n)$. By definition of $A_0^\omega(\mathbb{R}^n)$, there exists a compact subset $K \subset \mathbb{R}^n$ such that $h \in A_K^\omega(\mathbb{R}^n)$. Suppose that $\frac{1}{p(\cdot)} + \frac{1}{r(\cdot)} = 1$. Then by Hölder inequality for variable exponent Lebesgue spaces and by Lemma 2.13, there exists a $C > 0$ such that

$$\begin{aligned} |< f, h >| &= \left| \int_{\mathbb{R}^n} f(x)h(x)dx \right| \leq \int_{\mathbb{R}^n} |f(x)h(x)| dx \\ &\leq C \|f\|_{p(\cdot)} \|h\|_{r(\cdot)} \leq C \|f\|_{p(\cdot),w} \|h\|_{r(\cdot),\omega} < \infty. \end{aligned} \quad (1)$$

Hence the integral

$$< f, h > = \int_{\mathbb{R}^n} f(x)h(x)dx$$

is well defined. Now define the linear functional $< f, \cdot > : A_0^\omega(\mathbb{R}^n) \rightarrow \mathbb{C}$ for $f \in L_w^{p(\cdot)}(\mathbb{R}^n)$ such that

$$< f, h > = \int_{\mathbb{R}^n} f(x)h(x)dx.$$

It is known that the functional $< f, \cdot >$ is continuous from $A_0^\omega(\mathbb{R}^n)$ into $\mathbb{C}$ if and only if $< f, \cdot >|_{A_K^\omega}$ is continuous from $A_K^\omega(\mathbb{R}^n)$ into $\mathbb{C}$ for all compact subsets $K \subset \mathbb{R}^n$. By Lemma 2.13, there exists a $M_K > 0$ such that

$$\|h\|_{r(\cdot),w} \leq M_K \|h\|_\omega. \quad (2)$$

By (1) and (2),

$$\begin{aligned} |< f, h >| &\leq C \|f\|_{p(\cdot),w} \|h\|_{r(\cdot),\omega} \\ &\leq CM_K \|f\|_{p(\cdot),w} \|h\|_\omega = D_K \|h\|_\omega \end{aligned} \quad (3)$$

where $D_K = CM_K \|f\|_{p(\cdot),w}$. Then we have the inclusion $L_w^{p(\cdot)}(\mathbb{R}^n) \subset A_0^\omega(\mathbb{R}^n)'$. Define the unit map $I : L_w^{p(\cdot)}(\mathbb{R}^n) \rightarrow A_0^\omega(\mathbb{R}^n)'$. Let $h \in A_0^\omega(\mathbb{R}^n)$ be given. Then there exists a compact subset $K \subset \mathbb{R}^n$ such that $h \in A_K^\omega(\mathbb{R}^n)$. Take any semi-norm $q_h \in (q_h)$, $h \in A_0^\omega(\mathbb{R}^n)$ on $A_0^\omega(\mathbb{R}^n)'$. By using (3) we obtain

$$q_h(I(f)) = q_h(f) = |< f, h >| \leq B_K \|f\|_{p(\cdot),w},$$

where $B_K = CM_K \|h\|_\omega$. Then I is continuous map from $L_w^{p(\cdot)}(\mathbb{R}^n)$ into $A_0^\omega(\mathbb{R}^n)'$. The proof is completed. $\square$

3 Weighted Variable Exponent Amalgam Spaces $W(L_w^{p(\cdot)}, L_v^q)$

The space $\left(L_w^{p(\cdot)}(\mathbb{R}^n)\right)_{loc}$ consists of all (classes of) measurable functions f on $\mathbb{R}^n$ such that $f\chi_K \in L^{p(\cdot)}(\mathbb{R}^n)$ for any compact subset $K \subset \mathbb{R}^n$, where χ_K is the characteristic function of K . Since the general hypotheses for the amalgam space $W(L_w^{p(\cdot)}, L_v^q)$ are satisfied by Lemma 2.13 and Theorem 2.14, then $W(L_w^{p(\cdot)}, L_v^q)$ is well defined as follows as in [10].

Let us fix an open set $Q \subset \mathbb{R}^n$ with compact closure. The *variable exponent amalgam space* $W\left(L_w^{p(\cdot)}, L_v^q\right)$ consists of all elements $f \in \left(L_w^{p(\cdot)}(\mathbb{R}^n)\right)_{loc}$ such that $\mathcal{F}_f(z) = \|f\chi_{z+Q}\|_{p(\cdot),w}$ belongs to $L_v^q(\mathbb{R}^n)$; the norm of $W\left(L_w^{p(\cdot)}, L_v^q\right)$ is

$$\|f\|_{W\left(L_w^{p(\cdot)}, L_v^q\right)} = \|\mathcal{F}_f\|_{q,v}.$$

Given a discrete family $X = (x_i)_{i \in I}$ in $\mathbb{R}^n$ and a weighted space $L_w^q(\mathbb{R}^n)$, the *associated weighted sequence space* over X is the appropriate weighted ℓ^q -space ℓ_w^q , the *discrete w* being given by $w(i) = w(x_i)$ for $i \in I$, (see Lemma 3.5 in [12]).

