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MIXED PROBLEM FOR DEGENERATE NONLINEAR ULTRAPARABOLIC EQUATION

Nataliya Protsakh

ABSTRACT. This paper describes the solvability of mixed problem for nonlinear ultraparabolic equation with degeneration. The result is obtained in the cases of the weak and the strong degenerations. The conditions of solvability depend on the relations between the coefficients of the equation and the parameter of degeneration.

1. Introduction

Equations that on the certain domain or set change to a new object are called degenerate equations on this set. We consider the mixed problem for nonlinear ultraparabolic equation that degenerates on the initial hyperplane. This equation changes its type to parabolic when $t = 0$. The conditions of existence and the uniqueness of the solution for the problem depend on the relations between the coefficients of the equation and the parameter of degeneration. Note that problems for degenerate parabolic and hyperbolic equations were studied in works [1–6, 9, 10], the solvability of the mixed problem for nonlinear ultraparabolic equation without degeneration was studied in [7, 8].

2. Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^n$ with the boundary $\partial\Omega$; $y_0 < +\infty, D = \Omega \times (0, y_0), Q_T = D \times (0, T)$, $0 < T < +\infty; Q_{t,\tau} = D \times (t, \tau); D_\tau = Q_T \bigcap \{t = \tau\}$, $\tau \in [0, T]$, $\partial\Omega \subset C^2$, $\Sigma_T = \partial\Omega \times (0, y_0) \times (0, T)$.

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In the domain Q_T we consider the equation

$$\varphi(t)u_t - \lambda(x,y,t)u_y - \sum_{i,j=1}^{n} \big(a_{ij}(x,y,t)u_{x_i}\big)_{x_j} \tag{1}$$
$$+ \sum_{i=1}^{n} c_i(x,y,t)u_{x_i} + d(x,y,t)u + g(x,y,t)|u|^{p-2}u = f(x,y,t)$$

with the boundary conditions

$$u|_{\Sigma_T} = 0; \qquad u(x,y_0,t) = 0, \quad (x,t) \in \Omega \times (0,T). \tag{2}$$

We assume that coefficients of the equation (1) satisfy such conditions

(Φ): $\varphi \in C\big([0,T]\big)$, $\varphi(t) > 0$, $t \in (0,T]$; $\varphi(0) = 0$, $\varphi' \in C\big((0,T]\big)$; $\varphi'(t) \geqslant 0$, $t \in (0,T]$;

(A): $a_{ij} \in L^\infty(Q_T)$, $\sum_{i,j=1}^{n} a_{ij}(x,y,t)\,\xi_i\,\xi_j \geqslant a_0 \sum_{i=1}^{n} |\xi_i|^2$ for almost all $(x,\,y,\,t) \in Q_T$, $i,j \in \{1,\dots,n\}$, $\xi \in \mathbb{R}^n$, where a_0 is a positive constant;

(L): $\lambda \in C\,(\overline{Q}_T)$, $\lambda_y \in L^\infty(Q_T)$, for almost all $(x,\,y,\,t) \in Q_T$, $\lambda \geqslant 0$, $\lambda \not\equiv 0$;

(G): $g \in L^\infty(\overline{Q}_T)$, $g(x,\,y,\,t) \geqslant g_0 > 0$ for almost all $(x,\,y,\,t) \in Q_T$, where g_0 is a constant;

(P): $1 < p < \infty$.

Let us introduce the spaces $H_{0,\varphi}(Q_T)$ and $\widetilde{H}_{0,\varphi}(Q_T)$ as a closure of the set of infinitely differentiable functions in Q_T, such that $u|_{\Sigma_T} = 0, u|_{y=y_0} = 0$ under the norm

$$\|u; H_{0,\varphi}(Q_T)\|^2 = \int\limits_{\overline{Q}_T} \left[\varphi(t)|u|^2 + \varphi(t)|u_y|^2 + \varphi(t)|u_t|^2 + \sum_{i=1}^{n} |u_{x_i}|^2\right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

and

$$\|u; \widetilde{H}_{0,\varphi}(Q_T)\|^2 = \int\limits_{\overline{Q}_T} \left[\varphi(t)|u|^2 + \varphi(t)|u_y|^2 + \sum_{i=1}^{n} |u_{x_i}|^2\right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t,$$

respectively.

Definition 1. We say that the equation (1) has the strong degeneration if $\int_0^T \frac{\mathrm{d}t}{\varphi(t)} = \infty$, and (1) has the weak degeneration if $\int_0^T \frac{\mathrm{d}t}{\varphi(t)} < \infty$.

