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SOME FUNCTIONAL EQUATIONS CHARACTERIZING POLYNOMIALS

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ABSTRACT. We present a method of solving functional equations of the type

$$F(x) - F(y) = (x-y)\big[b_1 f(\alpha_1 x + \beta_1 y) + \cdots + b_n f(\alpha_n x + \beta_n y)\big],$$

where $f, F\colon P \to P$ are unknown functions acting on an integral domain P and parameteres $b_1, \ldots, b_n; \alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n \in P$ are given. We prove that under some assumptions on the parameters involved, all solutions to such kind of equations are polynomials. We use this method to solve some concrete equations of this type. For example, the equation

$$8\big[F(x) - F(y)\big] = (x-y)\left[f(x) + 3f\left(\frac{x+2y}{3}\right) + 3f\left(\frac{2x+y}{3}\right) + f(y)\right] \quad (1)$$

for $f, F\colon \mathbb{R} \to \mathbb{R}$ is solved without any regularity assumptions. It is worth noting that (1) stems from a well-known quadrature rule used in numerical analysis.

1. Introduction

We deal with functional equations of some specific form. In the monograph of T. Riedel and P. Sahoo [8], the following equation stemming from Simpson quadrature rule

$$F(y) - F(x) = (y-x)\left[\frac{1}{6}f(x) + \frac{2}{3}f\left(\frac{x+y}{2}\right) + \frac{1}{6}f(y)\right]$$

is considered. We present a general solution to the generalized version of this equation, i.e.,

$$F(x) - F(y) = (x-y)\big[b_1 f(\alpha_1 x + \beta_1 y) + \cdots + b_n f(\alpha_n x + \beta_n y)\big].$$

In the last section we enumerate particular cases of this equation in order to show the usefulness of the obtained results.

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We obtain it using the method based on the use of the lemma established by M. S a b l i k ([9, Lemma 2.3]) and improved by I. P a w l i k o w s k a (cf. [7]). We show at the end of the paper that our method is applicable also for some other functional equations.

In this paper, by a *polynomial function of order* n, we mean any solution to the functional equation $\Delta_h^{n+1} f(x) = 0$, where Δ_h^{n+1} stands for the $(n+1)$th iterate of the difference operator $\Delta_h f(x) = f(x+h) - f(x)$. Observe that a continuous polynomial function of order n is a polynomial of degree at most n (see [5, Theorem 4, p. 398] or [6, Theorem 15.9.4, p. 452]).

Let us now quote S a b l i k's result. First, we need some notations. Let G, H be Abelian groups and $SA^0(G,H) := H$, $SA^1(G,H) := \mathrm{Hom}(G,H)$ (i.e., the group of all homomorphisms from G into H), and for $i \in \mathbb{N}$, $i \geq 2$, let $SA^i(G,H)$ be the group of all i–additive and symmetric mappings from G^i into H. Furthermore, let $\mathscr{P} := \big\{(\alpha,\beta) \in \mathrm{Hom}(G,G)^2 : \alpha(G) \subset \beta(G)\big\}$. Finally, for $x \in G$ let $x^i = \underbrace{(x,\ldots,x)}_{i}$, $i \in \mathbb{N}$.

Lemma 1. *Fix $N \in \mathbb{N} \cup \{0\}$ and let $I_0, \ldots, I_N$ be finite subsets of $\mathscr{P}$. Suppose that H is uniquely divisible by $N!$ and let the functions $\varphi_i \colon G \to SA^i(G,H)$ and $\psi_{i,(\alpha,\beta)} \colon G \to SA^i(G,H)$ $((\alpha,\beta) \in I_i,\ i = 0,\ldots,N)$ satisfy*

$$\varphi_N(x)\big(y^N\big) + \sum_{i=0}^{N-1} \varphi_i(x)(y^i) = \sum_{i=0}^{N} \sum_{(\alpha,\beta)\in I_i} \psi_{i,(\alpha,\beta)}\big(\alpha(x) + \beta(y)\big)(y^i) \tag{2}$$

for every $x, y \in G$. Then φ_N is a polynomial function of order at most $k-1$, where

$$k = \sum_{i=0}^{N} \mathrm{card}\left(\bigcup_{s=i}^{N} I_s\right).$$

In the original S a b l i k's result, the order of φ_N was not greater than k, while I. P a w l i k o w s k a noticed that it can be lowered by 1.