The following theorem, based on Theorem 1 in [10], describes the basic properties of $W\left(L_w^{p(\cdot)}, L_v^q\right)$.

- Theorem 3.1.** (i) $W\left(L_w^{p(\cdot)}, L_v^q\right)$ is a Banach space with norm $\|\cdot\|_{W\left(L_w^{p(\cdot)}, L_v^q\right)}$.
(ii) $W\left(L_w^{p(\cdot)}, L_v^q\right)$ is continuously embedded into $\left(L_w^{p(\cdot)}(\mathbb{R}^n)\right)_{loc}$.
(iii) The space

$$\Lambda_0 = \left\{f \in L_w^{p(\cdot)}(\mathbb{R}^n) : \text{supp}(f) \text{ is compact}\right\}$$

is continuously embedded into $W\left(L_w^{p(\cdot)}, L_v^q\right)$.

(iv) $W\left(L_w^{p(\cdot)}, L_v^q\right)$ does not depend on the particular choice of Q , i.e. different choices of Q define the same space with equivalent norms.

By (iii) and Proposition 2.3 it is easy to see that $C_c(\mathbb{R}^n)$ is continuously embedded into $W\left(L_w^{p(\cdot)}, L_v^q\right)$.

Now by using the techniques in [14], we prove the following proposition.

Proposition 3.2. $W\left(L_w^{p(\cdot)}, L_v^q\right)$ is a BF-space on $\mathbb{R}^n$.

Proposition 3.3. $W\left(L_w^{p(\cdot)}, L_v^q\right)$ is strongly character invariant and the map $t \rightarrow M_t f$ is continuous from $\mathbb{R}^n$ into $W\left(L_w^{p(\cdot)}, L_v^q\right)$.

Proof. It is known that $L_w^{p(\cdot)}(\mathbb{R}^n)$ is strongly character invariant and the function $t \rightarrow M_t f$ is continuous from $\mathbb{R}^n$ into $L_w^{p(\cdot)}(\mathbb{R}^n)$ by Proposition 2.4. Hence the proof is completed by Lemma 1.5. in [24]. $\square$

Proposition 3.4. w_1, w_2, w_3, v_1, v_2 and v_3 be weight functions. Suppose that there exist constants $C_1, C_2 > 0$ such that

$$\forall h \in L_{w_1}^{p_1(\cdot)}(\mathbb{R}^n), \forall k \in L_{w_2}^{p_2(\cdot)}(\mathbb{R}^n), \quad \|hk\|_{p_3(\cdot), w_3} \leq C_1 \|h\|_{p_1(\cdot), w_1} \|k\|_{p_2(\cdot), w_2}$$

and

$$\forall u \in L_{v_1}^{q_1}(\mathbb{R}^n), \forall v \in L_{v_2}^{q_2}(\mathbb{R}^n), \quad \|uv\|_{q_3, v_3} \leq C_2 \|u\|_{q_1, v_1} \|v\|_{q_2, v_2}$$

Then there exists $C > 0$ such that

$$\|fg\|_{W(L_{w_3}^{p_3(\cdot)}, L_{v_3}^{q_3})} \leq C \|f\|_{W(L_{w_1}^{p_1(\cdot)}, L_{v_1}^{q_1})} \|g\|_{W(L_{w_2}^{p_2(\cdot)}, L_{v_2}^{q_2})}$$

for all $f \in W(L_{w_1}^{p_1(\cdot)}, L_{v_1}^{q_1})$ and $g \in W(L_{w_2}^{p_2(\cdot)}, L_{v_2}^{q_2})$. In other words

$$W(L_{w_1}^{p_1(\cdot)}, L_{v_1}^{q_1}) W(L_{w_2}^{p_2(\cdot)}, L_{v_2}^{q_2}) \subset W(L_{w_3}^{p_3(\cdot)}, L_{v_3}^{q_3}).$$