In the case of the strong degeneration the initial condition is

$$\lim_{t \to +0} \sqrt{\varphi(t)}u(x,y,t) = 0, \qquad (x,y) \in D. \tag{3}$$

Definition 2. Function $u \in H_{0,\varphi}(Q_T) \cap L^p(Q_T)$ is said to be the weak solution to the problem (1), (2), (3) (in the case of the strong degeneration) if it satisfies

the equality

$$\int\limits_{Q_T} L(u,v)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \stackrel{\text{def}}{=} \int\limits_{Q_T} \Bigg[\varphi(t)u_t v + \lambda(x,y,t)u_y v + \sum_{i,j=1}^{n} a_{ij}(x,y,t)u_{x_i}v_{x_j} + \sum_{i=1}^{n} c_i(x,y,t)u_{x_i}v + d(x,y,t)uv + g(x,y,t)|u|^{p-2}uv - f(x,y,t)v\Bigg]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0 \tag{4}$$

for all functions

$$v \in \big\{v\colon v \in L^2(Q_T)\cap L^p(Q_T),\ v_{x_i} \in L^2(Q_T),\ i=\overline{1,n},\ u|_{\Sigma_T}=0\big\}$$

and the initial condition (3) holds.

In the case when the equation (1) has the weak degeneration we set the initial condition

$$u(x,y,0)=0, \qquad (x,y)\in D. \tag{5}$$

Definition 3. Function $u \in \widetilde{H}_{0,\varphi}(Q_T)\cap L^p(Q_T)$ is said to be the weak solution to the problem (1), (2), (5) (in the case of the weak degeneration) if it satisfies the equality

$$\int\limits_{D_T} \varphi(T)uv\,\mathrm{d}x\,\mathrm{d}y + \int\limits_{Q_T} \Bigg[-\varphi(t)uv_t - \varphi'(t)uv + \lambda(x,y,t)u_y v + \sum_{i,j=1}^{n} a_{ij}(x,y,t)u_{x_i}v_{x_j} + \sum_{i=1}^{n} c_i(x,y,t)u_{x_i}v + d(x,y,t)uv + g(x,y,t)|u|^{p-2}uv - f(x,y,t)v\Bigg]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0 \tag{6}$$

for all functions

$$v \in \big\{v:\ v \in L^2(Q_T)\cap L^p(Q_T),\ v_{x_i},\ v_t \in L^2(Q_T),\ i=\overline{1,n},\ u|_{\Sigma_T}=0\big\}.$$

Denote by

$$c_0(t)=\sup_{Q_t}\sum_{i=1}^{n}\frac{|c_i(x,y,\tau)|^2}{\varphi'(\tau)}, \qquad \lambda_1(t)=\sup_{Q_t}\frac{|\lambda_y(x,y,\tau)|}{\varphi'(\tau)},$$

$$d_0(t)=\sup_{Q_t}\frac{|d(x,y,\tau)|}{\varphi'(\tau)}, \qquad c_1(t)=\sup_{Q_t}\sum_{i=1}^{n}\frac{|c_{iy}(x,y,\tau)|^2}{\varphi'(\tau)},$$

$$d_1(t)=\sup_{Q_t}\frac{|d_y(x,y,\tau)|}{\varphi'(\tau)}, \qquad c_2(t)=\sup_{Q_t}\sum_{i=1}^{n}\frac{|c_{it}(x,y,\tau)|^2}{\varphi'(\tau)},$$

$$\lambda_2(t)=\sup_{Q_t}\frac{|\lambda_t(x,y,\tau)|}{\varphi'(\tau)}, \qquad d_2(t)=\sup_{Q_t}\frac{|d_\tau(x,y,\tau)|}{\varphi'(\tau)},$$

$$\nu_0 = 1 + \inf_{[0,T]} \sup_{[0,\tau]} \left[\lambda_1(t) + 2d_0(t) + \frac{2c_0(t)}{a_0} \right], \qquad \nu_1 = \nu_0 - 2,$$

δ_i $(i = \overline{0,6})$ is a small positive number,

$$\max_{[0,T]} \left\{ c_1(t);\ \big(d_1(t)\big)^2;\ \operatorname{ess\,sup}_D g_{yy} \right\} \leqslant \omega_1(t),$$
$$\max_{[0,T]} \left\{ c_2(t);\ \big(d_2(t)\big)^2;\ \operatorname{ess\,sup}_D g_{tt} \right\} \leqslant \omega_2(t),$$

$$c_{0(1)}(t) = \sup_{Q_t} \sum_{i=1}^{n} |c_i(x,y,\tau)|^2,$$
$$\lambda_{1(1)}(t) = \sup_{Q_t} |\lambda_y(x,y,\tau)|,$$
$$d_{0(1)}(t) = \sup_{Q_t} |d(x,y,\tau)|,$$
$$c_{1(1)}(t) = \sup_{Q_t} \sum_{i=1}^{n} |c_{iy}(x,y,\tau)|^2,$$
$$d_{1(1)}(t) = \sup_{Q_t} |d_y(x,y,\tau)|.$$

