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ON A NON-CONVEX LAGRANGE OPTIMAL CONTROL PROBLEM

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ABSTRACT. In this paper, we are concerned with an iterative differential inclusion governed by the time-dependent maximal monotone operator with perturbation. The approach to solve our problem is based on the Yosida approximation technique. The theoretical result is applied to prove an existence result for a Lagrange optimal control problem without assumptions concerning convexity.

1. Introduction

This paper is devoted to the study of the existence of solutions for a new iterative differential inclusions governed by maximal monotone operators of the form:

$$\begin{cases} \dot{u}(t) \in -\mathcal{M}(t)u(t) + h\Big(t, u\Big(g_1\big(t,u(t)\big)\Big), \dots, u^{[n]}\Big(g_n\big(t,u(t)\big)\Big)\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e.. } t \in [0,T]; \\ u(0) = u_0. \end{cases} \tag{1}$$

The operator $\mathcal{M}(t) : D\big(\mathcal{M}(t)\big) \rightrightarrows \mathbb{R}$ is time-dependant maximal monotone, $h : [0,T] \times [0,T]^n \to \mathbb{R}$ is an iterative perturbation defined as a Carathéodory mapping, $g_i : [0,T]^2 \to [0,T]$ are deviating arguments assumed to be Carathéodory mappings, for all $1 \leq i \leq n$ and $u^{[i]}(t) = u\left(u^{[i-1]}(t)\right)$ indicates the ith iterate of u, for all $2 \leq i \leq n$, $0 \leq u(t) \leq T$ for all $t \in [0,T]$.

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It is recognized that iterative differential inclusions are a generalization of iterative differential equations which proved to be valuable in many fields such as electrodynamics (as in problems related to motions of charged particles with retarded interactions), biology (as in infection models problems), etc. Numerous results were found in the theory of iterative differential equations, for example: smoothness, equivariance, analycity, convexity and numerical solutions (see [11, 18, 23, 27, 29, 30, 32] and references therein). Several approaches have been used to prove existence results of initial value problems; we mention Schauder's fixed point [1, 17, 19, 28, 31], contraction principle [6, 13, 16], Picard's successive method [25, 26] and non-expansive operators [3, 22].

We refer also to the authors in [14] and [15] when interested in differential equations with deviating arguments on both the time and the state. They introduced an existence results for the delay-self-refereed differential equation

$$\dot{u}(t) = h\Big(t, u\Big(g\big(t, u(t)\big)\Big)\Big),$$

where g is a delay-refereed function such that $g\big(t, u(t)\big) \leq t$ (resp. $g\big(t, u(t)\big) \leq u(t)$).

In this paper, we aim to generalize the results cited above, by weakening the assumptions. Next, we solve the Lagrange optimal control problem associated with differential inclusion (1) defined as follow: Minimize

$$\mathcal{J}(u, z) = \int_0^T h_0\bigg(t, u\Big(g_1\big(t, u(t)\big)\Big), \ldots, u^{[n]}\Big(g_n\big(t, u(t)\big)\Big), z(t)\bigg)\,\mathrm{d}t, \tag{2}$$

on the class of admissible pairs

$$\mathcal{Y} = \big\{\,(u, z) : u \text{ is solution of (1) and } z(t) \in Z\big(u(t)\big)\big\}, \tag{3}$$

where $h_0 : [0, T] \times [0, T]^n \times \mathbb{R}^m \to \mathbb{R}$, $Z : [0, T] \rightrightarrows \mathbb{R}^m$ is a nonempty measurable multifunction and $z : [0, T] \to \mathbb{R}^m$ is the control.

Our goal is to establish a new existence result without any convexity assumption. Instead, we exchanged it by the so-called almost convexity concept (see [9] for the definition of the multifunction case and [8] for the definition of the Lagrangian case). In this approach, based on ideas given in [7], we assume the existence of a minimizer to the convexified optimal control problem associated and demonstrate the relation between the existence of original solution (solution of (2)–(3)) and relaxed solution (solution of convexified problem associated) due to the absence of convexity.

This paper is organized as follows: In Section 2, we give the notions and preliminaries needed, while in Section 3 we provide an existence result for problem (1). Finally, we discuss Lagrange optimal control problems with differential inclusion in Section 4.

