

INTERSECTIONS OF ROTATIONAL QUADRATIC HYPERSURFACES WITH A COMMON FOCUS

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ABSTRACT. This paper deals with rotational quadratic hypersurfaces in an n -dimensional Euclidean space. Namely, we explore some basic properties of the intersection of two rotational quadratic hypersurfaces that have a common focus, but are not necessarily confocal. We prove that any such intersection lies in at most two hyperplanes, and we specify the maximum number of its connected components.

1. Introduction

Let us consider a hyperplane Π in $\mathbb{R}^n$, a point $F \in \mathbb{R}^n$ and a number $e > 0$. Using these three objects, we can define a set $\Sigma \subset \mathbb{R}^n$ of all points satisfying the equation

$$|XF| = e|X\Pi|. \quad (1.1)$$

If we take any plane Λ perpendicular to Π and containing F , then $\Sigma \cap \Lambda$ is a quadratic curve — a conic section with focus F , directrix $\Pi \cap \Lambda$ and eccentricity e . Namely, if $F \notin \Pi$, then $0 < e < 1$ defines an ellipse, $e = 1$ a parabola, and $e > 1$ a hyperbola. If $F \in \Pi$, then $e < 1$ defines an empty set, $e = 1$ the normal line to Π containing F and $e > 1$ the union of two lines intersecting at F . Since all the considered planes can be obtained from one another by rotation, the set Σ can be constructed by rotating a quadratic curve. If $F \notin \Pi$ or if $F \in \Pi$ and $e > 1$, then Σ is of dimension $n - 1$ and we will call it a *rotational quadratic hypersurface*. The hyperplane Π is called the *directrix* of Σ and the point F is the *focus* of Σ .

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The normal line to Π containing F is the *axis* of the hypersurface. The above implies that Σ can be an ellipsoid with foci (a prolate spheroid), a paraboloid, a two-sheet hyperboloid or a double cone. In any case, Σ has one or two subsets that are the boundary of a convex subset of $\mathbb{R}^n$; we will refer to them as the *sheets* of Σ . Each sheet of Σ is included in a different half-space bounded by Π . If Σ is not a double cone, then none of the sheets intersects Π .

Quadratic curves, surfaces and hypersurfaces, their properties and potential intersections have been the subject of a vast number of scientific works concerning both the Euclidean and the non-Euclidean case (to name just a few: Odehnal et al. [8], Lopez de Medrano [2], Pevzner [10], Jukl [3]). The object of interest in this paper are pairs of rotational quadratic hypersurfaces in $\mathbb{R}^n$ that share one focus but do not need to be confocal. In contrast with the large number of results regarding confocal hypersurfaces (many of them mentioned by Odehnal [8] and Lopez de Medrano [2]), works regarding hypersurfaces with just one common focus are rare. Moreover, most authors deal only with conics in 2D, e.g., Lawrence [4], MacNeish [5], Neville [7]. Working on a synthetic proof of a theorem concerning focus-sharing sets of three ellipses, Bogdanov proved an auxiliary statement that every two prolate spheroids in $\mathbb{R}^3$ sharing a focus have a planar intersection [1]. Not aware of this fact, Martín and Narvaez [6] formulated a conjecture stating that two rotational quadratic surfaces in $\mathbb{R}^3$ with a common focus have an intersection consisting of planar components. However, although the paper presents many examples that confirm the conjecture, it does not bring a mathematical proof.

Although not as frequent as the confocal hypersurfaces, rotational quadratic hypersurfaces with one common focus do appear in practice or in other research fields. For example, the above-mentioned paper of Martín and Narvaez [6] shows an interesting application in architecture. Using a circle packing of a sphere, they show how any rotational hypersurface can be packed by planar conics. Based on this, it is possible to build interesting constructions such as the Archimedean Pavilion in Seville [9]. The idea of the procedure is illustrated in Fig. 1. Another straightforward architectural application is in constructing buildings that are composed of several parts of rotational quadratic surfaces. If they have a common focus, we can be sure that the individual parts can be separated by planar walls or doors.