3. Main results

a) *The problem* (1)–(3). Assume that the function f satisfies the condition **(F):**

$$\int\limits_{Q_T} \left[\frac{\omega_1(t)|f|^2 + |f_y|^2}{\big[\varphi(t)\big]^{\nu_0+2\delta_0}\varphi'(t)} + \frac{|f_t|^2}{\big[\varphi(t)\big]^{\nu_1+2\delta_0}\varphi'(t)} \right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t < +\infty,$$

$$\lim_{\varepsilon\to 0} \int\limits_{D_\varepsilon} \frac{|f(x,y,\varepsilon)|^2}{\big[\varphi(\varepsilon)\big]^{\nu_1+1+2\delta_0}}\,\mathrm{d}x\,\mathrm{d}y < +\infty,\ \ f\big|_{y=y_0} = 0$$

for almost all $(x,t) \in \Omega \times (0,T)$.

Theorem 1. *Under the hypotheses* (**A**), (**L**), (Φ), (**F**), (**G**), (**P**) *and assumptions that*

$$g_y(x,0,t) = 0$$

for all $(x,t) \in \Omega \times (0,T)$, $c_i,\ d_i,\ \lambda_j \in L^\infty([0,T])\, i = 0,1,2,\ j = 1,2, \omega_1(t), \omega_2(t)$ *are nonnegative monotone nonincreasing on* $[0,T]$ *functions, the weak solution to the problem* (1), (2), (3) exists.

P r o o f. Let $\{\varphi^k\}_{k=1}^{\infty}$ be the orthogonal basis of the space $H_0^1(\Omega)\cap L^p(\Omega)$, which is orthonormal in the space $L^2(\Omega)$, and $\psi^s = \cos\frac{(2s-1)\pi}{2y_0}y$ $(s \geqslant 1)$. Denote by

$$\{\varphi^{k,s}\}_{k,s=1}^{\infty} = \{\varphi^k(x)\psi^s(y)\}_{k,s=1}^{\infty}.$$

It is obvious that $\varphi^{k,s} \in H^1(D)$, $\varphi^{k,s}|_{\partial\Omega\times(0,y_0)} = 0, \varphi^{k,s}|_{y=y_0} = 0$. Let us consider functions

$$u^N(x,y,t) = \sum_{k,s=1}^{N} c_{ks}^N(t)\varphi^{k,s}(x,y), \qquad N \in \{1,2,\dots\},$$

where $c_{ks}^N(t)$ is the solution to the problem

$$\int_{D_t} L\big(u^N, \varphi^{k,s}\big)\,\mathrm{d}x\,\mathrm{d}y = 0, \tag{7}$$

$$c_{ks}^N(\varepsilon) = 0, \qquad k,s \subset \{1,\dots,N\}, \quad \varepsilon > 0, \quad t \in [\varepsilon, T]. \tag{8}$$

Due to Caratheodory theorem [1, p. 54] there exists absolutely continuous solution on $[\varepsilon,\tau_0]$, $\tau_0 < T$ to the problem (7), (8). Let ζ be a monotone nonincreasing nonnegative on $[0,T]$ function. Multiplying (7) on $c_{ks}^N(t)\zeta(t)$, summarizing up on s and k, integrating on t on the interval $[\varepsilon,\tau]$, we have

$$\int_{Q_{\varepsilon,\tau}} L\big(u^N, u^N\big)\zeta(t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0.$$

Estimating every term of this equality under the assumptions of the Theorem 1, it is easy to obtain the appraisal

$$\begin{aligned}
&\int_{D_\tau} \varphi(\tau)\big(u^N\big)^2\zeta(\tau)\,\mathrm{d}x\,\mathrm{d}y - \int_{Q_{\varepsilon,\tau}} \varphi(t)\big(u^N\big)^2\zeta'(t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad + \int_{\varepsilon}^{\tau}\int_{\Omega} \lambda(x,0,t)\big|u^N\big|^2\zeta(t)\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_{\varepsilon,\tau}} \Bigg[\Bigg(2a_0 - \frac{c_0(t)}{\delta_1}\Bigg)\sum_{i=1}^{n}\Big(u_{x_i}^N\Big)^2\zeta(t) \\
&\quad + \big(-1-\lambda_1(t)-\delta_1-2d_0(t)-\delta_2\big)\varphi'(t)\big|u^N\big|^2 + 2g_0\big|u^N\big|^p\Bigg]\zeta(t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\leqslant \int_{Q_{\varepsilon,\tau}} \frac{|f(x,y,t)|^2}{\varphi'(t)\delta_2}\zeta(t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,.
\end{aligned} \tag{9}$$