2. Preliminaries

Let $T > 0$, $[0,T] \subset \mathbb{R}$ be an interval, and $\mathbb{R}^n$ (resp. $\mathbb{R}^m$) be an n-dimensional (resp. m-dimensional) real space equipped with the norm $\|x\| = \sum_{i=1}^{n} |x_i|$ for $x = (x_1, x_2 \ldots, x_n) \in \mathbb{R}^n$. We denote by:

- $\mathcal{B}(\mathbb{R}^n)$ the Borel set of $\mathbb{R}^n$, we say that $h : \mathbb{R}^n \to \mathbb{R}$ is Borel if $h^{-1}(B) \in \mathcal{B}(\mathbb{R}^n)$ for all $B \in \mathcal{B}(\mathbb{R})$. It is well-known that every continuous function between two topological spaces is Borel;
- $\mathcal{C}([0,T])$ the Banach space of all continuous mappings $u : [0,T] \to [0,T]$ with the norm $\|u\|_{\mathcal{C}} = \sup_{t\in[0,T]} |u(t)|$;
- $\mathcal{AC}([0,T])$ the space of all absolutely continuous mappings $u : [0,T] \to [0,T]$;
- $L^p_{\mathbb{R}}([0,T])$ the space of all Lebesgue integrable mappings $u : [0,T] \to [0,T]$ equipped with the norm $\|u\|_{L^p_{\mathbb{R}}} = \left(\int_0^T |u(t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}}$;
- $\mathrm{co}(s)$ (resp. $\overline{\mathrm{co}}(S)$) the convex hull (resp. the closed convex hull) of a set S.

Let $f : \mathbb{R} \to \mathbb{R}$ be a mapping. We denote by:

- $\mathrm{epi} f(\cdot) = \{(v, v_0) \in \mathbb{R} \times \mathbb{R} : v_0 \geq f(v)\}$ the epigraph of f. Note that, $\mathrm{epi} f(\cdot)$ is closed if and only if $f(\cdot)$ is lower semicontinuous (lsc) (i.e., when a sequence (v_n) converges to v, then $\liminf_{n\to+\infty} f(v_n) \geq f(v)$).
- $f^{**}(v) = \inf\{v_0 : (v, v_0) \in \overline{\mathrm{co}}(\mathrm{epi} f(\cdot))\}$ the bipolar of f which is the pointwise supremum of all affine minorants of f. f^{**} is a convex mapping (see [4]).

Let $F : [0,T] \rightrightarrows \mathbb{R}^m$ be a nonempty multifunction. We denote by:

- $\mathrm{gph}(F) = \{(s,z) \in [0,T] \times \mathbb{R}^m : z \in F(s)\}$ the graph of the multifunction F;
- $\mathrm{co}F : [0,T] \rightrightarrows \mathbb{R}^m$ the multifunction defined by $(\mathrm{co}F)(s) = co(F(s))$.

Let $\mathcal{M}(t) : D(\mathcal{M}(t)) \subset \mathbb{R} \rightrightarrows \mathbb{R}$, $(t \in [0,T])$ be a maximal monotone operator and $\xi > 0$ be a constant. We denote by:

- $D(\mathcal{M}(t)) = \{z \in \mathbb{R} : \mathcal{M}(t)z \neq \emptyset\}$ the domain of $\mathcal{M}(t)$;
- $(\mathcal{M}(t)z)_0 = \inf\{|y| : y \in \mathcal{M}(t)z\}$ the element of minimal norm of $\mathcal{M}(t)z$;
- $J_\xi(t) = (I + \xi\mathcal{M}(t))^{-1}$ the resolvant of $\mathcal{M}(t)$;
- $\mathcal{M}_\xi(t) = \frac{1}{\xi}(I - J_\xi(t))$ the Yosida approximation of $\mathcal{M}(t)$;
- $\mathrm{dis}(\cdot,\cdot)$ the pseudo distance between two maximal monotone operators $\mathcal{M}, \mathcal{N}$ defined by:

$$\mathrm{dis}(\mathcal{M},\mathcal{N}) = \sup\left\{ \frac{(y-\overline{y})(\overline{x}-x)}{1+|y|+|\overline{y}|} : (x,y) \in \mathrm{gph}(\mathcal{M}), (\overline{x},\overline{y}) \in \mathrm{gph}(\mathcal{N}) \right\}.$$