In our considerations, we use Lemma 1 only for $G = H = \mathbb{R}$ and $N = 1$ and we consider only homomorphisms of the form $x \mapsto \alpha x$, $\alpha \in \mathbb{R}$. Consequently, let

$$\mathscr{P} := \big\{(\alpha,\beta) \in \mathbb{R}^2 : \alpha = 0 \quad \text{or} \quad \beta \neq 0\big\}$$

(the case $\alpha(\mathbb{R}) \not\subset \beta(\mathbb{R})$ may occur only if $\alpha \neq 0$, $\beta = 0$); in this setting, Lemma 1 can be rephrased in the form

Lemma 2. *Let I_0, I_1 be finite subsets of $\mathscr{P}$. Let the functions $\varphi_i, \psi_{i,(\alpha,\beta)} \colon \mathbb{R} \to \mathbb{R}$ $\big((\alpha,\beta) \in I_i,\ i = 0,1\big)$ satisfy*

$$\varphi_1(x)y + \varphi_0(x) = \sum_{(\alpha,\beta)\in I_0} \psi_{0,(\alpha,\beta)}(\alpha x + \beta y) + y \left[\sum_{(\gamma,\delta)\in I_1} \psi_{1,(\gamma,\delta)}(\gamma x + \delta y)\right] \tag{3}$$

for any $x, y \in \mathbb{R}$. Then φ_1 is a polynomial function of order at most equal to $\operatorname{card}(I_0 \cup I_1) + \operatorname{card} I_1 - 1$.

2. Preliminaries

Lemma 3. *Let P be an integral domain with unit element 1 uniquely divisible by 2 such that $n1 \neq 0$ for all $n \in \mathbb{N}$ and let the functions $f, F\colon P \to P$ satisfy the equation*

$$F(x) - F(y) = (x-y)\big[b_1 f(\alpha_1 x + \beta_1 y) + \cdots + b_n f(\alpha_n x + \beta_n y)\big] \tag{4}$$

for all $x, y \in P$ and some $b_1, \ldots, b_n, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in P$. Further, let f be a function of the form

$$f(x) = a_k(x) + a_{k-1}(x) + \cdots + a_1(x) + a_0, \qquad x \in P, \tag{5}$$

where $a_i, i \in \{1, \ldots, k\}$ stands for a monomial function of order i and $a_0 \in P$ is some constant.

Then each function $a_i, i \in \{1, \ldots, k\}$ also satisfies (4) with some function F_i.

Proof. We may assume $F(0) = 0$, so substituting $y = 0$ in (4), we obtain

$$F(x) = x\big[b_1 f(\alpha_1 x) + \cdots + b_n f(\alpha_n x)\big], \quad x \in P.$$

Using this formula in (4), we get

$$\begin{aligned} &x\big[b_1 f(\alpha_1 x) + \cdots + b_n f(\alpha_n x)\big] - y\big[b_1 f(\alpha_1 y) + \cdots + b_n f(\alpha_n y)\big] \\ &= (x-y)\big[b_1 f(\alpha_1 x + \beta_1 y) + \cdots + b_n f(\alpha_n x + \beta_n y)\big], \quad x, y \in P. \end{aligned} \tag{6}$$

It is easy to see that if f, g are solutions to this equation, then also $f + g$ satisfies this equation. Moreover, if f satisfies (6) and $\alpha \in P$ is given, then also αf is a solution. Now, substitute ax, ay in place of x and y. We obtain (after cancelling a on both sides) that

$$\begin{aligned} &x\big[b_1 f(\alpha_1 ax) + \cdots + b_n f(\alpha_n ax)\big] - y\big[b_1 f(\alpha_1 ay) + \cdots + b_n f(\alpha_n ay)\big] \\ &= (x-y)\big[b_1 f(\alpha_1 ax + \beta_1 ay) + \cdots + b_n f(\alpha_n ax + \beta_n ay)\big], \quad x, y \in P \end{aligned}$$

which means that the function $x \mapsto f(ax)$ also satisfies our equation (if f does).

Now, consider f of the form (5), then g_1 given by

$$\begin{aligned} g_1(x) :=& 2^k f(x) - f(2x) = (2^k - 2^{k-1}) a_{k-1}(x) \\ &+ (2^k - 2^{k-2}) a_{k-2}(x) + \cdots + (2^k - 1) a_0, \quad x \in P. \end{aligned}$$

satisfies (6). Thus, we may now put $g_2(x) := 2^{k-1} g_1(x) - g_1(2x), \quad x \in P$. The function g_2 is clearly of the form

$$g_2(x) = A_{k-2}^2 a_{k-2}(x) + \cdots + A_0^2 a_0, \quad x \in P$$

for some numbers $A_i^2, i = 0, \dots, k-2$. Repeating this procedure finitely many times, we obtain that

$$g_{k-1}(x) = A_1^{k-1} a_1(x) + A_0^{k-1} a_0, \qquad x \in P$$

satisfies the equation (6). However, since every constant is a solution of this equation, a_1 has to satisfy it. But

$$g_{k-2}(x) = A_2^{k-2} a_2(x) + A_1^{k-2} a_1(x) + A_0^{k-2} a_0, \qquad x \in P.$$

