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TATRA MOUNTAINS
Mathematical Publications

DOI: 10.2478/tmmp-2024-0012
Tatra Mt. Math. Publ. **88** (2024), 59–72

STABILITY AND HOPF BIFURCATION IN A MODIFIED SPROTT C SYSTEM

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ABSTRACT. In this article, a Modified Sprott C system is considered. The stability of equilibrium points and the occurrence of Hopf bifurcation in the system are investigated. It has been proved that the system displays a Hopf bifurcation at $\alpha = 0$. Additionally, by applying normal form theory, the stability, direction and increase (or decrease) of the period of bifurcating periodic solutions of the system are illustrated. It has been shown that the solutions of bifurcating periodic solutions at the bifurcation value $\alpha = 0$ are unstable. The type of Hopf bifurcation is subcritical and the periods of the bifurcating periodic solutions increase.

1. Introduction

Bifurcation, as a scientific term, is used to describe significant and qualitative changes that occur in the solution curves of a dynamical system when key system parameters are varied. It is often employed to describe the qualitative changes in the stability of solution curves in nonlinear dynamical systems [9]. Bifurcation occurs when a small change in parameter values causes a sudden qualitative or topological change in behaviour. Generally, at a bifurcation, the local stability properties of equilibrium points, periodic solutions, or other invariant sets undergo a change. When such a change occurs and leads to the appearance or disappearance of periodic solutions, it is referred to as Hopf bifurcation [3].

In general, the study of Hopf bifurcation involves investigating the existence, non-existence, direction and stability of the bifurcating periodic solutions. Hopf bifurcation refers to the occurrence or disappearance of a periodic solution from an equilibrium point as a parameter crosses a critical value. It has

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2020 Mathematics Subject Classification: 37G10.
Keywords: Hopf Bifurcation, Normal Form Theory, Periodic Solutions.
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played a crucial role in the development of the theory of dynamical systems across various dimensions. The uniqueness of this bifurcation lies in two aspects that differentiate it from other common types such as saddle-node, transcritical, pitchfork, etc. Firstly, Hopf bifurcation cannot occur in one dimension; it requires a minimum dimensionality of two. Secondly, it deals with the appearance or disappearance of a periodic solution [12]. There are two types of Hopf bifurcation: the supercritical Hopf bifurcation, where stable limit cycles are created around an unstable equilibrium point and the subcritical Hopf bifurcation, where an unstable limit cycle is created around a stable equilibrium point [7].

In 1994, Sprott conducted an exhaustive computer search and discovered 19 distinct simple three-dimensional autonomous systems. These systems are labelled as Sprott A, Sprott B, $\cdots$, Sprott S. Among these systems, Sprott $(A-E)$ each consisting of five terms, with two of them being nonlinear, while Sprott $(F-S)$ each consisting of six terms, with one of them being nonlinear. The systems can have a maximum of two equilibrium points [14]. The Sprott C system which is given by

$$\begin{aligned} \dot{x} &= yz, \\ \dot{y} &= x - y, \\ \dot{z} &= 1 - x^2, \end{aligned} \tag{1}$$

where x, y and z are variables [13–15]. Mota and Oliveira introduced a new parametrized system called the Sprott BC system, which serves as a bridge between types B and C of the Sprott systems [10]. They have proved that the system undergoes a Hopf bifurcation at a specific bifurcation value. By analysing the first Liapunov coefficient of the system at its equilibria, they demonstrated the presence of an unstable limit cycle. Furthermore, they showed that the system lacks a polynomial first integral, algebraic invariant surfaces, and a Darboux first integral. Dong et al. modified the system (1) by adding components to the three equations of the Sprott C system [2]. They demonstrated the occurrence of a Hopf bifurcation in the modified system. Wie and Yang introduced a new system that is topologically equivalent to the system (1) [16]. They illustrated that an unstable limit cycle can bifurcate from one of the equilibrium points. Additionally, they confirmed that the system exhibits chaotic behaviour for certain specific parameter values while remaining non-chaotic for others. In this paper, we have introduced a new system based on adding an extra term for Sprott C system and we call it the Modified Sprott C system which is given as follows

$$\begin{aligned} \dot{x} &= yz, \\ \dot{y} &= x - y, \\ \dot{z} &= 1 - x^2 + \alpha\, z, \end{aligned} \tag{2}$$

where $\alpha \in \mathbb{R}$. This system with $\alpha = 0$ is the Sprott C system [14]. In this paper, the stability of equilibrium points and the existence of Hopf bifurcation

for system (2) are investigated. Additionally, the stability and direction of the bifurcating periodic solutions of the system are also studied via applying normal form theory.