Proposition 2.1 ([5]). *Let $\mathcal{M}(t) : D\big(\mathcal{M}(t)\big) \rightrightarrows \mathbb{R}$, $(t \in [0,T])$ be a maximal monotone operator and $\xi > 0$. Then:*

(a) *$\mathcal{M}_\xi(t)$ is single valued, maximal monotone and Lipschitz with constant $\frac{1}{\xi}$ on $\mathbb{R}$.*

(b) *$\mathcal{M}_\xi(t)z \in \mathcal{M}(t)J_\xi(t)z, \forall z \in \mathbb{R}$.*

(c) *$\frac{1}{\xi}|J_\xi(t)z - z| = |\mathcal{M}_\xi(t)z| \leq |\mathcal{M}(t)z|_0, \forall z \in D\big(\mathcal{M}(t)\big)$.*

Definition 2.2 (Almost convex mapping). [8] Let $L : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a mapping, we say L is almost convex if for all μ, where $L^{**}(\mu) < L(\mu)$, there exist three functions

$$\lambda_1 : \mathbb{R} \to [0,1]; \qquad \lambda_2 : \mathbb{R} \to [1,+\infty[\,; \qquad \alpha : \mathbb{R} \to [0,1],$$

such that

$$L^{**}(\mu) = \big(1-\alpha(\mu)\big)L\big(\mu\lambda_1(\mu)\big) + \alpha(\mu)L\big(\mu\lambda_2(\mu)\big);$$

$$\mu = \big(1-\alpha(\mu)\big)\big(\mu\lambda_1(\mu)\big) + \alpha(\mu)\big(\mu\lambda_2(\mu)\big).$$

We observe that:

- $L^{**}(\cdot) = L(\cdot)$ in two cases: if $\lambda_1 = \lambda_2 = \alpha = I_d$ or if $\mu = 0$ with the convention $0 \cdot (\infty) = 0$.
- $L(\cdot)$ is almost convex when L is a convex and lsc mapping.

3. Existence results

Let us start by proving the existence of solutions to the iterative differential equation

$$\begin{cases} \dot{u}(t) = h\Big(t, u\Big(g_1\big(t,u(t)\big)\Big), \ldots, u^{[n]}\Big(g_n\big(t,u(t)\big)\Big)\Big) & \text{a.e. } t \in [0,T]; \\ u(0) = u_0. \end{cases} \tag{4}$$

Theorem 3.1. *Assume that:*

(H$_1$) *$h : [0,T] \times [0,T]^n \to \mathbb{R}$ is a Carathéodory mapping, (i.e., measurable on $[0,T]$ and continuous on $[0,T]^n$);*

(H$_2$) *$g_i : [0,T]^2 \to [0,T]$ are Carathéodory mappings, for $1 \leq i \leq n$;*

(H$_3$) *there exists a non-negative function $k \in L^1_{\mathbb{R}}([0,T])$ such that;*

$$|h(t,x)| \leq k(t), \ \forall (t,x) \in [0,T] \times [0,T]^n.$$

Then, the problem (4) admits a solution $u \in \mathcal{AC}([0,T])$, such that

$$\|u\|_{\mathcal{C}} \leq |u_0| + \|k\|_{L^1_{\mathbb{R}}}.$$

P r o o f. Put $k_1 = |u_0| + \|k\|_{L^1_{\mathbb{R}}}$ and define the following closed bounded convex subset

$$K = \left\{ v \in \mathcal{C}\big([0,T]\big) : \|v\|_{\mathcal{C}} \le k_1 \right\}.$$

For all $v \in K$, the problem

$$\begin{cases} \dot{u}(t) = h\Big(t, v\Big(g_1\big(t, v(t)\big)\Big), \dots, v^{[n]}\Big(g_n\big(t, v(t)\big)\Big)\Big); \\ u(0) = u_0, \end{cases}$$

admits a solution $u_v \in \mathcal{AC}([0,T])$ defined by

$$u_v(t) = u_0 + \int_0^t h\Big(s, v\Big(g_1\big(s, v(s)\big)\Big), \dots, v^{[n]}\Big(g_n\big(s, v(s)\big)\Big)\Big)\,\mathrm{d}s.$$