Proof. If $f \in W(L_{w_1}^{p_1(\cdot)}, L_{v_1}^{q_1})$ and $g \in W(L_{w_2}^{p_2(\cdot)}, L_{v_2}^{q_2})$, then we have

$$\begin{aligned} \|fg\|_{W(L_{w_3}^{p_3(\cdot)}, L_{v_3}^{q_3})} &= \left\| \|fg\chi_{z+Q}\|_{p_3(\cdot), w_3} \right\|_{q_3, v_3} \\ &= \left\| \|(f\chi_{z+Q})(g\chi_{z+Q})\|_{p_3(\cdot), w_3} \right\|_{q_3, v_3} \\ &\leq C_1 \left\| \|f\chi_{z+Q}\|_{p_1(\cdot), w_1} \|g\chi_{z+Q}\|_{p_2(\cdot), w_2} \right\|_{q_3, v_3} \\ &= C_1 \|\mathcal{F}_f \mathcal{F}_g\|_{q_3, v_3} \leq C_1 C_2 \|\mathcal{F}_f\|_{q_1, v_1} \|\mathcal{F}_g\|_{q_2, v_2} \\ &= C \|f\|_{W(L_{w_1}^{p_1(\cdot)}, L_{v_1}^{q_1})} \|g\|_{W(L_{w_2}^{p_2(\cdot)}, L_{v_2}^{q_2})} \end{aligned}$$

and the proof is complete. $\square$

Proposition 3.5. (i) If $p_1(\cdot) \leq p_2(\cdot)$, $q_2 \leq q_1$, $w_1 \prec w_2$ and $v_1 \prec v_2$, then

$$W(L_{w_2}^{p_2(\cdot)}, L_{v_2}^{q_2}) \subset W(L_{w_1}^{p_1(\cdot)}, L_{v_1}^{q_1}).$$

(ii) If $p_1(\cdot) \leq p_2(\cdot)$, $q_2 \leq q_1$, $w_1 \prec w_2$ and $v_1 \prec v_2$, then

$$W(L_{w_1}^{p_1(\cdot)} \cap L_{w_2}^{p_2(\cdot)}, L_{v_2}^{q_2}) \subset W(L_{w_1}^{p_1(\cdot)}, L_{v_1}^{q_1}).$$

Proof. (i) Let $f \in W \left(L_{w_2}^{p_2(\cdot)}, L_{v_2}^{q_2} \right)$ be given. Since $p_1(\cdot) \leq p_2(\cdot)$ and $w_1 \prec w_2$ then $L_{w_2}^{p_2(\cdot)}(z + Q) \hookrightarrow L_{w_1}^{p_1(\cdot)}(z + Q)$ and

$$\begin{aligned} \|f\chi_{z+Q}\|_{p_1(\cdot), w_1} &\leq C(\mu(z + Q) + 1) \|f\chi_{z+Q}\|_{p_2(\cdot), w_2} \\ &\leq C(\mu(Q) + 1) \|f\chi_{z+Q}\|_{p_2(\cdot), w_2} \end{aligned}$$

for all $z \in \mathbb{R}^n$ by Theorem 2.8 in [20], where μ is the Lebesgue measure. Hence by the solidity of $L_{v_2}^{q_2}(\mathbb{R}^n)$ we have

$$W \left(L_{w_2}^{p_2(\cdot)}, L_{v_2}^{q_2} \right) \subset W \left(L_{w_1}^{p_1(\cdot)}, L_{v_2}^{q_2} \right).$$

It is known by Proposition 3.7 in [12], that

$$W \left(L_{w_1}^{p_1(\cdot)}, L_{v_2}^{q_2} \right) \subset W \left(L_{w_1}^{p_1(\cdot)}, L_{v_1}^{q_1} \right)$$

if and only if $\ell_{v_2}^{q_2} \subset \ell_{v_1}^{q_1}$, where $\ell_{v_2}^{q_2}$ and $\ell_{v_1}^{q_1}$ are the associated sequence spaces of $L_{v_2}^{q_2}(\mathbb{R}^n)$ and $L_{v_1}^{q_1}(\mathbb{R}^n)$ respectively. Since $q_2 \leq q_1$ and $v_1 \prec v_2$, then $\ell_{v_2}^{q_2} \subset \ell_{v_1}^{q_1}$ [14]. This completes the proof.

(ii) The proof of this part is easy by (i). $\square$

The following Proposition was proved by [3].