The functions g_{k-2} and $x \mapsto A_1^{k-2} a_1(x) + A_0^{k-2} a_0$, satisfy (6), thus also a_2 is a solution to (6). In the same way, one can show that every a_i, $i = 1, 2, \dots, k$ satisfy the equation (6), and consequently, also the equation (4) (with $F_i(x) = x\big[b_1 a_i(\alpha_1 x) + \dots + b_n a_i(\alpha_n x)\big]$). □

Lemma 4. *Let P be an integral domain with unit element $\mathbf{1}$ such that $1+1 \neq 0$ and $\mathbf{1}+\mathbf{1}+\mathbf{1} \neq 0$, let $A_3 \colon P^3 \to P$ be a symmetric and 3-additive function. Further, let $a_3 \colon P \to P$ be the diagonalization of A_3 satisfying (4) for all $x, y \in P$ and some $b_1, \dots, b_n$, $\alpha_1, \dots, \alpha_n$, $\beta_1, \dots, \beta_n \in P$ such that for all $i = 1, \dots, n$ and $x, y, z \in P$ we have*

$$A_3(\alpha_i x, y, z) = \alpha_i A_3(x, y, z) \quad \text{and} \quad A_3(\beta_i x, y, z) = \beta_i A_3(x, y, z).$$

If $\xi, \eta, \gamma, \delta \in P$ are defined as follows

$$\begin{aligned} \xi &= b_1 \alpha_1 \beta_1^2 + \dots + b_n \alpha_n \beta_n^2, \\ \eta &= b_1 \alpha_1^2 \beta_1 + \dots + b_n \alpha_n^2 \beta_n, \\ \gamma &= b_1 \alpha_1^3 + \dots + b_n \alpha_n^3, \\ \delta &= b_1 \beta_1^3 + \dots + b_n \beta_n^3, \end{aligned} \tag{7}$$

then the following assertions hold true:

(i) *if $\gamma \neq \delta$, then $a_3 = 0$;*

(ii) *if $\gamma = \delta$ and $\xi \neq \eta$, then $a_3 = 0$;*

(iii) *if $\gamma = \delta$ and $\xi = \eta \neq 0$, then $a_3(x) = ax^3$ for some $a \in P$ and all $x \in P$ and if $a \neq 0$, then $\delta = 3\xi$;*

(iv) *if $\gamma = \delta \neq 0$ and $\xi = \eta = 0$, then $a_3 = 0$;*

(v) *if $\gamma = \delta = \xi = \eta = 0$, then a_3 may be any function of the assumed form.*

Conversely, in each of the above cases the function a_3 with the function F given by the formula

$$F(x) = x\big[b_1 a_3(\alpha_1 x) + \dots + b_n a_3(\alpha_n x)\big] + c, \qquad x \in P, \tag{8}$$

where $c \in P$ is some constant, is a solution to (4).

P r o o f. Let us rewrite the equation (4) for $f = a_3$

$$F(x)-F(y) = (x-y)[b_1a_3(\alpha_1x+\beta_1y)+\cdots+b_na_3(\alpha_nx+\beta_ny)], \quad x,y\in P. \tag{9}$$

Assuming without loss of generality $F(0) = 0$, take here $y = 0$ and substitute the obtained formula to (9)

$$x\big[b_1a_3(\alpha_1x)+\cdots+b_na_3(\alpha_nx)-b_1a_3(\alpha_1x+\beta_1y)-\cdots-b_na_3(\alpha_nx+\beta_ny)\big]$$
$$= y\big[b_1a_3(\alpha_1y)+\cdots+b_na_3(\alpha_ny)-b_1a_3(\alpha_1x+\beta_1y)-\cdots-b_na_3(\alpha_nx+\beta_ny)\big].$$

Using here the equality $a_3(x) = A_3(x,x,x)$, after some calculations we get for all $x,y\in P$

$$\begin{aligned}&3(b_1\alpha_1\beta_1^2+\cdots+b_n\alpha_n\beta_n^2)(y-x)A_3(x,y,y)\\ &\qquad+3(b_1\alpha_1^2\beta_1+\cdots+b_n\alpha_n^2\beta_n)(y-x)A_3(x,x,y)\\ &=(b_1\alpha_1^3+\cdots+b_n\alpha_n^3)y\big[a_3(y)-a_3(x)\big]\\ &\qquad+(b_1\beta_1^3+\cdots+b_n\beta_n^3)(x-y)a_3(y).\end{aligned}$$

In view of the notations from (4), we have

$$3(y-x)\big[\xi A_3(x,y,y)+\eta A_3(x,x,y)\big] = \big[\delta x+(\gamma-\delta)y\big]a_3(y)-\gamma ya_3(x). \tag{10}$$