2. Preliminaries

This section provides a review of normal form theory and how to apply it to determine the direction, stability and period of the bifurcating periodic solutions [4]. We consider the following nonlinear system

$$\dot{U} = AU + F(U), \tag{3}$$

where the variable $U \in \mathbb{R}^3$, AU is the linear part of the system, F is a non-linear analytic function, μ is the bifurcation parameter, $u_0(\mu)$ be an equilibrium point and $A(\mu)$ be the Jacobian matrix of (3) at $u_0(\mu)$. Let λ_1, λ_2 and λ_3 be eigenvalues of $A(\mu)$ satisfying $Re(\lambda_1) \geq Re(\lambda_2) \geq Re(\lambda_3)$, provided that $\lambda_1(\mu)$ and $\lambda_2(\mu)$ are a conjugate pair $(\lambda_1(\mu) = \overline{\lambda_2(\mu)})$ for μ in an open interval including μ_0. If $Re\big(\lambda_1(\mu_0)\big) = 0$, $Re\big(\lambda_1'(\mu_0)\big) \neq 0$, $Im\big(\lambda_1(\mu_0)\big) \neq 0$ and $Re(\lambda_3(\mu_0)) < 0$, then a Hopf bifurcation occurs at $\mu = \mu_0$ [4]. If the Jacobian matrix of the system at the bifurcation value $\mu = \mu_0$ is not in the following form

$$\begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

where $\omega = Im\big(\lambda_1(\mu_0)\big)$, we can transform it by the following matrix

$$T = \begin{pmatrix} Re(v_1) & -Im(v_1) & v_2 \end{pmatrix},$$

where v_1 is the eigenvector of $A(\mu_0)$ corresponding to $\lambda_1(\mu_0) = i\omega$ and v_2 is the eigenvector of $A(\mu_0)$ corresponding to $\lambda_3(\mu_0)$. Perform the change of variables

$$U = u_0(\mu_0) + TY,$$

and let

$$\dot{Y} = F(Y),$$

denote the system for Y. The Jacobian matrix $\frac{\partial F_i}{\partial Y_j(0)}$ $(i, j = 1, 2, 3)$ will have the real canonical form

$$\frac{\partial F}{\partial Y}(0) = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

In the following, we will follow the procedures by Hassard *et al.* [4] to work out the necessary quantities and all quantities are evaluated at $\mu = \mu_0$ and $Y = 0$.

$$g_{11} = \frac{1}{4}\left[\frac{\partial^2 F_1}{\partial y_1^2} + \frac{\partial^2 F_1}{\partial y_2^2} + i\left[\frac{\partial^2 F_2}{\partial y_1^2} + \frac{\partial^2 F_2}{\partial y_2^2}\right]\right],$$

$$g_{02} = \frac{1}{4}\left[\frac{\partial^2 F_1}{\partial y_1^2} - \frac{\partial^2 F_1}{\partial y_2^2} - 2\,\frac{\partial^2 F_2}{\partial y_1\,\partial y_2} + i\left[\frac{\partial^2 F_2}{\partial y_1^2} - \frac{\partial^2 F_2}{\partial y_2^2} + 2\,\frac{\partial^2 F_1}{\partial y_1\,\partial y_2}\right]\right],$$

$$g_{20} = \frac{1}{4}\left[\frac{\partial^2 F_1}{\partial y_1^2} - \frac{\partial^2 F_1}{\partial y_2^2} + 2\,\frac{\partial^2 F_2}{\partial y_1\,\partial y_2} + i\left[\frac{\partial^2 F_2}{\partial y_1^2} - \frac{\partial^2 F_2}{\partial y_2^2} - 2\,\frac{\partial^2 F_1}{\partial y_1\,\partial y_2}\right]\right],$$

$$\begin{aligned} G_{21} = \frac{1}{8}\Bigg[&\frac{\partial^3 F_1}{\partial y_1^3} + \frac{\partial^3 F_1}{\partial y_1\,\partial y_2^2} + \frac{\partial^3 F_2}{\partial y_1^2\,\partial y_2} + \frac{\partial^3 F_2}{\partial y_2^3} \\ &+ i\left[\frac{\partial^3 F_2}{\partial y_1^3} + \frac{\partial^3 F_2}{\partial y_1\,\partial y_2^2} - \frac{\partial^3 F_1}{\partial y_1^2\,\partial y_2} - \frac{\partial^3 F_1}{\partial y_2^3}\right]\Bigg]. \end{aligned}$$

