

AN APPLICATION OF ϕ -METRIC AND RELATED BEST PROXIMITY POINT RESULTS GENERALIZING WARDOWSKI'S FIXED POINT THEOREM

ABHISHIKTA DAS¹ — SUMIT SOM² — HEMANTA KALITA² —
TARAPADA BAG¹

¹Department of Mathematics, Siksha-Bhavana, Visva-Bharati, Santiniketan, INDIA

²Mathematics Division, School of Advanced Sciences and Languages, VIT Bhopal University,
Bhopal - Indore Highway, INDIA

ABSTRACT. In this article, an application of ϕ -metric is given via a geometrical example to show how it can help to measure distance for non-planar surfaces where the classical metric becomes incapable. We also introduce the concept of best proximity point and proximal contraction for a class of mappings in a ϕ -metric space and prove a best proximity point theorem for such class of contraction mappings from which the famous “Wardowski’s fixed point theorem” can be deduced as a particular case. To support our theorem, we provide an example in which Wardowski’s metric fixed point theorem cannot be applied.

1. Introduction

The theory of fixed points in metric space has various applications in non-linear analysis. Many authors studied the famous ‘Banach fixed point theorem’, its applications, and tried to generalize it by reforming the contraction principle in metric space or by generalizing the concept of metric space. As a result, many generalized metric spaces have been developed. One of them is ϕ -metric space recently introduced by Das et al. [1]. The ϕ -metric extends the idea of some generalized metric spaces (e.g., S-metric, b-metric) and can play the role of metrics

© 2024 Mathematical Institute, Slovak Academy of Sciences.

2020 Mathematics Subject Classification: 54E99, 47H10, 54H25.

Keywords: ϕ -metric space, ϕ -best proximity point, $\mathcal{F}$ -proximal contraction, Wardowski’s fixed point theorem.



Licensed under the Creative Commons BY-NC-ND 4.0 International Public License.

in many scenarios. In [1], it is discussed that the ‘triangle inequality’ is not necessarily affirmed on non-planar surfaces, due to which the usual metric may not be able to study the distance on such spaces. The ϕ -metric can help in this scenario. In this connection, in Section 3, we give a geometrical example as an application of ϕ -metric that shows how ϕ -metric can help to find out the distance between two points in non-planar surfaces where the classical metric fails.

The metric fixed point is essentially a sufficient condition for the existence of a point ζ which satisfies $\mathcal{T}\zeta = \zeta$ for a self-map $\mathcal{T}$ on the metric space $(\mathcal{X}, \mathcal{D})$. On the other hand, if a non-self mapping $\mathcal{T}$ is considered from $\mathcal{A}$ into $\mathcal{B}$ in $(\mathcal{X}, \mathcal{D})$, then $\mathcal{T}(\mathcal{A}) \cap \mathcal{A} \neq \emptyset$ is a necessary condition for existing a solution for $\mathcal{T}\zeta = \zeta$. But if $\mathcal{T}(\mathcal{A}) \cap \mathcal{A} = \emptyset$, then there must be a point in $\mathcal{A}$ that minimizes the distance between $\mathcal{T}(\mathcal{A})$ and $\mathcal{A}$. During the last few years it has been an interesting problem of finding such elements in metric space and in its hybrids. In 2010, Basha [3] first studied the theory of best proximity point in metric spaces. Later, the researchers developed more results on the best proximity point (for details, see [4–9]). Mathematicians have been doing different variants of contraction mappings [10–14] day by day using the theory of best proximity point, and the study is still ongoing.

Being encouraged by these literatures, in this article, we introduce the concept of the best proximity point in ϕ -metric space. The primary object of this article is to study the best proximity point theorems for contraction mappings in the setting of ϕ -metric space. We introduce the concept of ‘ ϕ -best proximity point’ and ‘ $\mathcal{F}$ -proximal contraction’ in ϕ -metric space that generalizes and improves existing results on $\mathcal{F}$ -contraction mappings of classical metric space.

$\mathcal{F}$ -contraction mappings is a special type of non-linear contraction introduced by Wardowski [2] in 2012 which ensures the existence of unique fixed point of self mappings in a metric space satisfying $\mathcal{F}$ -contraction condition. Being inspired by such a contraction mappings, we introduce ‘ $\mathcal{F}$ -proximal contraction’ in ϕ -metric space and study the existence of a unique ϕ -best proximity point for a self-mapping $\mathcal{T} : \mathcal{A} \rightarrow \mathcal{B}$ in $(\mathcal{X}, d_\phi)$. Using the example of ϕ -metric given in Section 3, we justify the result of Section 4. Since this particular ϕ -metric is not a metric, Wardowski’s fixed point theorem of classical metric space cannot be applied in this case.