Consider the operator

$$P : K \to \mathcal{C}([0,T]),$$

$$v \mapsto P(v) = u_v.$$

Let's show that $P(v) \in K$. For all $t \in [0,T]$

$$\begin{aligned} |u_v(t)| &\le |u_0| + \int_0^t \left| h\Big(s, v\Big(g_1\big(s, v(s)\big)\Big), \dots, v^{[n]}\Big(g_n\big(s, v(s)\big)\Big)\Big)\right| \mathrm{d}s \\ &\le |u_0| + \int_0^t k(s)\,\mathrm{d}s = |u_0| + \|k\|_{L^1_{\mathbb{R}}} = k_1. \end{aligned}$$

Then

$$\|u_v\|_{\mathcal{C}} \le k_1. \tag{5}$$

Let (v_ρ) be a sequence converging to v in K, since the mappings $v_\rho(\cdot)$ and $g_i(t,\cdot)$ are continuous, $(1 \le i \le n)$. Then $\big(v_\rho^{[i]}(g_i(\cdot, v_\rho(\cdot)))\big)_\rho$ converges to $v^{[i]}\big(g_i(\cdot, v(\cdot))\big)$ and

$$\begin{aligned} \lim_{\rho\to+\infty} |P(v_\rho) - P(v)| &= \lim_{\rho\to+\infty} |u_{v_\rho}(t) - u_v(t)| \\ &\le \lim_{\rho\to+\infty} \int_0^t \left| h\Big(s, v_\rho\Big(g_1\big(s, v_\rho(s)\big)\Big), \dots, v_\rho^{[n]}\Big(g_n\big(s, v_\rho(s)\big)\Big)\Big)\right. \\ &\quad \left. - h\Big(s, v\Big(g_1\big(s, v(s)\big)\Big), \dots, v^{[n]}\Big(g_n\big(s, v(s)\big)\Big)\Big)\right| \mathrm{d}s. \end{aligned}$$

Since $h(t,\cdot)$ is continuous, we get for all $t \in [0,T]$;

$$\lim_{\rho\to+\infty} h\Big(t, v_\rho\Big(g_1\big(t, v_\rho(t)\big)\Big), \dots, v_\rho^{[n]}\Big(g_n\big(t, v_\rho(t)\big)\Big)\Big) = h\Big(t, v\Big(g_1\big(t, v(t)\big)\Big), \dots, v^{[n]}\Big(g_n\big(t, v(t)\big)\Big)\Big).$$

We have also, for all $t \in [0,T]$,

$$\left|h\Big(t, v_\rho\Big(g_1\big(t, v_\rho(t)\big)\Big), \dots, v_\rho^{[n]}\Big(g_n\big(t, v_\rho(t)\big)\Big)\Big)\right| \le k(t).$$

By Lebesgue's theorem, we obtain

$$\lim_{\rho\to+\infty} \int_0^t \left|h\Big(s, v_\rho\Big(g_1\big(s, v_\rho(s)\big)\Big), \dots, v_\rho^{[n]}\Big(g_n\big(s, v_\rho(s)\big)\Big)\Big) - h\Big(s, v\Big(g_1\big(s, v(s)\big)\Big), \dots, v^{[n]}\Big(g_n\big(s, v(s)\big)\Big)\Big)\right| \mathrm{d}s = 0.$$

Hence, the continuity of P.

By the relation (5), $\{u_v(t), v \in K\}$ is relatively compact in $[0,T]$. Moreover, we have for all $t, \tau \in [0,T]$,

$$\begin{aligned} |u_v(t) - u_v(\tau)| &= \int_\tau^t \left|h\Big(s, v\Big(g_1\big(s, v(s)\big)\Big), \dots, v^{[n]}\Big(g_n\big(s, v(s)\big)\Big)\Big)\right| \mathrm{d}s \\ &\le \int_\tau^t k(s)ds, \end{aligned}$$

so $\{u_v : v \in K\}$ is equicontinuous since $k \in L^1_{\mathbb{R}}([0,T])$. Therefore, by applying Arzelà-Ascoli theorem, $P(K)$ is relatively compact in $\mathcal{C}([0,T])$. We deduce from Schauder's fixed point theorem (see in [20, Theorem 1.7]) that P admits a fixed point which is the solution of (4). □

Before announcing the main result, we need to demonstrate the next theorem.