Next, we calculate

$$h_{11} = \frac{1}{4}\left[\frac{\partial^2 F_3}{\partial y_1^2} + \frac{\partial^2 F_3}{\partial y_2^2}\right],$$

$$h_{20} = \frac{1}{4}\left[\frac{\partial^2 F_3}{\partial y_1^2} - \frac{\partial^2 F_3}{\partial y_2^2} - 2i\,\frac{\partial^2 F_3}{\partial y_1\,\partial y_2}\right].$$

By solving the following equations

$$\lambda_3\, w_{11} = -h_{11},$$

$$(\lambda_3 - 2i\,\omega)\, w_{20} = -h_{20},$$

the values of w_{11} and w_{20} are obtained. Furthermore, there is

$$G_{110} = \frac{1}{2}\left[\frac{\partial^2 F_1}{\partial y_1\,\partial y_3} + \frac{\partial^2 F_2}{\partial y_2\,\partial y_3} + i\left[\frac{\partial^2 F_2}{\partial y_1\,\partial y_3} - \frac{\partial^2 F_1}{\partial y_2\,\partial y_3}\right]\right],$$

$$G_{101} = \frac{1}{2}\left[\frac{\partial^2 F_1}{\partial y_1\,\partial y_3} - \frac{\partial^2 F_2}{\partial y_2\,\partial y_3} + i\left[\frac{\partial^2 F_2}{\partial y_1\,\partial y_3} + \frac{\partial^2 F_1}{\partial y_2\,\partial y_3}\right]\right],$$

$$g_{21} = G_{21} + 2\,G_{110}\,w_{11} + G_{101}\,w_{20}.$$

Based on the above analysis, one can compute the following quantities

$$C_1(0) = \frac{i}{2\omega}\left[g_{20}\,g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right] + \frac{1}{2}\,g_{21}.$$

So, we can calculate the following quantities

$$\mu_2 = \frac{-Re(C_1(0))}{\alpha'(0)},$$

$$\beta_2 = 2\ Re\big(C_1(0)\big),$$

$$\tau_2 = \frac{-(Im(C_1(0) + \mu_2\ w'(0))}{\omega},$$

where

$$\alpha'(0) = Re\big(\lambda_1'(\mu_0)\big) \quad \text{and} \quad w'(0) = Im\big(\lambda_1'(\mu_0)\big).$$

μ_2 determines the direction of the Hopf bifurcation; β_2 determines the stability of the bifurcating periodic solutions; τ_2 determines the increases (decreases) of the period of bifurcating periodic solutions.

- When $\mu_2 > 0$ ($\mu_2 < 0$), the Hopf bifurcation is supercritical (subcritical) and bifurcating periodic solutions exist for $\mu > \mu_0$ ($\mu < \mu_0$).
- When $\beta_2 < 0$ ($\beta_2 > 0$), the solutions of bifurcating periodic solutions are stable (unstable).
- When $\tau_2 > 0$ ($\tau_2 < 0$), the period of bifurcating periodic solutions increases (decreases).

For more details, see [1, 6, 8, 17–19].

3. Equilibrium points and their stability

In this section, the stability of equilibrium points of system (2) is studied. The system has three equilibrium points $S_{1,2} = (\pm 1, \pm 1, 0)$ and $S_3 = (0, 0, -\frac{1}{\alpha})$. Due to the symmetry of S_1 and S_2, it suffices to analyse only S_1.

Proposition 3.1.