The article is organized as follows. In Section 2, some definitions and results are collected from the existing literature that are used in our main results sections. In Section 3, an application of ϕ -metric space is discussed and illustrated by a geometrical example. Section 4 consists of the definition of the best proximity point and best proximity point theorems for $\mathcal{F}$ -proximal contraction mapping, which generalize the concept of fixed point theories for $\mathcal{F}$ -contraction mappings according to Wardowski [2].

2. Preliminaries

For the purpose of our main results of this article, we state the following definitions and results from the existing literature. First, we recall the notion of ϕ -metric space given by Das et al. [1] as follows.

DEFINITION 2.1 ([1]). Let $\mathcal{X}$ be a nonempty set. A function $d_\phi : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$ is said to be a ϕ -metric if it satisfies the following conditions:

($d_\phi 1$) $d_\phi(x, y) = 0$ if and only if $x = y$;

($d_\phi 2$) $d_\phi(x, y) = d_\phi(y, x)$;

($d_\phi 3$) $d_\phi(x, y) \leq d_\phi(x, z) + d_\phi(z, y) + \phi(x, y, z)$;

for all $x, y, z \in \mathcal{X}$ where $\phi : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$ is a function satisfying:

($\phi 1$) $\phi(x, y, z) = 0$ if $x = z$ or $y = z$;

($\phi 2$) $\phi(x, y, z) = \phi(y, x, z)$;

($\phi 3$) For all $\epsilon > 0$ there exists $\delta > 0$ such that $\phi(x, y, z) < \epsilon$ whenever $d_\phi(x, z) < \delta$ or $d_\phi(y, z) < \delta$

for all $x, y, z \in \mathcal{X}$. A set $\mathcal{X}$ together with the function d_ϕ , $(\mathcal{X}, d_\phi)$ is called a ϕ -metric space.

EXAMPLE. Let $(\mathcal{X}, d)$ be a metric space. Then, d_ϕ defined by $d_\phi(x, y) = (d(x, y))^2$ for all $x, y \in \mathcal{X}$ is a ϕ -metric on $\mathcal{X}$.

DEFINITION 2.2 ([1]). Let $(\mathcal{X}, d_\phi)$ be a ϕ -metric space.

- (i) A sequence $\{x_n\} \subseteq \mathcal{X}$ is said to be convergent and converges to x if for any $\epsilon > 0$ there exists a positive integer N satisfying $d_\phi(x_n, x) < \epsilon$ for all $n \geq N$. That is, $d_\phi(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$.
- (ii) $\{x_n\} \subseteq \mathcal{X}$ is said to be a Cauchy sequence if for any $\epsilon > 0$ there exists a positive integer N such that $d_\phi(x_n, x_m) < \epsilon$ for all $m, n \geq N$. That is, $d_\phi(x_n, x_m) \rightarrow 0$ as $m, n \rightarrow \infty$.

PROPOSITION 2.3 ([1]). *In the ϕ -metric space $(\mathcal{X}, d_\phi)$, every convergent sequence has a unique limit.*

DEFINITION 2.4 ([1]). A ϕ -metric space $(\mathcal{X}, d_\phi)$ is said to be complete if every Cauchy sequence converges to some point in $\mathcal{X}$.

There are so many contractions or expansion principles that have been developed in the context of metric spaces and generalized metric spaces. In 2012, Wardowski [2] proposed an interesting generalization of Banach contraction principle, known as $\mathcal{F}$ -contraction with a special class of mapping. In Section 4, we develop proximal contraction in the setting of ϕ -metric space which generalizes the concept of $\mathcal{F}$ -contraction mapping in metric spaces. In the following, we recall the definition of such contraction mappings.

DEFINITION 2.5 ([2]). Let $\mathcal{F}$ be a family of all functions $F : (0, \infty) \rightarrow \mathbb{R}$ which satisfies the following properties:

- (F1) F is strictly increasing;
- (F2) for every sequence $\{x_n\}$ in $(0, \infty)$, $\lim_{n \rightarrow \infty} x_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(x_n) = -\infty$;
- (F3) there exists $k \in (0, 1)$ such that $\lim_{x \rightarrow 0^+} x^k F(x) = 0$.