Theorem 3.2. *Let $\mathcal{M}(t) : D\big(\mathcal{M}(t)\big) \rightrightarrows \mathbb{R}$ be a maximal monotone operator such that $[0,T] \subset D\big(\mathcal{M}(t)\big)$. Suppose that* (H_1), (H_2) *and*

(H_4) *for all $x \in \mathbb{R}$ and all $\xi > 0$, $t \to J_\xi(t)x$ is Lebesgue measurable;*

(H_5) *there exists a non-negative function $\kappa \in L^2_{\mathbb{R}}([0,T])$ such that;*

$$\big|\big(\mathcal{M}(t)z\big)_0\big| + |h(t,x)| \le \kappa(t), \quad \forall\, (t,x,z) \in [0,T]^{n+2}.$$

Then, the problem (1) *admits a solution $u \in \mathcal{AC}([0,T])$.*

P r o o f.

Step 1. Let $(\xi_\rho)_\rho$ be a decreasing sequence in $]0,1[$ converging to 0 when ρ tend to $+\infty$. For all $\rho \in \mathbb{N}$, consider the mapping

$$f_\rho(t,x,z) = -\mathcal{M}_{\xi_\rho}(t)z + h\,(t,x)\,, \quad \forall (t,x,z) \in [0,T]^{n+2}.$$

By (c) of Proposition 2.1 and (H_5), we get

$$|f_\rho\,(t,x,z)\,| \leq \left|\big(\mathcal{M}(t)z\big)_0\right| + |h\,(t,x)\,| \leq \kappa(t).$$

According to (H_1), (H_4) and (a) of Proposition 2.1, f_ρ is a Carathéodory mapping. Theorem 3.1 allows us to conclude that for all $\rho \in \mathbb{N}$:

$$\begin{cases} \dot{u}_\rho(t) \;= f_\rho\Big(t, u_\rho\Big(g_1\big(t,u_\rho(t)\big)\Big), \ldots, u_\rho^{[n]}\Big(g_n\big(t,u_\rho(t)\big)\Big)\Big) & \text{a.e. } t \in [0,T]; \\ u_\rho(0) = u_0, \end{cases}$$

admits an absolutely continuous solution

$$u_\rho(t) \;=\; u_0 + \int_0^t f_\rho\Big(s, u_\rho\Big(g_1\big(s,u_\rho(s)\big)\Big), \ldots, u_\rho^{[n]}\Big(g_n\big(s,u_\rho(s)\big)\Big)\Big)\,\mathrm{d}s.$$

Step 2. Since $|u_\rho(t)| \leq |u_0| + \|\kappa\|_{L^1_{\mathbb{R}}}$, $\{u_\rho(t)\}$ is relatively compact in $[0,T]$. For all $t,\tau \in [0,T]$

$$\begin{aligned} |u_\rho(t) - u_\rho(\tau)| &\leq \int_\tau^t \left| f_\rho\Big(s, u_\rho\Big(g_1\big(s,u_\rho(s)\big)\Big), \ldots, u_\rho^{[n]}\Big(g_n\big(s,u_\rho(s)\big)\Big)\Big)\right| \mathrm{d}s \\ &\leq \int_\tau^t \kappa(s)ds, \end{aligned}$$

as $\kappa \in L^1_R([0,T])$, we get the equicontinuity of $\big(u_\rho(\cdot)\big)_\rho$. By Arzelà-Ascoli theorem, $\big(u_\rho(\cdot)\big)_\rho$ is relatively compact. Then, there exists a subsequence (again denoted by) $\big(u_\rho(\cdot)\big)_\rho$ converging to $u(\cdot)$ in $\mathcal{C}([0,T])$. Since $|\dot{u}_\rho(t)| \leq \kappa(t)$, the sequence $\big(\dot{u}_\rho(\cdot)\big)_\rho$ converges weakly to $\dot{u}(\cdot)$. Moreover, by the continuity of $h(t,\cdot)$ and $g_i(t,\cdot)$, $(1 \leq i \leq n)$; we get that the sequence