(I) *The equilibrium point S_1 is asymptotically stable if and only if $\alpha < 0$.*

(II) *The equilibrium point S_1 is unstable if and only if $\alpha > 0$.*

Proof. In order to transform the equilibrium point S_1 to the origin, $S_{10} = (0, 0, 0)$, the transformations $x - 1 \to x_1$, $y - 1 \to x_2$ and $z \to x_3$ are used. Then system (2) changes into

$$\begin{aligned} \dot{x}_1 &= x_3 + x_2\ x_3, \\ \dot{x}_2 &= x_1 - x_2, \\ \dot{x}_3 &= \alpha\ x_3 - x_1^2 - 2\ x_1, \end{aligned} \tag{4}$$

from system (4), the Jacobian matrix at the original S_{10} is

$$J(S_{10}) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ -2 & 0 & \alpha \end{pmatrix},$$

and the characteristic equation is

$$\lambda^3 + a_1\ \lambda^2 + a_2\ \lambda + a_3 = 0, \tag{5}$$

where $a_1 = 1 - \alpha,\ a_2 = 2 - \alpha$ and $a_3 = 2$.

(I) Suppose that S_1 is asymptotically stable and $\alpha \neq 0$ ($\alpha = 0$ is a bifurcation value). Let $\alpha > 0$, then the roots of equation (5) are $\lambda_{1,2} = \frac{\alpha}{2} \pm \frac{\sqrt{\alpha^2 - 8}}{2}$ and $\lambda_3 = -1$.

- If $\alpha \in (0, 2\sqrt{2})$, then the real parts of $\lambda_{1,2}$ is $\frac{\alpha}{2}$ which is positive. Thus, S_{10} is unstable [11], which is contradiction.
- If $\alpha \in [2\sqrt{2}, \infty)$, then the values of $\lambda_{1,2}$ are always positive (see Figure 1). Thus, S_{10} is unstable [11], which is contradiction.

Therefore, the value of α is negative.

Conversely, suppose that $\alpha < 0$, this implies that $a_1 = 1 - \alpha > 0,\ a_1 a_2 - a_3 = \alpha\ (\alpha - 3) > 0$ and already $a_3 = 2 > 0$. Thus, by Hurwitz's Theorem all roots of equation (5) have negative real parts [5]. Therefore, S_{10} is asymptotically stable (see Figure 2-(a)).

(II) Suppose that S_1 is unstable and $\alpha \neq 0$ ($\alpha = 0$ is a bifurcation value). Let α is negative, then by Proposition 3.1-(I), S_{10} is asymptotically stable, which is contradiction. Therefore, the value of α should be positive.

Conversely, suppose that $\alpha > 0$, from the roots of equation (5), we note that:

- When $\alpha \in (0, 2\sqrt{2})$, the values of $\lambda_{1,2}$ are complex and their real parts are positive. Therefore, S_{10} is unstable [11].
- When $\alpha \in [2\sqrt{2}, \infty)$, $\lambda_{1,2}$ are positive real values. Therefore, S_{10} is unstable [11] (see Figure 2-(b)) □

Proposition 3.2. *The equilibrium point S_3 is unstable.*

Proof. We change coordinates by letting $y_1 = x,\ y_2 = y$ and $y_3 = z + \frac{1}{\alpha}$, then $\dot{y}_1 = \dot{x},\ \dot{y}_2 = \dot{y}$ and $\dot{y}_3 = \dot{z}$, system (2) becomes

$$\begin{aligned} \dot{y}_1 &= y_2\ y_3 - \frac{1}{\alpha}\ y_2, \\ \dot{y}_2 &= y_1 - y_2, \end{aligned} \tag{6}$$

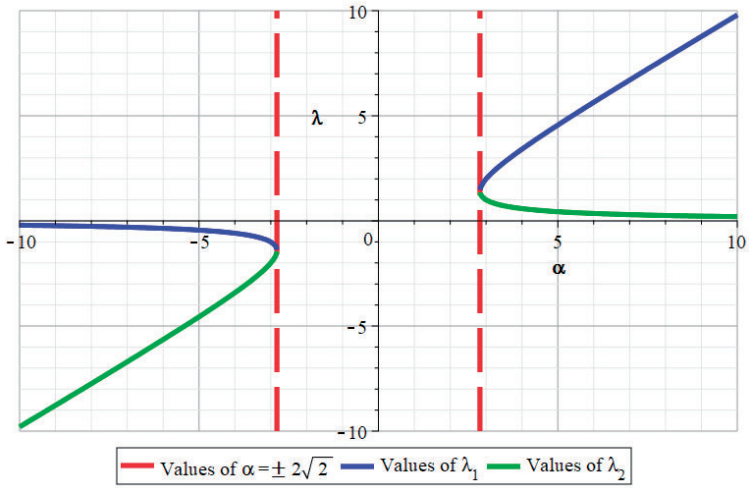


FIGURE 1. The plot of $\lambda_{1,2} = \frac{\alpha}{2} \pm \frac{\sqrt{\alpha^2-8}}{2}$ for $\alpha \in (-\infty, -2\sqrt{2}) \cup [2\sqrt{2}, \infty)$.