EXAMPLE ([2]). Let $F_1 : (0, \infty) \rightarrow \mathbb{R}$ be a function defined by $F_1(x) = \ln x \quad \forall x \in (0, \infty)$. Then, F_1 satisfies properties (F1) – (F3).

DEFINITION 2.6 ([2]). Let $(\mathcal{X}, d)$ be a metric space. A self mapping T on $\mathcal{X}$ is said to be an $\mathcal{F}$ -contraction mapping on $(\mathcal{X}, d)$ if there exists $(F, \tau) \in \mathcal{F} \times (0, \infty)$ such that

$$t + F(d(Tx, Ty)) \leq F(d(x, y)) \quad (1)$$

holds for all $x, y \in \mathcal{X}$ satisfying $d(Tx, Ty) > 0$.

THEOREM 2.7 ([2]). Let $(\mathcal{X}, d)$ be a complete metric space and T be an $\mathcal{F}$ -contraction mapping on $\mathcal{X}$. Then, T has a unique fixed point $\alpha \in \mathcal{X}$ and for every $x \in \mathcal{X}$ the sequence $\{T^n x\}$ converges to α .

3. An application of ϕ -metric

A familiar example of ‘measuring the distance between two objects’ is to find the Euclidean distance between two points in xy -plane. The notion of ‘metric’ is a common way of computing the distance between any two elements of a set. There are also so many examples of metric spaces other than Euclidean spaces.

The Euclidean distance

$$d_u(P, Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

between two points

$$P(x_1, y_1, z_1) \quad \text{and} \quad Q(x_2, y_2, z_2) \quad \text{in 3-D}$$

is defined by the length of the line segment joining the points P and Q .

However, we do not always measure the distance between two points in 3-D using only this formula. For a detail discussion, we consider the Earth where we consider the Earth as a subset of 3-D space and the surface of the Earth is considered to be the surface of a sphere. For example, suppose we connect the north and south poles along the Earth’s surface. Then, the shortest path connecting them along the Earth’s surface should not be a straight line. So, the question is whether we can use the Euclidean metric ‘ d_u ’ of 3-D space to calculate the distance in such cases.

AN APPLICATION OF ϕ -METRIC

We think that the concept of ϕ -metric can be helpful for such a geometry. As an example, consider the following abstract, see Fig. 1 on page 128.

Our goal is to define the distance between the points lying on the curve which is formed by joining a circular arc ADB with the two line segments AC & BC where the surface of this figure is the domain of its points and the strip

$$\mathcal{M} = [-1, 1] \times [-1, -2].$$

It is obvious that the Euclidean metric is not sufficient to calculate the distance between any two points on this geometry.

Let Y, Z and W be the set containing the points on the line segment joining $A(-1, 0)$ & $C(0, -1)$, on the line segment joining $B(1, 0)$ & $C(0, -1)$, and on the curve ADB , which is the upper circular arc of the circle

$$x^2 + y^2 = 1 \quad \text{and} \quad \mathcal{X} = Y \cup Z \cup W.$$

Let d_u be the usual Euclidean metric on Y and Z and for all $(x_1, y_1), (x_2, y_2)$ in $\mathcal{X} \cup \mathcal{M}$, we define a function

$$\text{by} \quad d_\phi : (\mathcal{X} \cup \mathcal{M}) \times (\mathcal{X} \cup \mathcal{M}) \rightarrow [0, \infty)$$

$$d_\phi((x_1, y_1), (x_2, y_2)) = \begin{cases} d_u((x_1, y_1), (x_2, y_2)) & \text{if } (x_1, y_1), (x_2, y_2) \text{ are both in } Y \text{ or } Z; \\ d_u((x_1, y_1), C) + d_u(C, (x_2, y_2)) & \text{if } (x_1, y_1) \in Y, (x_2, y_2) \in Z; \\ d_u((x_1, y_1), A) + \text{arc}(AE), \text{ where } \angle AOE = \tan^{-1} \left(\frac{y_2}{|x_2|} \right) & \text{if } (x_1, y_1) \in Y, E = (x_2, y_2) \in W; \\ d_u((x_1, y_1), B) + \text{arc}(BE), \text{ where } \angle BOE = \tan^{-1} \left(\frac{y_2}{|x_2|} \right) & \text{if } (x_1, y_1) \in Z, F = (x_2, y_2) \in W; \\ \pi - (\text{arc}(AE) + \text{arc}(BF)), \text{ where } \angle AOE = \tan^{-1} \left(\frac{y_1}{|x_1|} \right), & \angle BOE = \tan^{-1} \left(\frac{y_2}{|x_2|} \right); \\ |x_1 - x_2|^2 + |y_1 - y_2|^2 & \text{if } E = (x_1, y_1), F = (x_2, y_2) \in W; \\ |x_1 - x_2|^2 + |y_1 - y_2|^2 & \text{if } (x_1, y_1), (x_2, y_2) \in \mathcal{M}; \\ |x_1|^2 + |y_1 + 1|^2 + \alpha & \text{if } (x_1, y_1) \in \mathcal{M} \text{ \& } (x_2, y_2) \in \mathcal{X}, \end{cases}$$