Now, substitute $-y$ in place of y in (10). Then

$$-3(x+y)\big[\xi A_3(x,y,y)-\eta A_3(x,x,y)\big] = -\big[\delta x-(\gamma-\delta)y\big]a_3(y)+\gamma ya_3(x). \tag{11}$$

Adding the equations (10) and (11), we arrive at

$$3\xi\big[(y-x)-(x+y)\big]A_3(x,y,y)+3\eta\big[(y-x)+(x+y)\big]A_3(x,x,y) = 2y\big[\gamma-\delta\big]a_3(y),$$

i.e.,

$$-3\xi xA_3(x,y,y)+3\eta yA_3(x,x,y) = (\gamma-\delta)ya_3(y), \qquad x,y\in P. \tag{12}$$

Further, in (12) we put $2x$ in place of x. Thus

$$-12\xi xA_3(x,y,y)+12\eta yA_3(x,x,y) = (\gamma-\delta)ya_3(y), \qquad x,y\in P. \tag{13}$$

Multiplying (12) by 4, we have

$$-12\xi xA_3(x,y,y)+12\eta yA_3(x,x,y) = 4(\gamma-\delta)ya_3(y), \qquad x,y\in P. \tag{14}$$

After comparing equations (14) and (13), we observe that

$$(\gamma-\delta)a_3(y) = 0, \qquad y\neq 0.$$

Clearly, this equation is also satisfied for $y = 0$.

Now, if $\gamma\neq\delta$, then $a_3(y) = 0$ for all $y\in P$ and (i) is proved.

Assume now that $\gamma = \delta$. Then equation (12) takes the form

$$\xi xA_3(x,y,y) = \eta yA_3(x,x,y), \qquad x,y\in P. \tag{15}$$

Take here $y = x$

$$(\xi-\eta)a_3(x) = 0, \qquad x\in P, \quad x,y\in P.$$

Similarly as above, if $\xi \neq \eta$, then $a_3(x) = 0$, $x \in P$ and (ii) holds.

This means that we may assume that $\xi = \eta$. Let us now consider the case $\xi = \eta = 0$. Then by (10) we have

$$\gamma x a_3(y) = \gamma y a_3(x), \quad x, y \in P.$$

Taking here $y = 2x$, we get

$$\gamma a_3(x) = 0, \qquad x \in P, \quad x \neq 0.$$

If $\gamma \neq 0$, then $a_3(x) = 0$ for $x \in P$ and (iv) holds. If, on the other hand, $\gamma = 0$, then

$$\gamma = \delta = \eta = \xi = 0.$$

In this case the equation (10) reduces to a trivial one, so (v) holds.

This means that from this point we may assume that $\gamma = \delta$ and $\xi = \eta \neq 0$ (as in (iii)). Using these facts in (15), we obtain

$$xA_3(x, y, y) = yA_3(x, x, y), \qquad x, y \in P. \tag{16}$$

Taking here $x + y$ in place of x, we may write

$$(x - y)A_3(x, y, y) - yA_3(x, x, y) = -xa_3(y), \qquad x, y \in P.$$

Thus in view of (16), we obtain

$$xa_3(y) = yA_3(x, y, y), \qquad x, y \in P. \tag{17}$$

Interchanging x with y in this equation, we arrive at

$$ya_3(x) = xA_3(x, x, y), \qquad x, y \in P. \tag{18}$$

Now, multiply (17) by x, and (18) by y^2 to get

$$x^2 a_3(y) = xyA_3(x, y, y), \qquad x, y \in P \tag{19}$$

and

$$y^3 a_3(x) = xy^2 A_3(x, x, y), \qquad x, y \in P. \tag{20}$$

On the other hand, multiply (16) by y and use (19)

$$x^2 a_3(y) = y^2 A_3(x, x, y),$$

which multiplied by x gives us

$$x^3 a_3(y) = xy^2 A_3(x, x, y), \qquad x, y \in P.$$

Comparing this equation with (20), we obtain

$$x^3 a_3(y) = y^3 a_3(x), \qquad x, y \in P$$

which after substituting $y = 1$ yields

$$a_3(x) = a_3(1)x^3, \qquad x \in P. \tag{21}$$

Now, it is enough to show that if $a_3(1) \neq 0$, then $\delta = 3\xi$. To this end, use (21) in (17)

$$a_3(\mathbf{1})xy^3 = yA_3(x,y,y),$$

i.e.,

$$A_3(x,y,y) = a_3(\mathbf{1})xy^2, \qquad x,y \in P \tag{22}$$

and by symmetry of A_3

$$A_3(x,x,y) = a_3(\mathbf{1})x^2y, \qquad x,y \in P. \tag{23}$$

By (22) and (23) and using the equalities $\gamma = \delta$ and $\eta = \xi$ in (10), we get for all $x, y \in P$

$$3\xi(y-x)a_3(\mathbf{1})(xy^2 + x^2y) = \delta a_3(\mathbf{1})[xy^3 - x^3y], \qquad x,y \in P.$$

Thus, $3\xi = \delta$ which proves (iii).