Note, that under the conditions on the function ζ it follows that $\zeta'(t) \leqslant 0$. Choosing $\delta_1 = \sup_{Q_T}\frac{2c_0(t)}{a_0}$, δ_2 as a small positive number and taking into account the view of ν_0, we get $\sup_{[0,T]}\big(1+\lambda_1(t)+\delta_1+2d_0(t)+\delta_2\big) < \nu_0+\delta_0, \nu_0 > 0$.

Denote by

$$y(\tau) = \int\limits_{Q_{\varepsilon,\tau}} \varphi'(t) \left|u^N\right|^2 \zeta(t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

Then

$$\int\limits_{D_\tau} \varphi(\tau) \left(u^N\right)^2 \zeta(\tau)\,\mathrm{d}x\,\mathrm{d}y \;= \frac{\varphi(\tau)y'(\tau)}{\varphi'(\tau)}$$

and from (9) we get the estimation

$$\frac{\varphi(\tau)y'(\tau)}{\varphi'(\tau)} \le (\nu_0+\delta_0)y(\tau) + \int\limits_{Q_{\varepsilon,\tau}} \frac{|f|^2}{\delta_2\varphi'(t)}\zeta(t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

The solving of this inequality gives

$$\begin{aligned} y(\tau) &\le \frac{\left[\varphi(\tau)\right]^{\nu_0+\delta_0}}{\delta_2} \int\limits_{\varepsilon}^{\tau} \varphi'(\theta)\left[\varphi(\theta)\right]^{-(\nu_0+2\delta_0)}\left[\varphi(\theta)\right]^{\delta_0-1} \int\limits_{Q_{\varepsilon,\theta}} \frac{|f|^2}{\varphi'(t)}\zeta(t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\theta \\ &\leqslant M_1\left[\varphi(\tau)\right]^{\nu_0+2\delta_0} \int\limits_{Q_T} \frac{1}{\left[\varphi(t)\right]^{\nu_0+2\delta_0}} \frac{|f|^2\zeta(t)}{\varphi'(t)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t, \end{aligned}$$

where $M_1 = \frac{1}{\delta_0\delta_1}$. Applying the last estimation to (9) we obtain

$$\begin{aligned} &\int\limits_{D_\tau} \varphi(\tau)\left(u^N\right)^2\zeta(\tau)\,\mathrm{d}x\,\mathrm{d}y \;+ \int\limits_{\varepsilon}^{\tau}\int\limits_{\Omega} \lambda(x,0,t)\left(u^N\right)^2\zeta(t)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int\limits_{Q_{\varepsilon,\tau}} \left[\sum_{i=1}^{n}\left(u^N_{x_i}\right)^2 + \left|u^N\right|^p\right]\zeta(t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \qquad (10) \\ &\quad \leqslant M_1\left[\varphi(\tau)\right]^{\nu_0+2\delta_0} \int\limits_{Q_T} \frac{1}{\left[\varphi(t)\right]^{\nu_0+2\delta_0}} \frac{|f|^2\zeta(t)}{\varphi'(t)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t, \end{aligned}$$

where the constant M_1 does not depend on ε and N.

Since

$$\left(\cos\frac{(2s-1)\pi}{2y_0}y\right)_{yy} = -\omega_s\cos\frac{(2s-1)\pi}{2y_0}y, \quad \text{where} \quad \omega_s = \left(\frac{(2s-1)\pi}{2y_0}\right)^2,$$

then we multiply (5) by $c^N_{k,s}(t)\omega_s$, summarize up on s,k, integrate on t from ε to τ and change $\sum_{k,s=1}^{N} c^N_{k,s}(t)\omega_s\varphi^{k,s}(x,y)$ on the $-u^N_{yy}$ from the previous expression. We obtain

$$-\int\limits_{Q_{\varepsilon,\tau}} L\left(u^N, u^N_{yy}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0.$$