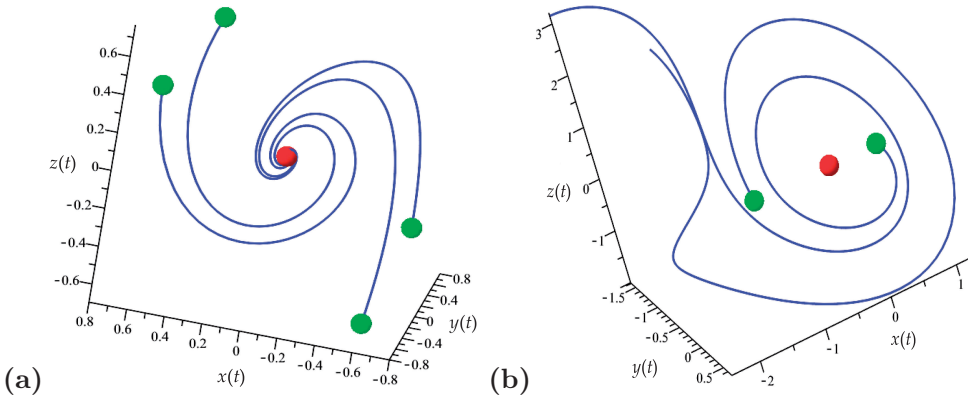


FIGURE 2. The phase portrait of system (4)) when **(a)** $\alpha = -1$ **(b)** $\alpha = 0.1$.

$$\dot{y}_3 = \alpha\ y_3 - {y_1}^2,$$

the equilibrium point now the equilibrium point is $S_{30} = (0, 0, 0)$. The characteristic equation of evaluated (6) at S_{30} is

$$\lambda^3 + (1-\alpha)\ \lambda^2 + \left(\frac{1-\alpha^2}{\alpha}\right)\lambda + 1 = 0. \tag{7}$$

The roots of equation (7) are $\lambda_{1,2} = \frac{-\alpha \pm \sqrt{\alpha^2-4\alpha}}{2\alpha}$ and $\lambda_3 = \alpha$. Since $\alpha \neq 0$, then $\alpha > 0$ or $\alpha < 0$.

- When $\alpha > 0$, λ_3 is positive (see Figure 3). Therefore, S_{30} is unstable (see Figure 4-(a)) [3].
- When $\alpha < 0$, λ_2 is positive (see Figure 3). Therefore, S_{30} is unstable (see Figure 4-(b)) [3]. □

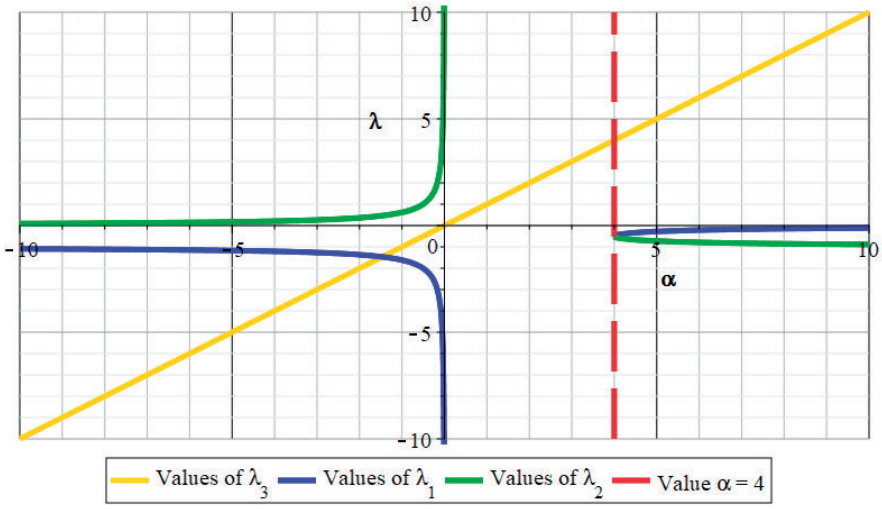


FIGURE 3. The plots of $\lambda_{1,2} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\alpha}}{2\alpha}$ for $\alpha \in (-\infty, 0) \cup [4, \infty)$ and $\lambda_3 = \alpha$.