where

$$\alpha = \begin{cases} d_u(C, (x_2, y_2)) & \text{if } (x_2, y_2) \in Y \cup Z, \\ d_\phi(C, (x_2, y_2)) & \text{if } (x_2, y_2) \in W. \end{cases}$$

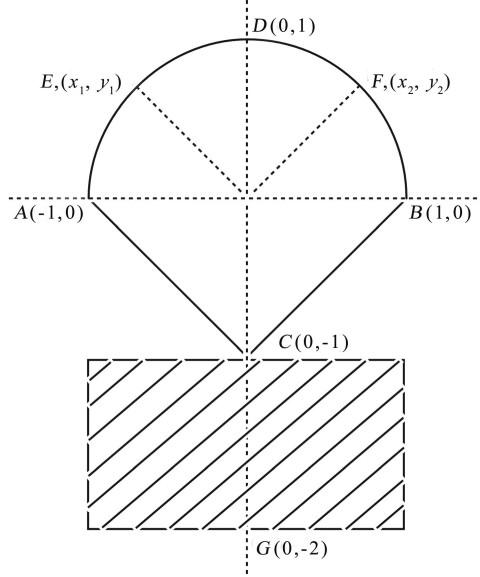


FIGURE 1.

Clearly, $d_\phi(x, y) \geq 0$ for all $x, y \in \mathcal{X}$ and $d_\phi(x, y) = 0 \iff x = y$ and $d_\phi(x, y) = d_\phi(y, x)$. So, d_ϕ satisfies the first two metric axioms. Note, however, that $d_\phi(A, B) = \pi$, $d_\phi(A, C) = \sqrt{2} = d_\phi(B, C)$ which implies that

$$d_\phi(A, B) > d_\phi(A, C) + d_\phi(B, C).$$

Hence, d_ϕ is not a metric on $\mathcal{X}$. In this scenario, the ϕ -metric can save this purpose. For, let $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2)$ and define a function ϕ on $(\mathcal{X} \cup \mathcal{M})^3$ by

$$\phi(x, y, z) = \begin{cases} d_\phi(x, z)d_\phi(y, z) & \text{if } x, y, z \in \mathcal{X}; \\ 2 \sum_{i=1}^2 |x_i - z_i||z_i - y_i| & \text{if } x, y, z \in \mathcal{M}; \\ 2\{|x_1||y_1| + |x_2 + 1||y_2 + 1|\} & \text{if } x, y \in \mathcal{M} \text{ \& } z \in \mathcal{X}; \\ 2\{|x_1 - z_1||z_1| + |x_2 - z_2||z_2 + 1|\} & \text{if } x, z \in \mathcal{M} \text{ \& } y \in \mathcal{X}; \\ 0 & \text{if } x \in \mathcal{M}, y \in Y \text{ \& } z \in Z \cup W. \end{cases}$$

Clearly, the function ϕ satisfies conditions $(\phi 1)$ – $(\phi 3)$ of Definition 2.1. By a routine calculation, we can verify that

$$(d_\phi 3) : d_\phi(x, y) \leq d_\phi(x, z) + d_\phi(y, z) + \phi(x, y, z)$$

holds for all $x, y, z \in \mathcal{X} \cup \mathcal{M}$. Therefore, d_ϕ is a ϕ -metric on $\mathcal{X} \cup \mathcal{M}$. Hence using the ϕ -metric, we can also calculate the distance between any two points on a curved surface.

4. $\mathcal{F}$ -proximal contraction mappings in ϕ -metric spaces

This section presents the best proximity theorems in ϕ -metric spaces for a class of non-self mappings known as $\mathcal{F}$ -proximal contraction mappings. We show that these results can be treated as generalized version of some well-known metric fixed point theories.

We begin our main results with the definition of the ϕ -best proximity point for two non-empty subsets in a ϕ -metric space.