$$\left(h\Big(\cdot, u_\rho\Big(g_1\big(\cdot,u_\rho(\cdot)\big)\Big), \ldots, u_\rho^{[n]}\Big(g_n\big(\cdot,u_\rho(\cdot)\big)\Big)\Big)\right)_\rho$$

converges to

$$h\Big(\cdot, u\Big(g_1\big(\cdot,u(\cdot)\big)\Big), \ldots, u^{[n]}\Big(g_n\big(\cdot,u(\cdot)\big)\Big)\Big)$$

and for all $t \in [0,T]$

$$\left| h\Big(t, u_\rho\Big(g_1\big(t,u_\rho(t)\big)\Big), \dots, u_\rho^{[n]}\Big(g_n\big(t,u_\rho(t)\big)\Big)\Big)\right| \leq \kappa(t).$$

Using Lebesgue's theorem, we conclude that

$$\left| h\Big(t, u\big(g_1(t,u(t))\big), \dots, u^{[n]}\Big(g_n\big(t,u(t)\big)\Big)\Big)\right| \leq \kappa(t),$$

and

$$\left(h\Big(\cdot, u_\rho\Big(g_1\big(\cdot,u_\rho(\cdot)\big)\Big), \dots, u_\rho^{[n]}\Big(g_n\big(\cdot,u_\rho(\cdot)\big)\Big)\Big)\right)_\rho$$

converges to

$$h\Big(\cdot, u\Big(g_1\big(\cdot,u(\cdot)\big)\Big), \dots, u^{[n]}\Big(g_n\big(\cdot,u(\cdot)\big)\Big)\Big) \quad \text{in } L^2_{\mathbb{R}}([0,T]).$$

Step 3. According to (b) of Proposition 2.1, we obtain for all $t \in [0,T]$,

$$\begin{aligned}
&\dot{u}_\rho(t) - h\Big(t, u_\rho\Big(g_1\big(t,u_\rho(t)\big)\Big), \dots, u_\rho^{[n]}\Big(g_n\big(t,u_\rho(t)\big)\Big)\Big)\\
&= -\mathcal{M}_{\xi_\rho}(t)u_\rho(t)\\
&\in -\mathcal{M}(t)J_{\xi_\rho}(t)u_\rho(t),
\end{aligned}$$

and

$$|J_{\xi_\rho}(t)u_\rho(t) - u(t)| \leq |J_{\xi_\rho}(t)u_\rho(t) - u_\rho(t)| + |u_\rho(t) - u(t)|,$$

by (c) of Proposition 2.1 and hypothesis (H_5), we obtain for all $\rho \in \mathbb{N}$, $\xi_\rho < 1$ and $t \in [0,T]$

$$|J_{\xi_\rho}(t)u_\rho(t) - u_\rho(t)| = \xi_\rho |\mathcal{M}_{\xi_\rho}(t)u_\rho(t)| \leq \xi_\rho |\left(\mathcal{M}_{\xi_\rho}(t)u_\rho(t)\right)_0| \leq \xi_\rho \kappa(t).$$

Hence,

$$\lim_{\rho\to+\infty} |J_{\xi_\rho}(t)u_\rho(t) - u_\rho(t)| = 0,$$

so

$$\lim_{\rho\to+\infty} |J_{\xi_\rho}(t)u_\rho(t) - u(t)| = 0 \tag{6}$$

and

$$|J_{\xi_\rho}(t)u_\rho(t) - u(t)| \leq \kappa(t) + 2\left(|u_0| + \|\kappa\|_{L^1_{\mathbb{R}}}\right),$$

as $\kappa \in L^2_{\mathbb{R}}([0,T])$ and by Lebesgue's theorem, we conclude $\left(J_{\xi_\rho}(\cdot)u_\rho(\cdot)\right)_\rho$ converges to $u(\cdot)$ in $L^2_{\mathbb{R}}([0,T])$.