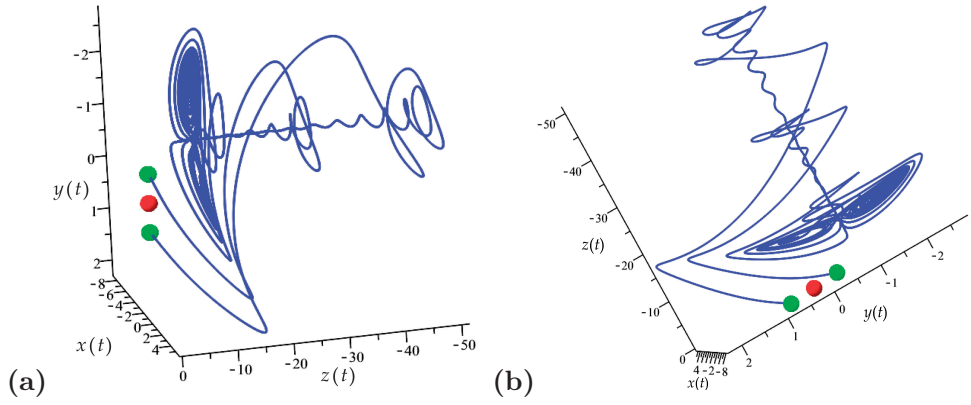


FIGURE 4. The phase portrait of system (6) when **(a)** $\alpha = 0.1$ **(b)** $\alpha = -0.1$.

4. The occurrence of Hopf bifurcation

In this section, the occurrence of Hopf bifurcation, the direction, stability and period of bifurcating periodic solutions for system (4) are studied via finding the values of μ_2, β_2 and τ_2, which are introduced in [4].

Proposition 4.1. *When α passes through the critical value $\alpha_0 = 0$, system (4) undergoes a Hopf bifurcation at the equilibrium S_{10}.*

Proof. Since $\alpha = 0$, then equation (5) changes into

$$(\lambda + 1)(\lambda^2 + 2) = 0.$$

Therefore, equation (5) has a pair of purely imaginary conjugate roots $\lambda_{1,2} = \pm\sqrt{2}\ i$ with a real root $\lambda_3 = -1$. Thus, we can choose α as the bifurcation parameter and the critical value is $\alpha_0 = 0$. Implicitly differentiating $\lambda = \lambda(\alpha_0)$ in equation (5), we have

$$\lambda'(\alpha_0) = \frac{-1}{3\,\lambda^2 + 2\lambda + 2},$$

hence

$$Re\big(\lambda'(\alpha_0)\big)\big|_{\lambda=i\sqrt{2}} = \frac{1}{2} \quad \text{and} \quad Im\big(\lambda'(\alpha_0)\big)\big|_{\lambda=i\sqrt{2}} = 0.$$

Since λ_1 and λ_2 are a conjugate pair

$$\big(\lambda_1(\alpha) = \overline{\lambda}_2(\alpha)\big),\ \ Re\big(\lambda_1{}'(\alpha_0)\big) \neq 0,\ \ Im\big(\lambda_1(\alpha_0)\big) \neq 0,\ \ \text{and}\ \ \big(Re\lambda_3(\alpha_0)\big) = -1 < 0,$$

then a Hopf bifurcation occurs. □

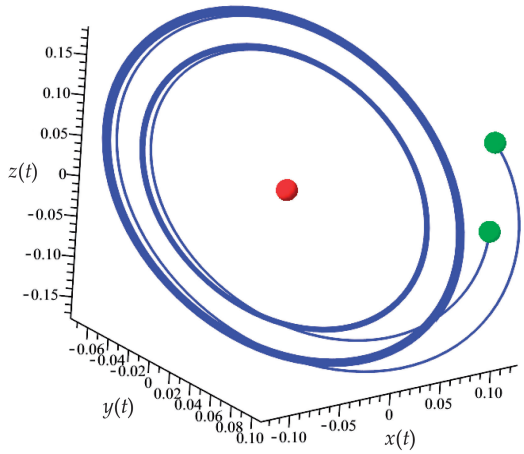


Figure 5. The phase portrait of system (4) when $\alpha = 0$.

Now, we apply normal form theory to find the direction, stability and period of bifurcating periodic solutions for system (4).