Transforming and estimating every term of this equality we obtain the estimation:

$$\int\limits_{D_\tau} \varphi(\tau)\Big(u_y^N\Big)^2 \,\mathrm{d}x\,\mathrm{d}y \,+\int\limits_{\varepsilon}^{\tau}\int\limits_{\Omega} \lambda(x,y_0,t)\Big(u_y^N\Big)^2 \,\mathrm{d}x\,\mathrm{d}t \,+2(p-1)g_0$$
$$\times \int\limits_{Q_{\varepsilon,\tau}} \big|u^N\big|^{p-2}\Big(u_y^N\Big)^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \,+\int\limits_{Q_{\varepsilon,\tau}}\Bigg[\bigg(a_0-\frac{c_0(t)}{\delta_3}\bigg)\sum_{i=1}^{n}\Big(u_{x_iy}^N\Big)^2$$
$$+\big(-1-\lambda_1(t)-\delta_4-\delta_5-\delta_3-d_1(t)\delta_4-2d_0(t)\big)\varphi'(t)\,\big|u_y^N\big|^2\Bigg]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$
$$\leqslant \int\limits_{Q_{\varepsilon,\tau}}\Bigg[\bigg(\frac{(a^1)^2}{a_0}+\frac{c_1(t)}{\delta_5}\bigg)\sum_{i=1}^{n}\Big(u_{x_i}^N\Big)^2+\frac{d_1(t)}{\delta_4}\varphi'(t)\big|u^N\big|^2$$
$$+\frac{\big(f_y(x,y,t)\big)^2}{\delta_4\varphi'(t)}+\frac{2}{p}g_{yy}\big|u^N\big|^p\Bigg]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t,$$

where $a^1=\sup_{Q_T}|a_{iy}|$, $\delta_3=\frac{2\sup\limits_{Q_T}c_0(t)}{a_0}$, δ_4, δ_5 are small positive numbers. Solving this inequality similarly to (9), we have

$$\int\limits_{D_\tau} \varphi(\tau)\Big(u_y^N\Big)^2 \,\mathrm{d}x\,\mathrm{d}y \,+\int\limits_{\varepsilon}^{\tau}\int\limits_{\Omega} \lambda(x,y_0,t)\Big(u_y^N\Big)^2 \,\mathrm{d}x\,\mathrm{d}t \,+\int\limits_{Q_{\varepsilon,\tau}}\sum_{i=1}^{n}\Big(u_{x_iy}^N\Big)^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$
$$\leqslant M_2\big[\varphi(\tau)\big]^{\nu_0+2\delta_0}\int\limits_{Q_T}\frac{1}{\big[\varphi(t)\big]^{\nu_0+2\delta_0}}\frac{\omega_1(t)|f|^2+|f_y|^2}{\varphi'(t)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,, \qquad (11)$$

where the constant M_2 does not depend on ε, N. Setting $\tau=\varepsilon$ in (7) we get

$$\int\limits_{D_\varepsilon}\varphi(\varepsilon)\Big(u_t^N\Big)^2\,\mathrm{d}x\,\mathrm{d}y=\int\limits_{D_\varepsilon} f(x,y,\varepsilon)u_t^N\,\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant\int\limits_{D_\varepsilon}\bigg[\frac{1}{2}\frac{|f(x,y,\varepsilon)|^2}{\varphi(\varepsilon)}+\frac{1}{2}\big|u_t^N\big|^2\varphi(\varepsilon)\bigg]\mathrm{d}x\,\mathrm{d}y. \qquad (12)$$

So,

$$\frac{1}{2}\int\limits_{D_\varepsilon}\varphi(\varepsilon)\Big(u_t^N\Big)^2\mathrm{d}x\,\mathrm{d}y\leqslant\frac{1}{2}\int\limits_{D_\varepsilon}\frac{|f(x,y,\varepsilon)|^2}{\varphi(\varepsilon)}\,\mathrm{d}x\,\mathrm{d}y.$$

Differentiation (7) on t, multiplying every equation by $c_{kst}^N(t)e^{-\nu t}$, summation up on k,s, integration on $[\varepsilon,\tau]$, $\tau<T$ and estimation of every term of the obtained equality implies the inequality