DEFINITION 4.1. Let A and B be two non-empty subsets of a ϕ -metric space $(\mathcal{X}, d_\phi)$ and $T : A \rightarrow B$ be a mapping. An element $a \in A$ is said to be a ϕ -best proximity point for the mapping T if

$$d_\phi(a, Ta) = d_\phi(A, B) = \inf \{d_\phi(x, y) : (x, y) \in A \times B\}.$$

In the following, we define some other notations.

For any non-empty subsets A and B of a ϕ -metric space $(\mathcal{X}, d_\phi)$, we define

$$d_\phi(\mathcal{X}, B) = \inf \{d_\phi(x, y) : y \in B\} \quad \forall x \in \mathcal{X};$$

$$A_0 = \{x \in A : d_\phi(x, y) = d_\phi(A, B) \text{ for some } y \in B\};$$

$$B_0 = \{y \in B : d_\phi(x, y) = d_\phi(A, B) \text{ for some } x \in A\}.$$

Next, we define $\mathcal{F}$ -proximal contraction in ϕ -metric space as follows.

DEFINITION 4.2. Let $(\mathcal{X}, d_\phi)$ be a ϕ -metric space and A, B be two non-empty subsets of $\mathcal{X}$. A mapping $T : A \rightarrow B$ is said to satisfy $\mathcal{F}$ -proximal contraction condition if there exists $(F, t) \in \mathcal{F} \times (0, \infty)$ such that

$$\begin{aligned} d_\phi(u_1, T\zeta_1) = d_\phi(A, B) \quad \& \quad d_\phi(u_2, T\zeta_2) = d_\phi(A, B) \\ \implies F(d_\phi(u_1, u_2)) + t &\leq F(d_\phi(\zeta_1, \zeta_2)) \end{aligned} \quad (2)$$

for all $u_1 \neq u_2, \zeta_1 \neq \zeta_2 \in A$ whenever $d_\phi(u_1, u_2) > 0$.

Remark 1. If we take $A = B = \mathcal{X}$ and T is a self mapping on $\mathcal{X}$, then proximal contraction condition (2) becomes

$$F(d_\phi(T\zeta_1, T\zeta_2)) + t \leq F(d_\phi(\zeta_1, \zeta_2)) \quad (3)$$

for all $\zeta_1, \zeta_2 \in \mathcal{X}$ provided $d_\phi(T\zeta_1, T\zeta_2) > 0$. Then, T is said to be an $\mathcal{F}$ -contraction mapping [2].

The following theorem is a new ϕ -best proximity point result in ϕ -metric space.

THEOREM 4.3. Let $(\mathcal{X}, d_\phi)$ be a complete ϕ -metric space and A & B be two non-empty subsets of $\mathcal{X}$ such that A_0 is non-empty and closed. If $T : A \rightarrow B$ is an $\mathcal{F}$ -proximal contraction mapping such that $T(A_0) \subseteq B_0$, then T has exactly one ϕ -best proximity point in A .

Proof. Let $a_0 \in A_0$.

Since $Ta_0 \in T(A_0) \subseteq B_0$, there exists $a_1 \in A_0$ such that

$$d_\phi(a_1, Ta_0) = d_\phi(A, B).$$

Similarly, since $Ta_1 \in T(A_0) \subseteq B_0$, there exists $a_2 \in A_0$ such that

$$d_\phi(a_2, Ta_1) = d_\phi(A, B).$$

Continuing this process, we can construct a sequence $\{a_n\}$ in A_0 such that

$$d_\phi(a_{n+1}, Ta_n) = d_\phi(A, B) \quad \text{for all } n \in \mathbb{N}. \quad (4)$$

We denote

$$\alpha_n = d_\phi(a_{n+1}, a_n), \quad n = 0, 1, 2, \dots$$

If there exists $l \in \mathbb{N}$ such that $a_{l+1} = a_l$, then a_l is the best proximity point for T . So, suppose then that no such $l \in \mathbb{N}$ exists. Therefore, $\alpha_n > 0$ for all $n \in \mathbb{N}$. Since by (4),

$$d_\phi(a_n, Ta_{n-1}) = d_\phi(A, B) \quad \& \quad d_\phi(a_{n+1}, Ta_n) = d_\phi(A, B) \quad \forall n \geq 1,$$

there exists $(F, t) \in \mathcal{F} \times (0, \infty)$ such that

$$F(d_\phi(a_n, a_{n+1})) + t \leq F(d_\phi(a_{n-1}, a_n)),$$

i.e.,

$$F(\alpha_n) \leq F(\alpha_{n-1}) - t \quad \text{for } n = 1, 2, \dots$$

Thus, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} F(\alpha_n) &\leq F(\alpha_{n-1}) - t \leq F(\alpha_{n-2}) - 2t \dots \\ &\leq F(\alpha_0) - nt. \end{aligned}$$

Letting $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty,$$

which yields $(\mathcal{F}2)$, $\lim_{n \rightarrow \infty} \alpha_n = 0$.