At first, we find the eigenvectors. Let the eigenvectors correspond to the eigenvalues λ_1 and λ_3 be v_1 and v_2, respectively.
The eigenvector v_1 corresponding to $\lambda_1 = i\sqrt{2}$ is

$$v_1 = \begin{pmatrix} 1 + i\ \sqrt{2} \\ 1 \\ -2 + i\ \sqrt{2} \end{pmatrix}.$$

The eigenvector v_2 corresponding to $\lambda_3 = -1$ is

$$v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Second, system (4) has been transformed into the canonical form by the following linear transformation.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = T \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad \text{where} \qquad T = \big(Re(v_1) - Im(v_1)v_2\big) = \begin{pmatrix} 1 & -\sqrt{2} & 0 \\ 1 & 0 & 1 \\ -2 & -\sqrt{2} & 0 \end{pmatrix}.$$

Therefore, the following system is obtained

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} -\sqrt{2}\ v \\ \sqrt{2}\ u \\ -\ w \end{pmatrix} + \begin{pmatrix} F_1(u,v,w) \\ F_2(u,v,w) \\ F_3(u,v,w) \end{pmatrix}, \tag{8}$$

where

$$\begin{aligned} F_1(u,v,w) &= \frac{-1}{3}\ u^2 + \frac{2}{3}\ v^2 - \sqrt{2}\ uv - \frac{2}{3}\ uw - \frac{\sqrt{2}}{3}\ vw, \\ F_2(u,v,w) &= \frac{5\sqrt{2}}{3}\ u^2 + \frac{\sqrt{2}}{3}\ v^2 + \frac{2\sqrt{2}}{3}\ uw + \frac{2}{3}\ vw, \\ F_3(u,v,w) &= \frac{1}{3}\ u^2 - \frac{2}{3}\ v^2 + \sqrt{2}\ uv + \frac{2}{3}\ uw + \frac{\sqrt{2}}{3}\ vw. \end{aligned}$$

In the following, we will follow the procedures proposed by Hassard *et al.* [4] to work out the necessary quantities. Also, all quantities are evaluated at $\alpha = 0$ and $(u, v, w) = (0, 0, 0)$.

$$g_{11} = \frac{1}{4}\left[\frac{\partial^2 F_1}{\partial u^2} + \frac{\partial^2 F_1}{\partial v^2} + i\left[\frac{\partial^2 F_2}{\partial u^2} + \frac{\partial^2 F_2}{\partial v^2}\right]\right] = \frac{1}{6} + \frac{7\sqrt{2}}{12}\, i,$$

$$\begin{aligned} g_{02} &= \frac{1}{4}\left[\frac{\partial^2 F_1}{\partial u^2} - \frac{\partial^2 F_1}{\partial v^2} - 2\,\frac{\partial^2 F_2}{\partial u\,\partial v} + i\left[\frac{\partial^2 F_2}{\partial u^2} - \frac{\partial^2 F_2}{\partial v^2} + 2\,\frac{\partial^2 F_1}{\partial u\,\partial v}\right]\right] \\ &= \frac{-1}{2} - \frac{\sqrt{2}}{4}\, i, \end{aligned}$$

$$\begin{aligned} g_{20} &= \frac{1}{4}\left[\frac{\partial^2 F_1}{\partial u^2} - \frac{\partial^2 F_1}{\partial v^2} + 2\frac{\partial^2 F_2}{\partial u\,\partial v} + i\left[\frac{\partial^2 F_2}{\partial u^2} - \frac{\partial^2 F_2}{\partial v^2} - 2\,\frac{\partial^2 F_1}{\partial u\,\partial v}\right]\right] \\ &= \frac{-1}{2} + \frac{3\sqrt{2}}{4}\, i, \end{aligned}$$

$$\begin{aligned} G_{21} = \frac{1}{8}\Bigg[&\frac{\partial^3 F_1}{\partial u^3} + \frac{\partial^3 F_1}{\partial u\,\partial v^2} + \frac{\partial^3 F_2}{\partial u^2\,\partial v} + \frac{\partial^3 F_2}{\partial v^3} \\ &+ i\left[\frac{\partial^3 F_2}{\partial u^3} + \frac{\partial^3 F_2}{\partial u\,\partial v^2} - \frac{\partial^3 F_1}{\partial u^2\,\partial v} - \frac{\partial^3 F_1}{\partial v^3}\right]\Bigg] = 0. \end{aligned}$$