$$\int\limits_{D_\tau} \varphi(\tau)(u_t^N)^2\,\mathrm{d}x\,\mathrm{d}y + \int\limits_{\varepsilon}^{\tau}\int\limits_{\Omega} \lambda(x,0,t)\left|u_t^N\right|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$- \int\limits_{Q_{\varepsilon,\tau}} \Big(\big(-1+\lambda_1(t)+2\delta_5+\delta_6+\delta_3+d_2(t)\delta_5+2d_0(t)\big)\Big)\varphi'(t)\big|u_t^N\big|^2$$
$$+ \left(a_0 - \frac{c_0(t)}{\delta_3}\right)\sum_{i=1}^{n}\Big(u_{x_it}^N\Big)^2\Bigg)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + 2g_0(p-1)\int\limits_{Q_{\varepsilon,\tau}}\left|u^N\right|^{p-2}\left|u_t^N\right|^2\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$
$$\leqslant \int\limits_{Q_{\varepsilon,\tau}}\Bigg(\left(\frac{(a^1)^2}{a_0}+\frac{c_2(t)}{\delta_5}\right)\sum_{i=1}^{n}\Big(u_{x_i}^N\Big)^2$$
$$+ \frac{d_2(t)}{\delta_5}\varphi'(t)\big(u^N\big)^2 + \frac{\lambda_2(t)\varphi'(t)\left|u_y^N\right|^2}{\delta_5} + 2g_{tt}(x,y,t)\left|u^N\right|^p\Bigg)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$
$$+ \frac{2}{p}\int\limits_{D_\tau} g_t(x,y,\tau)\left|u^N\right|^p\mathrm{d}x\,\mathrm{d}y + \int\limits_{Q_{\varepsilon,\tau}}\frac{|f_t(x,y,t)|^2}{\delta_5\varphi'(t)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + \int\limits_{D_\varepsilon}\varphi(\varepsilon)\left(u_t^N\right)^2\mathrm{d}x\,\mathrm{d}y\,.$$

Solving this inequality, we find

$$\int\limits_{Q_{\varepsilon,\tau}}\varphi'(t)\Big(u_t^N\Big)^2\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \leqslant M_3\big[\varphi(\tau)\big]^{\nu_1+2\delta_0}\Bigg[\int\limits_{D_\varepsilon}\frac{|f(x,y,\varepsilon)|^2}{\big[\varphi(\varepsilon)\big]^{\nu_1+2\delta_0}\varphi(\varepsilon)}\,\mathrm{d}x\,\mathrm{d}y \tag{13}$$
$$+\int\limits_{Q_{\varepsilon,\tau}}\left(\frac{1}{\big[\varphi(t)\big]^{\nu_0+2\delta_0}}\frac{\big(\omega_1(t)+\omega_2(t)\big)|f|^2+|f_y|^2}{\varphi'(t)}+\frac{|f_t|^2}{\big[\varphi(t)\big]^{\nu_1+2\delta_0}\varphi'(t)}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\Bigg].$$

So, the estimations (10), (11), (13) imply the appraisal

$$\big\|u^N; H_{0,\varphi}(Q_{\varepsilon,\tau_0})\cap L^p(Q_{\varepsilon,\tau_0})\big\|^2$$
$$\leqslant M_3\big[\varphi(\tau)\big]^{\nu_1+2\delta_0}\Bigg[\int\limits_{Q_{\varepsilon,\tau}}\frac{1}{\big[\varphi(t)\big]^{\nu_0+2\delta_0}}\frac{\big(\omega_1(t)+\omega_2(t)\big)|f|^2+|f_y|^2}{\varphi'(t)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$
$$+\int\limits_{D_\varepsilon}\frac{|f(x,y,\varepsilon)|^2}{\big[\varphi(\varepsilon)\big]^{\nu_1+2\delta_0}\varphi(\varepsilon)}\,\mathrm{d}x\,\mathrm{d}y+\int\limits_{Q_{\varepsilon,\tau}}\frac{|f_t|^2}{\big[\varphi(t)\big]^{\nu_1+2\delta_0}\varphi'(t)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\Bigg], \tag{14}$$

where the constant M_3 does not depend on ε, N. The similar estimation can be obtained in the domain $Q_{\tau_0,T}$. Taking into account (1) we obtain such conclusion:

$$\varphi(t)u_t^N \in L^2\big(\varepsilon,T;H^{-1}(D)\big).$$

So,

$$u \in C\big((0,T];H^{-1}(D)\big).$$

From the obtained estimation it follows that

$$\sqrt{\varphi(t)}u_t^N \to \sqrt{\varphi(t)}u_t \text{*–weakly in } L^\infty\big(0,T;L^2(D)\big),$$
$$u^N \to u \text{–weakly in } H_{0,\varphi}(Q_{\varepsilon,T}) \cap L^p(Q_{\varepsilon,T}).$$

Besides,

$$\big|u^N\big|^{p-2}u^N \to \chi \text{ weakly in } L^{p'}(Q_{\varepsilon,T}).$$

Since the embedding $H^1(Q) \subset L^2(Q)$ is compact, we obtain that $u^N \to u$ strongly in $L^2(Q_{\varepsilon,T})$ and almost everywhere. So, Lemma 1.3 [8, c. 25] implies that $\chi = |u|^{p-2}u$.