Again, by $(\mathcal{F}3)$, there exists $k \in (0, 1)$ such that

$$\lim_{n \rightarrow \infty} \alpha_n^k F(\alpha_n) = 0.$$

Therefore,

$$\alpha_n^k F(\alpha_n) \leq \alpha_n^k F(\alpha_0) - nt\alpha_n^k$$

implies

$$\alpha_n^k F(\alpha_n) - \alpha_n^k F(\alpha_0) \leq -nt\alpha_n^k$$

or

$$\lim_{n \rightarrow \infty} (nt\alpha_n^k) = 0$$

or

$$\lim_{n \rightarrow \infty} n\alpha_n^k = 0.$$

AN APPLICATION OF ϕ -METRIC

So, there exists $N_1 \in \mathbb{N}$ such that

$$n\alpha_n^k < 1 \quad \forall \quad n \geq N_1, \quad \text{i.e.,} \quad \alpha_n < \frac{1}{n^{\frac{1}{k}}} \quad \forall n \geq N_1. \quad (5)$$

Next, we show that $\{a_n\}$ is a Cauchy sequence in A . Choose $m, n \in \mathbb{N}$ such that $m > n \geq N_1$. Then,

$$\begin{aligned} d_\phi(a_m, a_n) &\leq d_\phi(a_m, a_{n+1}) + d_\phi(a_{n+1}, a_n) + \phi(a_m, a_n, a_{n+1}) \\ &\leq \{d_\phi(a_m, a_{n+2}) + d_\phi(a_{n+2}, a_{n+1}) + \phi(a_m, a_{n+1}, a_{n+2})\} \\ &\quad + d_\phi(a_n, a_{n+1}) + \phi(a_m, a_n, a_{n+1}) \\ &\quad \dots \quad \dots \quad \dots \\ &\leq \{d_\phi(a_m, a_{m-1}) + \dots + d_\phi(a_{n+1}, a_{n+2}) + d_\phi(a_{n+1}, a_n)\} \\ &\quad + \{\phi(a_m, a_n, a_{n+1}) + \phi(a_m, a_{n+1}, a_{n+2}) + \dots + \phi(a_m, a_{m-2}, a_{m-1})\} \\ &= \{\alpha_{m-1} + \alpha_{m-2} + \dots + \alpha_n\} + \sum_{i=n}^{m-2} \phi(a_m, a_i, a_{i+1}) \\ &< \left\{ \frac{1}{(m-1)^{\frac{1}{k}}} + \frac{1}{(m-2)^{\frac{1}{k}}} + \dots + \frac{1}{n^{\frac{1}{k}}} \right\} + \sum_{i=n}^{m-2} \phi(a_m, a_i, a_{i+1}) \quad \forall m, n \geq N_1 \\ &\leq \sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}} + \sum_{i=n}^{m-2} \phi(a_m, a_i, a_{i+1}) \quad \forall m, n \geq N_1. \end{aligned}$$

Let $\epsilon > 0$.

Then, there exists $\delta > 0$ such that $\phi(a_r, a_l, a_{l+1}) < \epsilon$ whenever

$$d_\phi(a_r, a_{l+1}) < \delta \quad \text{or} \quad d_\phi(a_l, a_{l+1}) < \delta.$$

Let $\beta = \min\{\epsilon, \delta\}$.

Since $0 < k < 1$, so $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is a convergent series and hence there exists $N_2 \in \mathbb{N}$ such that $\left| \frac{1}{i^{\frac{1}{k}}} \right| < \beta \quad \forall i \geq N_2$. Therefore,

$$\alpha_i = d_\phi(a_i, a_{i+1}) < \beta \quad \forall i \geq N_2.$$

Let $N = \max\{N_1, N_2\}$.