On the other hand, it is easy to show that all functions of the forms (i)–(v) satisfy equation (10) (and (9) with F given by (8)). □

Lemma 5. *Let P be an integral domain with a unit element $\mathbf{1}$ such that $\mathbf{1}+\mathbf{1} \neq 0$ and let $A_2\colon P^2 \to P$ be a symmetric and 2-additive function. Further, let $a_2 : P \to P$ be the diagonalization of A_2 satisfying the equation (4) for all $x, y \in P$ and some $b_1, \dots, b_n$, $\alpha_1, \dots, \alpha_n$, $\beta_1, \dots, \beta_n \in P$ such that for all $i = 1, \dots, n$ and $x, y \in P$ we have*

$$A_2(\alpha_i x, y) = \alpha_i A_2(x,y) \quad \text{and} \quad A_2(\beta_i x, y) = \beta_i A_2(x,y).$$

If $\gamma, \delta, \eta \in P$ are defined as follows

$$\begin{aligned} \gamma &= b_1\alpha_1^2 + \cdots + b_n\alpha_n^2, \\ \delta &= b_1\beta_1^2 + \cdots + b_n\beta_n^2, \\ \eta &= b_1\alpha_1\beta_1 + \cdots + b_n\alpha_n\beta_n, \end{aligned} \tag{24}$$

then the following assertions hold true:

(i) *if $\gamma \neq \delta$, then $a_2 = 0$;*

(ii) *if $\gamma = \delta$ and $\gamma + 2\eta \neq 0$, then $a_2(x) = ax^2$ for some $a \in P$ and all $x \in P$;*

(iii) *if $\gamma = \delta$, $\gamma + 2\eta = 0$ and $a_2 \neq 0$, then $\gamma = \delta = \eta = 0$ and a_2 may be any function.*

Conversely, in each of the above cases, the function a_2 with the function F given by the formula

$$F(x) = x\big[b_1a_2(\alpha_1 x) + \cdots + b_na_2(\alpha_n x)\big] + c, \qquad x \in P, \tag{25}$$

where $c \in P$ is some constant, is a solution to (4).

Proof. Similarly as in the proof of Lemma 4 from (4), we obtain

$$\begin{aligned} &x\big[b_1a_2(\alpha_1 x) + \cdots + b_na_2(\alpha_n x) - b_1a_2(\alpha_1 x + \beta_1 y) - \cdots - b_na_2(\alpha_n x + \beta_n y)\big] \\ &= y\big[b_1a_2(\alpha_1 y) + \cdots + b_na_2(\alpha_n y) - b_1a_2(\alpha_1 x + \beta_1 y) - \cdots - b_na_2(\alpha_n x + \beta_n y)\big]. \end{aligned}$$

Thus using here the form of a_2 we get

$$\begin{aligned}&-x\big[(b_1\beta_1^2+\cdots+b_n\beta_n^2)a_2(y)+2(b_1\alpha_1\beta_1+\cdots+b_n\alpha_n\beta_n)A_2(x,y)\big]\\&=y\big[(b_1\alpha_1^2+\cdots+b_n\alpha_n^2-b_1\beta_1^2-\cdots-b_n\beta_n^2)a_2(y)-(b_1\alpha_1^2+\cdots+b_n\alpha_n^2)a_2(x)\\&\qquad-2(b_1\alpha_1\beta_1+\cdots+b_n\alpha_n\beta_n)A_2(x,y)\big],\end{aligned}$$

or, equivalently,

$$\gamma y\big[a_2(x)-a_2(y)\big]-\delta(x-y)a_2(y)=2\eta(x-y)A_2(x,y),\qquad x,y\in P. \tag{26}$$

Interchanging x with y in this equation, we have

$$\gamma x\big[a_2(x)-a_2(y)\big]-\delta(x-y)a_2(x)=2\eta(x-y)A_2(x,y),\qquad x,y\in P. \tag{27}$$

Comparing the equations (27) and (26), we are able to write

$$(\gamma-\delta)(x-y)\big[a_2(x)-a_2(y)\big]=0\qquad\text{for all}\quad x,y\in P.$$

Now, substitute $y=0$, then since $a_2(0)=0$,

$$(\gamma-\delta)a_2(x)=0,\qquad x\neq 0,$$

which is clearly satisfied also for $x=0$.