Our motivation for studying this problem came from Euclidean graph theory, namely, from studying the so-called LED trees [11]. These are Euclidean rooted trees, where all leaves have the same depth. This implies that for any vertex, the lengths of all its leaf paths must be equal. An example of such a tree is shown in Fig. 2, and potential applications can be found, for example, in computational phylogeny or network optimization. To see how this relates to the problem studied in this paper, let us consider, for example, the vertex X

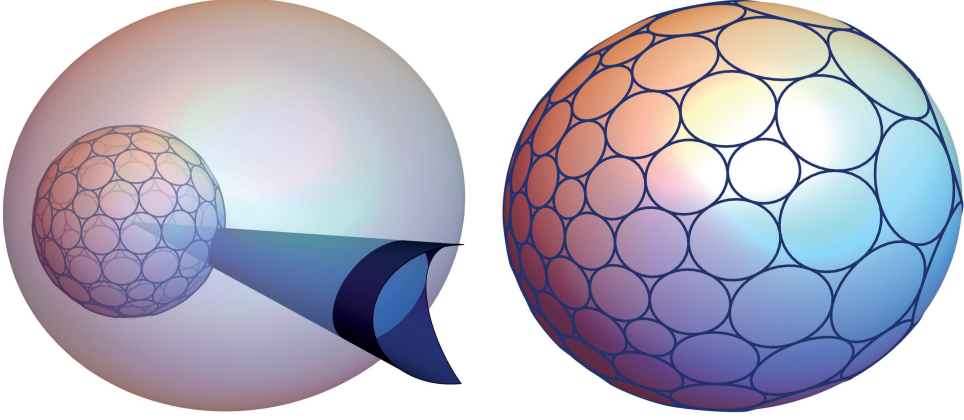


FIGURE 1. An example of an ellipse packing of a rotational ellipsoid. We start from an arbitrary circle packing of a sphere that is centered in a focus of the ellipsoid. Then we use cones with the same focus and containing the individual circles of the packing. The intersection of the cone and the ellipsoid is guaranteed to be planar (it is, in fact, an ellipse). Similar packing could be obtained in the same way for any other rotational quadratic hypersurface.

in Fig. 2. When studying the structure of the set of all LED trees, a natural question that arises is how we can move X , if the positions of all other vertices are fixed and we want to keep the equal depth of all the leaves. Let $a = |RX| + |XZ|$. If we want to preserve the length of all leaf paths of the root R , the movement of X is limited to the rotational ellipsoid with foci R and Z and the semi-major axis $a/2$. But, at the same time, the lengths of all leaf paths of X must be kept the same. If h_Y and h_Z are the heights of Y and Z , respectively, then this condition restricts the position of X to a hyperboloid sheet with foci Y and Z and semi-major axis $|h_Y - h_Z|/2$. As a result, the vertex X must lie at the intersection of a rotational ellipsoid and a hyperboloid sheet with a common focus Z . The specific problem that we studied was the connectedness of the set of all n -dimensional LED trees with given positions of leaves and a given isomorphism type. The connectedness of the intersection of two rotational hypersurfaces with a common focus is one of the arguments that can be used in order to prove a sufficient condition for path connectedness.

This paper brings two main results. Generalizing the idea of Bogdanov [1, Lemma 1], we show that for any two rotational quadratic hypersurfaces in $\mathbb{R}^n$ with a common focus, their intersection lies in at most two hyperplanes.

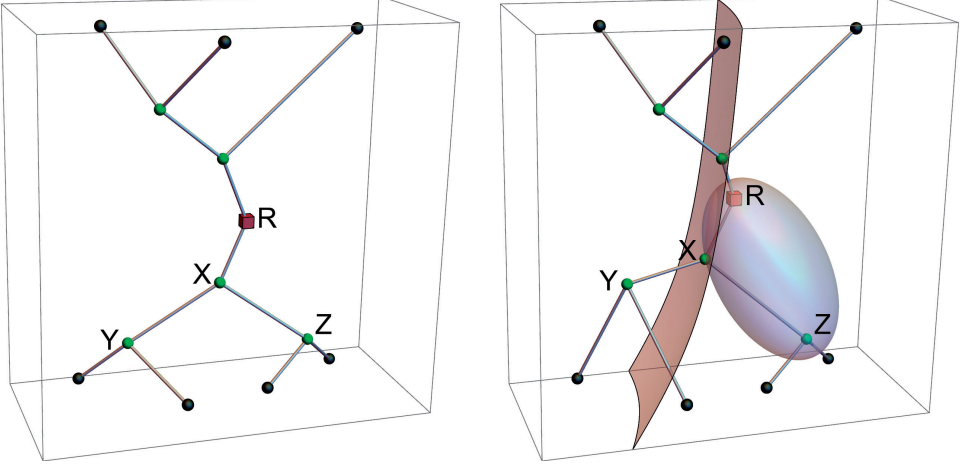


FIGURE 2. An example of a three dimensional LED (Leaves of Equal Depth) tree. Each leaf has to have the same depth, or, equivalently, all leaf path corresponding to a chosen vertex must have the same length. This condition implies that, given the positions of all other vertices, the vertex X has to lie in the intersection of a rotational ellipsoid and a rotational hyperboloid sheet sharing one focus (the vertex Z). Similar conditions hold true for the other inner vertices.