Let us prove that u is the weak solution of (1)–(3). Extend functions $\{u^N\}_{N=1}^\infty$ to the domain Q_T by zero and choose $\varepsilon = \frac{1}{N}$. Then from (7), multiplying it on $c_{ks}(t)$, and summarizing up on k, s from 1 to N_0, we find

$$\int\limits_{Q_{\frac{1}{N},T}} L\Big(u^N, v^{N_0}\Big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\ = 0.$$

Let $N \to \infty$. Taking into account convergences of the $\{u^N\}_{N=1}^\infty$, we obtain that u satisfies (4). Let us prove that the initial condition (3) fulfills. Consider the sequence of functions

$$g^N(t) = \int\limits_{D_\tau} \varphi(\tau)\big|u^N\big|^2\,\mathrm{d}x\,\mathrm{d}y, \qquad N \geqslant 1.$$

We have

$$\begin{aligned}\int\limits_{D_t} \varphi(t)\big(u^N(x,y,t)\big)^2\mathrm{d}x\,\mathrm{d}y \leqslant{}& \frac{\varphi^{\delta_0}(t)}{2\delta_0}\int\limits_{Q_t}\frac{\varphi'(\tau)}{\big(\varphi(\tau)\big)^{\delta_0}}\left(u^N\right)^2\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau\\ &+ 2t\int\limits_{Q_t}\varphi(\tau)\left(u_\tau^N\right)^2\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau \qquad (15)\\ \leqslant{}& \frac{M_4}{(\delta_0)^2}\varphi^{2\delta_0}(t)\int\limits_{Q_t}\frac{|f|^2}{\varphi^{\nu_0+2\delta_0}(\tau)\varphi'(\tau)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau\\ &+ 2t\int\limits_{Q_t}\varphi(\tau)\big(u_\tau^N\big)^2\mathrm{d}x\,dy\,d\tau.\end{aligned}$$

Let $t \to 0$ in (15). Fix $\varepsilon > 0$. Since the Lebesgue integral is absolutely continuous and the estimation (15) holds, this implies the existence of such $\delta_0 \in (0,T]$, that the righthand side of (15) is less than ε. Besides the function $u^N \in C\big((0,T];L^2(D)\big)$ and $u^N(x,y,0) = 0$. Hence, $\sqrt{\varphi(t)}u^N \to 0$ as $t \to 0$. $\square$

Theorem 2. *Under the hypotheses* (**A**), (**L**), (**Φ**), (**G**), (**P**), *and* $c_0,\ d_0,\ \lambda_1 \in L^\infty\big([0,T]\big)$, *the problem* (1), (2), (3) can not have more than one weak solution.

P r o o f. Assume that there exist two solutions u^1, u^2 to the problem (1), (2), (3). Their difference $u = u^1 - u^2$ satisfies the equality

$$\int\limits_{Q_\tau}\Bigg[\varphi(t)u_t u - \lambda(x,y,t)u_y u + \sum_{i,j=1}^{n} a_{ij}(x,y,t)u_{x_i}u_{x_j} + \sum_{i=1}^{n} c_i(x,y,t)u_{x_i}u \qquad (16)$$
$$+d(x,y,t)(u)^2 + g(x,y,t)\Big(\big|u^1\big|^{p-2}u^1 - \big|u^2\big|^{p-2}u^2\Big)u\Bigg]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\ = 0.$$

Estimating every term of this equality using the conditions of the Theorem 2 and noting that

$$\int\limits_{Q_\tau} g(x,y,t)\Big(\big|u^1\big|^{p-2}u^1 - |u^2|^{p-2}u^2\Big)u\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\ \geq 0,$$

from (16) we obtain the estimation

$$\int\limits_{D_\tau}\varphi(\tau)(u)^2\,\mathrm{d}x\,\mathrm{d}y + \int\limits_0^\tau\!\!\int\limits_\Omega \lambda(x,0,t)(u)^2\mathrm{d}x\,\mathrm{d}t\ + \int\limits_{Q_\tau}\sum_{i=1}^{n}(u_{x_i})^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\ \leqslant 0.$$

Hence $u = 0$ almost everywhere in Q_T, that is $u_1 = u_2$. □

b) *The problem* (1), (2), (5). Assume that the righthand side of the equation (1) satisfies the condition

(F1):

$$\int\limits_{Q_T}\big(|f|^2 + |f_y|^2 + |f_t|^2\big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t < +\infty,$$
$$f(x,y_0,t) = 0 \qquad \text{for almost all} \quad (x,t) \in \Omega \times (0,T).$$

Theorem 3. *Let the conditions* (**A**), (**L**), (**Φ**), (**G**), (**P**), (**F1**) *and* $c_{i(1)},\ d_{i(1)},\ \lambda_{j(1)} \in L^\infty\big([0,T]\big)$ $i = 0,1$, $j = 1,2$, $g_{yy} \in L^\infty(Q_T)$ *be fulfilled. Then the solution of the problem* (1), (2), (5) exists.