Now, we consider two cases:

1° $\gamma\neq\delta$.

In this case, we obtain $a_2(x)=0$ for every $x\in P$, which proves (i).

2° $\gamma=\delta$.

From (26), we have

$$\gamma\big[ya_2(x)-xa_2(y)\big]=2\eta(x-y)A_2(x,y),\qquad x,y\in P. \tag{28}$$

If we substitute here $x+y$ and $x-y$ in place of x and y, respectively, then, after some calculations, we arrive at

$$a_2(x)(2\gamma y+4\eta y)+a_2(y)(2\gamma y-4\eta y)=4\gamma xA_2(x,y),$$

thus

$$(\gamma+2\eta)ya_2(x)+(\gamma-2\eta)ya_2(y)=2\gamma xA_2(x,y),\qquad x,y\in P. \tag{29}$$

Now, interchange x with y. We have

$$(\gamma+2\eta)xa_2(y)+(\gamma-2\eta)xa_2(x)=2\gamma yA_2(x,y),\qquad x,y\in P. \tag{30}$$

Multiplying (29) by y and (30) by x and comparing the obtained equations, we may write

$$(\gamma+2\eta)y^2a_2(x)+(\gamma-2\eta)y^2a_2(y)=(\gamma+2\eta)x^2a_2(y)+(\gamma-2\eta)x^2a_2(x),$$

which gives us

$$a_2(x)\big[y^2(\gamma+2\eta)-x^2(\gamma-2\eta)\big]=a_2(y)\big[x^2(\gamma+2\eta)-y^2(\gamma-2\eta)\big],\ x,y\in P.$$

Substituting $y=\mathbf{1}$ in this equation, we have

$$a_2(x)\big[(\gamma+2\eta)-x^2(\gamma-2\eta)\big]=a_2(\mathbf{1})\big[x^2(\gamma+2\eta)-(\gamma-2\eta)\big],\ x\in P. \tag{31}$$

Now, we are going to show that if $a_2 \neq 0$, then $\gamma - 2\eta = 0$. To this end, put $y = 2x$ in (28)

$$\gamma\big[2xa_2(x) - xa_2(2x)\big] = -2\eta x A_2(x, 2x)$$

i.e.,

$$(\gamma - 2\eta)a_2(x) = 0, \qquad x \in P, \quad x \neq 0.$$

Consequently,

$$\gamma - 2\eta = 0 \tag{32}$$

and using this equality in (31), we obtain

$$(\gamma + 2\eta)a_2(x) = x^2(\gamma + 2\eta)a_2(1), \qquad x \in P. \tag{33}$$

Consider the case $\gamma + 2\eta \neq 0$, then

$$a_2(x) = a_2(1)x^2, \qquad x \in P$$

which proves (ii). On the other hand, if $\gamma + 2\eta = 0$, then together with (32) we have $\gamma = \eta = 0$ and the equation 33 is trivial, which gives (iii).

To prove the converse assertion it is enough to use the obtained forms of a_2 and F given by (8) and check that the equation (4) is satisfied. □

Lemma 6. *Let P be an integral domain and let $a_1 \colon P \to P$ be an additive function satisfying equation (4) for all $x, y \in P$ and some $b_1, \dots, b_n$, $\alpha_1, \dots, \alpha_n$, $\beta_1, \dots, \beta_n \in P$ such that for all $i = 1, \dots, n$ and $x \in P$ we have*

$$a_1(\alpha_i x) = \alpha_i a_1(x) \quad \text{and} \quad a_1(\beta_i x) = \beta_i a_1(x).$$

If $\gamma, \delta \in P$ are defined as follows

$$\begin{aligned} \gamma &= b_1\alpha_1 + \cdots + b_n\alpha_n, \\ \delta &= b_1\beta_1 + \cdots + b_n\beta_n, \end{aligned} \tag{34}$$

then the following assertions hold true:

(i) *if $\gamma \neq \delta$, then $a_1 = 0$;*

(ii) *if $\gamma = \delta \neq 0$, then $a_1(x) = ax$ for some $a \in P$ and all $x \in P$;*

(iii) *if $\gamma = \delta = 0$, then a_1 may be any function.*

Conversely, in each of the above cases function a_1 with

$$F(x) = x\big[b_1a_1(\alpha_1 x) + \cdots + b_n a_1(\alpha_n x)\big] + c \tag{35}$$

where $c \in P$ is some constant, is a solution of (4).