Then we provide an upper bound for the number of connected components of the intersection.

2. A proper common focus

THEOREM 2.1. *Let Σ_1 and Σ_2 be two rotational quadratic hypersurfaces defined by the equalities*

$$|XF| = e_1|X\Pi_1|, \quad |XF| = e_2|X\Pi_2|, \quad (2.1)$$

where $\Pi_1 \neq \Pi_2$. Then $\Sigma_1 \cap \Sigma_2$ lies in at most two hyperplanes.

Proof. According to (2.1), for any point $X \in \Sigma_1 \cap \Sigma_2$ we have

$$|XF| = e_1 d_1(X) \quad \wedge \quad |XF| = e_2 d_2(X), \quad (2.2)$$

where d_1 and d_2 are the distance functions of Π_1 and Π_2 . This implies

$$e_1 d_1(X) = e_2 d_2(X). \quad (2.3)$$

The distance function d of any hyperplane Π can be used to define a signed distance function δ of Π . The sign of $\delta(X)$ is given by the half-space bounded by Π , where X lies. Unlike d , the function δ is affine. Clearly, $-\delta$ is also a signed distance function of Π . Now, let δ_1 and δ_2 be signed distance functions of Π_1 and Π_2 . Any point X that satisfies (2.3) is also a solution of one of the two equations

$$e_1\delta_1(X) = e_2\delta_2(X), \quad e_1\delta_1(X) = -e_2\delta_2(X). \quad (2.4)$$

Since δ_1 and δ_2 are affine functions, the solution of each of these equations is a hyperplane. Thus, $\Sigma_1 \cap \Sigma_2$ lies in at most two hyperplanes. $\square$

THEOREM 2.2. *Let Σ_1 and Σ_2 be two rotational quadratic hypersurfaces in $\mathbb{R}^n$, $n \geq 3$, defined by the equalities (2.1), where $\Pi_1 \neq \Pi_2$. If c_1 and c_2 are the numbers of sheets of Σ_1 and Σ_2 , then $\Sigma_1 \cap \Sigma_2$ has at most c_1c_2 connected components. If Σ_1 and Σ_2 are not double cones, then each connected component of the intersection lies in a hyperplane.*

Proof. Let Π_1^1 and Π_1^2 be the two half-spaces bounded by Π_1 . As we have said earlier, c_1 (i.e., one or two) of them include a sheet of Σ_1 . Similarly, c_2 of the half-spaces Π_2^1 and Π_2^2 bounded by Π_2 include a sheet of Σ_2 . Further, let us consider the regions $\Pi^{ij} = \Pi_1^i \cap \Pi_2^j$, where $i = 1, 2$ and $j = 1, 2$. What we have said above implies that c_1c_2 of them can include an intersection of a sheet of Σ_1 and a sheet of Σ_2 . Now, let Ω_1 and Ω_2 be the two hyperplanes that solve (2.4). Since $\delta_1(X)$ and $\delta_2(X)$ are both zero for any $X \in \Pi_1 \cap \Pi_2$, we can see that $\Pi_1 \cap \Pi_2$ is included in both Ω_1 and Ω_2 . If $\Pi_1 \cap \Pi_2 = \emptyset$, then Ω_1 and Ω_2 are parallel to Π_1 and Π_2 . Either way, the equation (2.3) implies that only one of them can cross any Π^{ij} .

Let us first assume that Σ_1 is not a double cone. In that case, the first equation in (2.2) together with (2.4) imply the part of $\Sigma_1 \cap \Sigma_2$ contained in Π^{ij} is an intersection of a hyperplane and a single hypersurface that is the boundary of a convex set. In any dimension, such an intersection is the boundary of a convex set in the intersecting hyperplane. For $n \geq 3$, it is connected. Further, Σ_1 does not intersect Π_1 . This means that each part of $\Sigma_1 \cap \Sigma_2$ is contained in the interior of the corresponding region Π^{ij} . Thus, $\Sigma_1 \cap \Sigma_2$ can have up to c_1c_2 connected components, each of them lying in a hyperplane.