So, for all $m > n \geq N$, $d_\phi(a_{n+1}, a_n) < \beta$ and hence $\phi(a_m, a_n, a_{n+1}) < \epsilon$. Since $\epsilon > 0$ is arbitrarily chosen, so

$$\lim_{n \rightarrow \infty} d_\phi(a_{n+1}, a_n) = 0 \implies \lim_{n \rightarrow \infty} \phi(x_m, x_n, x_{n+1}) = 0.$$

This yields,

$$\lim_{m, n \rightarrow \infty} d_\phi(a_m, a_n) = 0$$

which implies $\{a_n\}$ is a Cauchy sequence in A_0 .

Since A_0 is a closed subset of the complete ϕ -metric space $(\mathcal{X}, d_\phi)$, thus there exists $a \in A_0$ such that $\{a_n\}$ converges to a .

On the other hand, $Ta \in T(A_0) \subseteq B_0$ implies there exists $c \in A_0$ such that

$$d_\phi(c, Ta) = d_\phi(A, B). \quad (6)$$

Using (4), (6) and $\mathcal{F}$ -proximal contraction condition, we can write

$$\begin{aligned} & F(d_\phi(a_{n+1}, c)) + t \leq F(d_\phi(a_n, a)) \\ \implies & \lim_{n \rightarrow \infty} F(d_\phi(a_{n+1}, c)) + t \leq -\infty \quad (\text{by } (\mathcal{F}2)) \\ \implies & \lim_{n \rightarrow \infty} d_\phi(a_{n+1}, c) = 0 \quad (\text{by } (\mathcal{F}2)) \\ \implies & a = c. \end{aligned}$$

Hence, from (6) we conclude that a is a ϕ -best proximity point for T in A .

Now, assume a and x are two different ϕ -best proximity points for T in A . Then,

$$d_\phi(a, Ta) = d_\phi(A, B) \quad \& \quad d_\phi(x, Tx) = d_\phi(A, B)$$

which implies

$$F(d_\phi(a, x)) + t \leq F(d_\phi(a, x)), \quad \text{i.e., } t \leq 0.$$

This is a contradiction. Hence, the proof is complete. $\square$

COROLLARY 4.4. *Let $(\mathcal{X}, d_\phi)$ be a complete ϕ -metric space and T be an $\mathcal{F}$ -contraction mapping on $\mathcal{X}$. Then, T has a unique fixed point $\alpha \in \mathcal{X}$.*

Proof. If we take $A = B = \mathcal{X}$ and T is a self mapping on $\mathcal{X}$, then proximal contraction condition (2) reduces to contraction condition (3).

Hence, the conclusion follows from Theorem 4.3 and the unique best proximity point for T of Theorem 4.3 becomes a unique fixed point for T . $\square$

Remark 2. Corollary 4.4 is a generalization of Theorem 2.7 of Wardowski [2]. If we take $\phi(x, y, z) = 0 \quad \forall \quad x, y, z \in \mathcal{X}$, then $(\mathcal{X}, d_\phi)$ becomes a metric space and Theorem 2.7 of Wardowski [2] immediately follows from Corollary 4.4.

The following example is in support of the main theorem.

EXAMPLE. We consider the ϕ -metric space of Section 3 and take

$$P = [-1, 1] \times \{-1\} \quad \& \quad Q = [-1, 1] \times \{-2\}.$$

Then, both P and Q are two closed subsets of $\mathcal{X} \cup \mathcal{M}$ such that $P_0 = P$ and $Q_0 = Q$. Here, $d_\phi(P, Q) = 1$, and clearly, Q is approximatively compact with respect to P .

Next, we define $T : P \rightarrow Q$ by $T((x, -1)) = (\frac{x}{2}, -2)$ for all $(x, -1) \in P$. Clearly, $T(P_0) \subset Q_0$. By a routine calculation, we can check that

$$\begin{aligned} d_\phi(u_1, T\zeta_1) &= d_\phi(P, Q) \quad \& \quad d_\phi(u_2, T\zeta_2) = d_\phi(P, Q) \\ \text{implies} \quad F(d_\phi(u_1, u_2)) + t &\leq F(d_\phi(\zeta_1, \zeta_2)) \end{aligned}$$

for all $u_1 \neq u_2, \zeta_1 \neq \zeta_2 \in P$ where $F(x) = \ln x$, $x > 0 \in \mathcal{F}$ and $t \in (0, 1.38)$. Therefore, T is a $\mathcal{F}$ -proximal contraction for the mapping $F(x) = \ln x$, $x > 0$ lies in $\mathcal{F}$ & $t \in (0, 1.38)$ and T satisfies the hypothesis of Theorem 4.3. Hence, T has a unique ϕ -best proximity point which is $(0, -1)$.