Next, the values of h_{11} and h_{20} are found by the following

$$h_{11} = \frac{1}{4}\left[\frac{\partial^2 F_3}{\partial u^2} + \frac{\partial^2 F_3}{\partial v^2}\right] = \frac{-1}{6},$$

$$h_{20} = \frac{1}{4}\left[\frac{\partial^2 F_3}{\partial u^2} - \frac{\partial^2 F_3}{\partial v^2} - 2i\,\frac{\partial^2 F_3}{\partial u\,\partial v}\right] = \frac{1}{2} - \frac{\sqrt{2}}{2}\, i.$$

By solving the following equations

$$\lambda_3\, w_{11} = -h_{11},$$

$$(\lambda_3 - 2i\,\omega)\, w_{20} = -h_{20},$$

the values of w_{11} and w_{20} are obtained, as follows

$$w_{11} = \frac{-1}{6},$$

$$w_{20} = \frac{-1}{6} - \frac{\sqrt{2}}{6}\, i.$$

Furthermore, there is

$$G_{110} = \frac{1}{2}\left[\frac{\partial^2 F_1}{\partial u\ \partial w} + \frac{\partial^2 F_2}{\partial v\ \partial w} + i\left[\frac{\partial^2 F_2}{\partial u\ \partial w} - \frac{\partial^2 F_1}{\partial v\ \partial w}\right]\right] = \frac{\sqrt{2}}{2}\ i,$$

$$G_{101} = \frac{1}{2}\left[\frac{\partial^2 F_1}{\partial u\ \partial w} - \frac{\partial^2 F_2}{\partial v\ \partial w} + i\left[\frac{\partial^2 F_2}{\partial u\ \partial w} + \frac{\partial^2 F_1}{\partial v\ \partial w}\right]\right] = \frac{-2}{3} + \frac{\sqrt{2}}{6}\ i,$$

$$g_{21} = G_{21} + 2\ G_{110}\ w_{11} + G_{101}\ w_{20} = \frac{1}{6} - \frac{\sqrt{2}}{12}\ i.$$

Based on the above analysis, one can compute the following quantities

$$C_1\ (0) = \frac{i}{2\omega}\ [g_{20}\ g_{11} - 2\ |g_{11}|^2 - \frac{1}{3}\ |g_{02}|^2] + \frac{1}{2}\ g_{21} = \frac{1}{6} - \frac{2\sqrt{2}}{3}\ i$$

so, the following quantities can be calculated

$$\mu_2 = \frac{-Re(C_1(0))}{\alpha'(0)} = -\frac{1}{3},$$

$$\beta_2 = 2\ Re\big(C_1(0)\big) = \frac{1}{3},$$

$$\tau_2 = \frac{-(Im(C_1(0) + \mu_2\ w'(0))}{\omega} = \frac{2}{3},$$

where $\alpha'(0) = \frac{1}{2}$ and $w'(0) = 0$.

Therefore, by above calculations and Proposition 4.1 the following theorem is obtained.

Theorem 4.2. *System* (4) *exhibits a Hopf bifurcation at the equilibrium* S_{10} *as* α *passes through* α_0*, with the following properties*

- **(I)** *Since* $\mu_2 < 0$*, then the Hopf bifurcation is Subcritical and bifurcating periodic solutions exist for* $\alpha < \alpha_0$*.*
- **(II)** *Since* $\beta_2 > 0$*, then the solution of bifurcating periodic solutions are unstable.*
- **(III)** *Since* $\tau_2 > 0$*, then the period of bifurcating periodic solutions increases.*

5. Conclusions

The Modified Sprott C system (2) has been introduced and investigated. It has been demonstrated that the equilibrium point $S_3 = (0, 0, -\frac{1}{\alpha})$; $\alpha \neq 0$ is unstable, while $S_1 = (1, 1, 0)$ is asymptotically stable for $\alpha < 0$ and unstable for $\alpha > 0$. Additionally, it has been proved that the bifurcating periodic solutions

at the bifurcation value $\alpha = 0$ are unstable. The Hopf bifurcation is of the subcritical type, and the period of the bifurcating periodic solutions increases.

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Received December 28, 2022
Revised May 12, 2024
Accepted June 13, 2024
Publ. online July 30, 2024

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