If both Σ_1 and Σ_2 are double cones, then we have $F \in \Pi_1 \cap \Pi_2$. This implies that both Ω_1 and Ω_2 contain F and that the part of $\Sigma_1 \cap \Sigma_2$ included in Π^{ij} is a connected set containing F . Hence, the whole $\Sigma_1 \cap \Sigma_2$ is a connected set. $\square$

To be specific, Theorem 2.2 says that two ellipsoids and two paraboloids with a common focus have a connected intersection lying in a hyperplane. The same is true for an ellipsoid and a paraboloid. The intersection of an ellipsoid

or a paraboloid with a hyperboloid or a double cone can have up to two connected components lying in two hyperplanes. Finally, the intersection of two hyperboloids or a hyperboloid and a double cone can have up to four connected components, and each of them can lie in one of two hyperplanes.

3. An improper common focus

In this part, we discuss the case when two hypersurfaces do not really have a common focus, but they are the limit of a sequence of pairs of hypersurfaces, for which the common focus exists. These situations are simple to resolve, however, we add them to fully cover the scope of the conjecture of Martín and Narvaez [6].

Let us first consider two ellipsoids with a common focus. If we move this focus along any line towards infinity, while fixing the other foci of the ellipsoids and making their eccentricities converge to 1, the limit surfaces will be two equally oriented paraboloids with parallel axes. The properties of their intersection follow from Theorems 2.1 and 2.2.

COROLLARY 3.1. *Let Σ_1 and Σ_2 be two different equally oriented rotational paraboloids with parallel axes. Then $\Sigma_1 \cap \Sigma_2$ is either an empty set or a connected subset of a hyperplane.*

Proof. Let F_1, F_2 and Π_1, Π_2 be the foci and the directrices of Σ_1 and Σ_2 , respectively. Any $X \in \Sigma_1 \cap \Sigma_2$ satisfies

$$|XF_1| = |X\Pi_1| \quad \text{and} \quad |XF_2| = |X\Pi_2|. \quad (3.1)$$

Since Σ_1 and Σ_2 are equally oriented and their axes are parallel, we have $\Pi_1 \parallel \Pi_2$ and $|X\Pi_2| = |X\Pi_1| + d$. Hence, $|XF_2| = |X\Pi_1| + d$. This, subtracted from the first equation in (3.1), yields

$$|XF_2| - |XF_1| = d,$$

which represents a sheet of a rotational hyperboloid with foci F_1, F_2 . Solving for (3.1) is thus equivalent to looking for the intersection of a paraboloid and a hyperboloid sheet with a common focus F_1 . This situation is covered by Theorems 2.1 and 2.2. $\square$

Now, let us have an ellipsoid and a paraboloid with a common focus. Let us move this focus along a line towards infinity so that we fix the other focus of the ellipsoid and let its eccentricity converge to 1. At the same time, we move the directrix of the paraboloid so that its distance from the moving focus is constant. The moving ellipsoid will converge to a paraboloid and the moving paraboloid will converge to a cylinder.

PROPOSITION 3.2. *Let Σ_1 be a rotational paraboloid and Σ_2 be a cylinder, whose axis is parallel to the axis of Σ_1 . Then $\Sigma_1 \cap \Sigma_2$ is either empty or is a connected subset of a hyperplane.*

Proof. Let F and Π be the focus and the directrix of Σ_1 and ξ the axis of Σ_2 . Since the axes of Σ_1 and Σ_2 are parallel, we have $\xi \perp \Pi$. Let $P = \Pi \cap \xi$ and let r be the radius of Σ_2 . Then any $X \in \Sigma_1 \cap \Sigma_2$ has to satisfy

$$|XF| = |X\Pi| \quad \wedge \quad |XP|^2 = |X\Pi|^2 + r^2. \quad (3.2)$$

Substituting the first equation into the second, we get

$$|XP|^2 = |XF|^2 + r^2.$$

This equation represents a hyperplane and thus solving (3.2) is equivalent to finding an intersection of a paraboloid (or a cylinder) with a hyperplane. This proves our statement. $\square$

Finally, starting from two paraboloids with a common focus and moving this focus along a line towards infinity similarly to above, we get two parallel cylinders, for which the statements of Theorems 2.1 and 2.2 are clearly true.

4. A concluding remark

Considering a general Euclidean space $\mathbb{E}^n$, equation (1.1) defines a hypersurface under the same conditions as in $\mathbb{R}^n$. All the steps of the proofs presented in the previous sections hold true as well. Thus, all our statements are valid also for hypersurfaces in $\mathbb{R}^n$ that are not rotational with respect to its standard inner product, but they can be represented by equations (2.1) with respect to another inner product.

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