P r o o f. Similarly as in the proof of Lemma 4 and Lemma 5 from (4) we obtain

$$\begin{aligned} &x\big[b_1a_1(\alpha_1 x) + \cdots + b_n a_1(\alpha_n x) - b_1a_1(\alpha_1 x + \beta_1 y) - \cdots - b_n a_1(\alpha_n x + \beta_n y)\big] \\ &= y\big[b_1a_1(\alpha_1 y) + \cdots + b_n a_1(\alpha_n y) - b_1a_1(\alpha_1 x + \beta_1 y) - \cdots - b_n a_1(\alpha_n x + \beta_n y)\big]. \end{aligned}$$

Thus, using here additivity of a_1 and in view of the notations (6), we get

$$\gamma y a_1(x) = a_1(y)\big[(\gamma-\delta)y + \delta x\big], \qquad x, y \in P. \tag{36}$$

If we substitute $x+y$ in place of x in (36), then, after some calculations, we may write

$$\gamma y a_1(x) = \delta x a_1(y), \qquad x, y \in P. \tag{37}$$

Comparing the obtained equation and the equation (36), we get

$$(\gamma-\delta)y a_1(y) = 0, \qquad y \in P.$$

Now, we consider two cases:

1° $\gamma \neq \delta$.

In this case we obtain $a_1(x) = 0$ for every $x \in P$, which gives (i).

2° $\gamma = \delta$.

From (37) we have

$$\gamma y a_1(x) = \gamma x a_1(y), \qquad x, y \in P.$$

Substituting $y = \mathbf{1}$ in this equation, we obtain

$$\gamma a_1(x) = \gamma a_1(1)x, \qquad x \in P.$$

If $\gamma \neq 0$, then $a_1(x) = a_1(1)x$ for $x \in P$ and (ii) holds. On the other hand, if $\gamma = 0$, then $\delta = 0$ and our equation reduces to a trivial one.

Now, it is enough to show that all the functions of the forms (i)–(iii) satisfy equation (4) with F given by (35). □

3. Results

Now, we are in position to use the above lemmas to solve some equations. We start with the main result of the paper.

Theorem 1. *If the functions $F, f\colon \mathbb{R} \to \mathbb{R}$ satisfy the equation*

$$8\big[F(y) - F(x)\big] = (y-x)\left[f(x) + 3f\left(\frac{x+2y}{3}\right) + 3f\left(\frac{2x+y}{3}\right) + f(y)\right] \tag{38}$$

for all $x, y \in \mathbb{R}$, then there exist $a, b, c, d, e \in \mathbb{R}$, such that

$$f(x) = ax^3 + bx^2 + cx + d, \qquad x \in \mathbb{R}$$

and

$$F(x) = \frac{1}{4}ax^4 + \frac{1}{3}bx^3 + \frac{1}{2}cx^2 + dx + e, \qquad x \in \mathbb{R}.$$

On the other hand, the functions f and F given by the above formulas satisfy (38).

P r o o f. Assume that F, f satisfy (38). Put $x + y$ instead of y in this equation. Then, we get

$$8F(x+y) - 8F(x) = yf(x) + 3yf\left(\frac{3x+2y}{3}\right) + 3yf\left(\frac{3x+y}{3}\right) + yf(x+y),$$

i.e.,

$$8F(x) + yf(x) = 8F(x+y) - 3yf\left(\frac{3x+2y}{3}\right) - 3yf\left(\frac{3x+y}{3}\right) - yf(x+y).$$

Use Lemma 2 for

$$I_0 = \{(1,1)\}, \quad I_1 = \left\{\left(1, \frac{2}{3}\right), \left(1, \frac{1}{3}\right), (1,1)\right\}.$$

Then, f is a polynomial function of order at most 5. Thus

$$f(x) = a_0 + a_1(x) + a_2(x) + a_3(x) + a_4(x) + a_5(x), \qquad x \in \mathbb{R},$$

where $a_i \colon \mathbb{R} \to \mathbb{R}$ is a diagonalization of a symmetric i-additive function $A_i \colon \mathbb{R}^i \to \mathbb{R}$, i.e.,

$$a_i(x) = A_i(\underbrace{x, \ldots, x}_{i}), \qquad x \in \mathbb{R}.$$

Moreover, from Lemma 3 we know that every function a_i, $i \in \{1, \ldots, 5\}$ satisfies (38).

Substitute $x = 0$ in (38). We obtain

$$8F(y) = y\left[f(0) + 3f\left(\frac{2y}{3}\right) + 3f\left(\frac{y}{3}\right) + f(y)\right] + 8F(0), \qquad y \in \mathbb{R},$$

which used in (38) gives for all $x, y \in \mathbb{R}$

$$y\left[f(0) + 3f\left(\frac{2y}{3}\right) + 3f\left(\frac{y}{3}\right) + f(y)\right] - x\left[f(0) + 3f\left(\frac{2x}{3}\right) + 3f\left(\frac{x}{3}\right) + f(x)\right]$$
$$= (y-x)\left[f(x) + 3f\left(\frac{x+2y}{3}\right) + 3f\left(\frac{2x+y}{3}\right) + f(y)\right],$$

or, equivalently,

$$y\left[f(0)+3f\left(\frac{2y}{3}\right)+3f\left(\frac{y}{3}\right)-f(x)-3f\left(\frac{x+2y}{3}\right)-3f\left(\frac{2x+y}{3}\right)\right]$$
$$=x\left[f(0)+3f\left(\frac{2x}{3}\right)+3f\left(\frac{x}{3}\right)-f(y)-3f\left(\frac{x+2y}{3}\right)-3f\left(\frac{2x+y}{3}\right)\right].$$