Remark 3. Since the distance function defined in Section 3 is a ϕ -metric but not a metric, the Wardowski's metric fixed point theorem cannot be applied for such cases.

Conclusion

This manuscript contains an extension of the concept of best proximity results for several types of proximal contraction mappings of metric spaces in the setting of ϕ -metric space. Here, we prove the existence and uniqueness of the best proximity point for a special class of proximal contraction mapping called $\mathcal{F}$ -proximal contraction mapping. Another section of this article consists of a geometrical explanation to show how ϕ -metric can help to measure the distance in case of non-planar surfaces. Lastly, we use this example in support of our ϕ -best proximity theorem and show that Wardowski's metric fixed point theorem cannot be applied in such spaces. Hence, the ϕ -best proximity point theorems of our manuscript also properly generalize the existing literature in metric spaces.

Acknowledgement. The author AD is thankful to University Grant Commission (UGC), New Delhi, India for awarding her senior research fellowship [Grant No.1221/(CSIR NET JUNE 2019)]. AD & TB are grateful to Department of Mathematics, Siksha-Bhavana, Visva-Bharati. The authors are grateful to the Editor-in-Chief, Editors and Reviewers of the journal TMMP for their valuable comments which helped them to revise the manuscript in the present form.

Conflict of interest. The authors declare no conflict of interest.

REFERENCES

- [1] DAS, A.—KUNDU, A.—BAG, T.: *A new approach to generalize metric functions*, International Journal of Nonlinear Analysis and Applications **14** (2023), 279–298.
- [2] WARDOWSKI, D.: *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl. **94** (2012), 6 pp.
- [3] BASHA, S. S.: *Extensions of Banach’s contraction principle*, Numer. Funct. Anal. Optim. **31** (2010), no. 4–6, 569–576.
- [4] BASHA, S. S.: *Best proximity points: ptimal solutions*, J. Optim. Theory Appl. **151** (2011), 210–216.
- [5] ALTUN, I.—ASLANTAS, M.—SAHIN, H.: *Best proximity point results for p -proximal contractions*, Acta Math. Hungar. **162** (2020), 393–402.
- [6] GABELEH, M.—MARKIN, J.: *A note on the paper “Best proximity point results for p -proximal contractions”*, Acta Math. Hungar. **164** (2021), 326–329.
- [7] SOM, S.: *A remark on the paper “A note on the paper Best proximity point results for p -proximal contractions”*, Acta Math. Hungar. **166** (2022), 103–106.
- [8] SOM, S.: *Comments on the paper “Best proximity point results for p -proximal contractions”*, Acta Math. Hungar. **168** (2022), 516–519.
- [9] GABELEH, M.: *Common best proximity pairs in strictly convex Banach spaces*, Georgian Math. J. **24** (2017), 363–372.
- [10] ASLANTAS, M.: *Best proximity point results on strong b -metric spaces*, Journal of New Theory **34** (2021), 28–36.
- [11] RAJ, V. S.—VEERAMANI, RAJ.: *Best proximity pair theorems for relatively non-expansive mappings*, Appl. Gen. Topol. **10** (2009), 21–28.
- [12] BASHA, S. S.: *Best proximity points: Global optimal approximate solutions*, J. Global Optim. **49** (2011), 15–21.
- [13] GABELEH, M.: *Convergence of Picard’s iteration using projection algorithm for non-cyclic contractions*, Indag. Math. (N.S.) **30** (2019), 227–239.
- [14] GABELEH, M.—VETRO, C.: *A new extension of Darbo’s fixed point theorem using relatively Meir–Keeler condensing operators*, Bull. Aust. Math. Soc. **98** (2018), 286–297.

Received December 8, 2023

Revised April 8, 2024

Accepted May 11, 2024

Publ. online June 18, 2024

Abhishikta Das

Tarapada Bag

Department of Mathematics, Siksha-Bhavana

Visva-Bharati

Santiniketan 731235

INDIA

E-mail: abhishikta.math@gmail.com

tarapadavb@gmail.com

Sumit Som

Hemanta Kalita

Mathematics Division

School of Advanced Sciences and Languages

VIT Bhopal University

Bhopal - Indore Highway, Kothrikalan

Sehore, Madhya Pradesh, 466114

INDIA

E-mail: somkakdwip@gmail.com

hemanta30kalita@gmail.com