Proposition 3.6. Let B be any solid space. If $q_2 \leq q_1$ and $v_1 \prec v_2$, then we have

$$W(B, L_{v_1}^{q_1} \cap L_{v_2}^{q_2}) = W(B, L_{v_2}^{q_2}).$$

Corollary 3.7. (i) If $p_1^*, p_2^* < \infty$, $L_{w_1}^{p_1(\cdot)}(\mathbb{R}^n) \subset L_{w_2}^{p_2(\cdot)}(\mathbb{R}^n)$, $q_2 \leq q_1$, $q_4 \leq q_3$, $q_4 \leq q_2$, $v_1 \prec v_2$, $v_3 \prec v_4$ and $v_2 \prec v_4$, then

$$W \left(L_{w_1}^{p_1(\cdot)}, L_{v_3}^{q_3} \cap L_{v_4}^{q_4} \right) \subset W \left(L_{w_2}^{p_2(\cdot)}, L_{v_1}^{q_1} \cap L_{v_2}^{q_2} \right).$$

(ii) If $p_1(x) \leq p_3(x)$, $p_2(x) \leq p_4(x)$, $q_2 \leq q_1$, $q_4 \leq q_3$, $q_4 \leq q_2$, $w_1 \prec w_3$, $w_2 \prec w_4$, $v_1 \prec v_2$, $v_3 \prec v_4$ and $v_2 \prec v_4$, then

$$W \left(L_{w_3}^{p_3(\cdot)} \cap L_{w_4}^{p_4(\cdot)}, L_{v_3}^{q_3} \cap L_{v_4}^{q_4} \right) \subset W \left(L_{w_1}^{p_1(\cdot)} \cap L_{w_2}^{p_2(\cdot)}, L_{v_1}^{q_1} \cap L_{v_2}^{q_2} \right).$$

Proposition 3.8. If $1 \leq q \leq \infty$ and $v \in L^q(\mathbb{R}^n)$, then $L_w^{p(\cdot)}(\mathbb{R}^n) \subset W \left(L_w^{p(\cdot)}, L_v^q \right)$.

Proof. If $1 \leq q < \infty$ and $v \in L^q(\mathbb{R}^n)$, we have

$$\begin{aligned} \|f\|_{W(L_w^{p(\cdot)}, L_v^q)} &= \left\| \|f\chi_{z+Q}\|_{p(\cdot), w} \right\|_{q, v} \\ &= \left\{ \int_{\mathbb{R}^n} \|f\chi_{z+Q}\|_{p(\cdot), w}^q v^q(z) dz \right\}^{\frac{1}{q}} \\ &\leq \left\{ \int_{\mathbb{R}^n} \|f\|_{p(\cdot), w}^q v^q(z) dz \right\}^{\frac{1}{q}} \\ &= \|f\|_{p(\cdot), w} \|v\|_q. \end{aligned}$$

Hence $L_w^{p(\cdot)}(\mathbb{R}^n) \subset W(L_w^{p(\cdot)}, L_v^q)$. Similarly, for $q = \infty$, we obtain

$$\|f\|_{W(L_w^{p(\cdot)}, L_v^\infty)} = \left\| \|f\chi_{z+Q}\|_{p(\cdot), w} v \right\|_\infty \leq \|f\|_{p(\cdot), w} \|v\|_\infty.$$

Then $L_w^{p(\cdot)}(\mathbb{R}^n) \subset W(L_w^{p(\cdot)}, L_v^\infty)$. $\square$

Proposition 3.9. Let $1 \leq q_0, q_1 < \infty$. If $p_0(\cdot)$ and $p_1(\cdot)$ are variable exponents with $1 < p_{j,*} \leq p_j^* < \infty$, $j = 0, 1$. Then, for $\theta \in (0, 1)$, we have

$$\left[W(L_{w_0}^{p_0(\cdot)}, L_{v_0}^{q_0}), W(L_{w_1}^{p_1(\cdot)}, L_{v_1}^{q_1}) \right]_{[\theta]} = W(L_w^{p_\theta(\cdot)}, L_v^{q_\theta})$$

where $\frac{1}{p_\theta(x)} = \frac{1-\theta}{p_0(x)} + \frac{\theta}{p_1(x)}$, $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, $w = w_0^{1-\theta} w_1^\theta$ and $v = v_0^{1-\theta} v_1^\theta$.

Proof. By Theorem 2.2 in [11] the interpolation space $\left[W(L_{w_0}^{p_0(\cdot)}, L_{v_0}^{q_0}), W(L_{w_1}^{p_1(\cdot)}, L_{v_1}^{q_1}) \right]_{[\theta]}$ is $W\left(\left[L_{w_0}^{p_0(\cdot)}, L_{w_1}^{p_1(\cdot)}\right]_{[\theta]}, \left[L_{v_0}^{q_0}, L_{v_1}^{q_1}\right]_{[\theta]}\right)$. We know that $\left[L_{v_0}^{q_0}, L_{v_1}^{q_1}\right]_{[\theta]} = L_v^{q_\theta}$ and by Corollary A.2. in [7] that $\left[L_{w_0}^{p_0(\cdot)}, L_{w_1}^{p_1(\cdot)}\right]_{[\theta]} = L_w^{p_\theta(\cdot)}$. This completes the proof. $\square$

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