Substituting here $3x$ and $3y$ in place of x and y, respectively, we may write

$$y\big[f(0)-f(3x)+3f(2y)+3f(y)-3f(x+2y)-3f(2x+y)\big] \tag{39}$$
$$=x\big[f(0)-f(3y)+3f(2x)+3f(x)-3f(x+2y)-3f(2x+y)\big].$$

Further, take $y=2x$ in (39). Then

$$f(6x)-3f(5x)+3f(4x)-2f(3x)+3f(2x)-3f(x)+f(0)=0,\quad x\neq 0, \tag{40}$$

which is also true for $x=0$.

Since a_5 satisfies (38), then it also satisfies (40) and using 5-additivity of a_5, we get

$$(6^5-3\cdot 5^5+3\cdot 4^5-2\cdot 3^5+3\cdot 2^5-3)a_5(x)=0,\qquad x\in\mathbb{R},$$

i.e.,

$$1080a_5(x)=0,\qquad x\in\mathbb{R}.$$

This means that $a_5(x)=0$ for $x\in\mathbb{R}$. Similarly, we show that $a_4(x)=0$ for all $x\in\mathbb{R}$. Indeed, equation (40) for a_4 takes form

$$72a_4(x)=0,\qquad x\in\mathbb{R}.$$

Now, by Lemmas 4, 5, 6 we obtain that

$$f(x)=ax^3+bx^2+cx+d$$

for some $a,b,c,d\in\mathbb{R}$ and all $x\in\mathbb{R}$. Moreover, by (39) we obtain the desired formula for F. To finish the proof, it suffices to check that these functions satisfy our equation. □

Remark 1. For the sake of simplicity, we proved the above theorem for functions acting on reals. However, it is easy to see that the same proof may be applied for functions acting on an integral domain P, uniquely divisible by 5! and such that $n\mathbf{1}\neq 0$ for all $n\in\mathbb{N}$, where $\mathbf{1}$ is a unit element of P.

Let us point out that our method can be applied to solve some other functional equations. We give an example (see [4]).

Theorem 2. *The functions $f,F\colon\mathbb{R}\to\mathbb{R}$ satisfy*

$$F(y)-F(x)=(y-x)\left[\frac{1}{6}f(x)+\frac{2}{3}f\left(\frac{x+y}{2}\right)+\frac{1}{6}f(y)\right],\qquad x,y\in\mathbb{R} \tag{41}$$

if and only if

$$f(x) = ax^3 + bx^2 + cx + d, \qquad x \in \mathbb{R}$$

and

$$F(x) = \frac{1}{4}ax^4 + \frac{1}{3}bx^3 + \frac{1}{2}cx^2 + dx + e, \qquad x \in \mathbb{R},$$

where $a, b, c, d, e \in \mathbb{R}$ *are some numbers.*

P r o o f. Substitute $y = y + x$ in (41). Then we have

$$-\frac{1}{6}yf(x) - F(x) = -F(x+y) + y\left[\frac{2}{3}f\left(x + \frac{y}{2}\right) + \frac{1}{6}f(x+y)\right], \qquad x, y \in \mathbb{R}.$$

From Lemma 2 we know that f must be a polynomial function of order at most 3. Using Lemmas 3, 4, 5 and 6 we obtain that f must be an ordinary polynomial and putting in $x = 0$ (41), we obtain the desired form of F.

On the other hand, it is easy to show that f and F of this form satisfy (41). □

Some geometric problems considered recently by C. A l s i n a, M. S a b l i k and J. S i k o r s k a in [2] lead to the functional equation

$$2F(y) - 2F(x) = (y - x)\left[f\left(\frac{x+y}{2}\right) + \frac{f(x) + f(y)}{2}\right],$$

solved also by B. E b a n k s and M. S a b l i k (see [10]). Further, in the paper [3] S h. H a r u k i considered the equation

$$f(x) - f(y) = (x - y)\left(\frac{g(x) + g(y)}{2}\right)$$

and, on the other hand, J. A c z é l in [1] solved the equation

$$f(x) - f(y) = (x - y)g(x + y).$$

It is easy to observe that all these equations may be solved with use of our method also.

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