P r o o f. Let $\{\varphi^k\}_{k=1}^\infty$, $\{\psi^s\}_{s=1}^\infty$ be the same as in the Theorem 1,

$$u^N(x,y,t) = \sum_{k,s=1}^{N} c_{ks}^N(t)\varphi^{k,s}(x,y), \qquad N \in \{1,2,\dots\},$$

where $c_{ks}^N(t)$ are the solutions to the problem (7)–(8). Multiplying (7) by $c_{ks}^N(t)$, summarizing up on s and k, integrating on t on the interval $[\varepsilon,\tau]$ we obtain the

equality (9). Then for approximations u^N, $N \geqslant 1$, we find the estimation

$$\left\|u^N; \widetilde{H}_{0,\varphi}(Q_{\varepsilon,\tau_0}) \cap L^p(Q_{\varepsilon,\tau_0})\right\|^2 \leqslant M_5 \left[\varphi(\tau)\right]^{\nu_0+2\delta_0} \int\limits_{Q_T} \frac{|f|^2 + |f_y|^2}{\varphi'(t)\left[\varphi(t)\right]^{\nu_0+2\delta_0}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,,$$

where the constant M_5 does not depend on ε, N. (The proof of this appraisal is similar to finding formulae (10), (11)). This estimation implies convergences

$$\sqrt{\varphi(t)}u^N \to \sqrt{\varphi(t)}u\text{*–weakly in } L^\infty\big(0,T;L^2(D)\big),$$
$$u^N \to u\text{–weakly in } \widetilde{H}_{0,\varphi}(Q_{\varepsilon,T}) \cap L^p(Q_{\varepsilon,T}).$$

Besides

$$\left|u^N\right|^{p-2} u^N \to \chi_1 \text{ weakly in } L^{p'}(Q_{\varepsilon,T}).$$

Taking into account these convergences and using method of monotonicity from [8, p. 171] we prove that $\chi_1 = |u|^{p-2}u$. Consider the sequence of functions

$$g^N(t) = \int\limits_{D_\tau} \left|u^N\right|^2 \mathrm{d}x\,\mathrm{d}y.$$

We have

$$\int\limits_{D_t} \big(u^N(x,y,t)\big)^2 \,\mathrm{d}x\,\mathrm{d}y \leqslant \left(\int\limits_0^t \frac{\varphi(\tau)}{\varphi'(\tau)} \frac{1}{\varphi(\tau)}\,\mathrm{d}\tau\right) \int\limits_{D_t} \left(\int\limits_0^t \varphi'(\tau)\big(u^N\big)^2 \,\mathrm{d}\tau\right) \mathrm{d}x\,\mathrm{d}y$$
$$\leqslant M_6 \max_{[0,T]} \frac{\varphi(\tau)}{\varphi'(\tau)}, \tag{17}$$

where M_6 does not depend on ε, N. Fix $\varepsilon > 0$. Since $\frac{\varphi(\tau)}{\varphi'(\tau)}$ is the monotone nondecreasing function, $\frac{\varphi(0)}{\varphi'(0)} = 0$, there exists such δ that for all $\tau \in [0,\delta]$ the righthand side of (17) is less than ε. Besides, the function u^N belongs to the space $C\big((0,T];L^2(D_t)\big)$ and $u^N(x,y,0) = 0$. Hence, $u^N \to 0$ as $t \to 0$. □

Theorem 4. *Let the conditions* (**A**), (**L**), (**Φ**), (**G**), (**P**), *and* $c_{0(1)}, d_{0(1)}, \lambda_{0(1)} \in L^\infty\big([0,T]\big)$ *be fulfilled. Then the problem* (1), (2), (5) cannot have more than one solution.

Proof. The proof of this theorem is similar to the proof of Theorem 2. □

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National Forestry Engineering University of Ukraine
Department of Mathematics
Generala Chuprynky str. 103
79057-L'viv
UKRAINE

Pidstryhach Institute for Applied Problems of
Mechanics and Mathematics of NAS of Ukraine
Naukova 3b str., dep. 26
79060-L'viv
UKRAINE
E-mail: protsakh@ukr.net