Step 4. Put

$$\tilde{u}_\rho(t) = -\dot{u}_\rho(t) + h\Big(t, u_\rho\Big(g_1\big(t, u_\rho(t)\big)\Big), \dots, u_\rho^{[n]}\Big(g_n\big(t, u_\rho(t)\big)\Big)\Big)$$

and

$$\tilde{u}(t) = -\dot{u}(t) + h\Big(t, u\Big(g_1\big(t, u(t)\big)\Big), \dots, u^{[n]}\Big(g_n\big(t, u(t)\big)\Big)\Big).$$

Based on the monotony of $\mathcal{M}(t)$, for all $z \in D\big(\mathcal{M}(t)\big)$, we get

$$\begin{aligned}
&\Big(\tilde{u}_\rho(t) - \big(\mathcal{M}(t)z\big)_0\Big)\big(J_{\xi_\rho}(t)u_\rho(t) - z\big)\\
&= \Big(\tilde{u}_\rho(t) - \big(\mathcal{M}(t)z\big)_0\Big)\big(J_{\xi_\rho}(t)u_\rho(t) - u(t) + u(t) - z\big)\\
&= \Big(\tilde{u}_\rho(t) - \big(\mathcal{M}(t)z\big)_0\Big)\big(J_{\xi_\rho}(t)u_\rho(t) - u(t)\big)\\
&\quad + \Big(\tilde{u}_\rho(t) - \big(\mathcal{M}(t)z\big)_0\Big)\big(u(t) - z\big)\\
&\geq 0,
\end{aligned}$$

by (6), we get

$$\Big(\tilde{u}(t) - \big(\mathcal{M}(t)z\big)_0\Big)\big(u(t) - z\big) \geq 0.$$

According to Lemma 2 in [21], this yields that

$$\tilde{u}(t) \in \mathcal{M}(t)u(t).$$

Therefore,

$$\dot{u}(t) \in -\mathcal{M}(t)u(t) + h\Big(t, u\Big(g_1\big(t, u(t)\big)\Big), \dots, u^{[n]}\Big(g_n\big(t, u(t)\big)\Big)\Big). \qquad \square$$

We are now in a position to prove our main result.

Theorem 3.3. *Let $\mathcal{M}(t) : D\big(\mathcal{M}(t)\big) \rightrightarrows \mathbb{R}$ be a maximal monotone operator such that $[0,T] \subset D\big(\mathcal{M}(t)\big)$. Suppose that* (H_1), (H_2), (H_4) *hold and*

(H_6) *there exists a non-negative function*

$$\beta \in L^2_{\mathbb{R}}([0,T]) \quad \text{such that} \quad \|\beta\|_{L^1_{\mathbb{R}}} < \frac{1}{n+1}$$

and for all $(t,x,z) \in [0,T]^{n+2}$

$$|\mathcal{M}(t)z| + |h(t,x)| \leq \beta(t)(1 + \|x\| + |z|),$$

where

$$|\mathcal{M}(t)z| = \sup\{|y|; y \in \mathcal{M}(t)z\}.$$

Then, the problem (1) *admits a solution $u \in \mathcal{AC}([0,T])$.*

P r o o f.

Step 1. Suppose that $u : [0,T] \to [0,T]$ is an absolutely continuous solution of (1), for all $t \in [0,T]$

$$\begin{aligned}
|u(t)| &\leq |u_0| + \int_0^t |\dot{u}(s)| ds \\
&\leq |u_0| + \int_0^t \beta(s) \left(1 + |u(s)| + \sum_{i=1}^{n} \left|u^{[i]}\Big(g_i\big(s,u(s)\big)\Big)\right|\right) ds \\
&\leq |u_0| + \int_0^t \beta(s) \left(1 + |u(s)| + \sum_{i=1}^{n} \sup_{0\leq g_i\left(s,u(s)\right)\leq T} \left|u^{[i]}\Big(g_i\big(s,u(s)\big)\Big)\right|\right) ds \\
&\leq |u_0| + \int_0^t \beta(s) \left(1 + |u(s)| + \sum_{i=1}^{n} \sup_{0\leq s\leq T} \left|u^{[i]}(s)\right|\right) ds \\
&\leq |u_0| + \|\beta\|_{L^1_{\mathbb{R}}} + (n+1)\|u\|_{\mathcal{C}}\|\beta\|_{L^1_{\mathbb{R}}},
\end{aligned}$$

then,

$$\|u\|_{\mathcal{C}} \leq \frac{|u_0| + \|\beta\|_{L^1_{\mathbb{R}}}}{1 - (n+1)\|\beta\|_{L^1_{\mathbb{R}}}} = \zeta, \tag{7}$$

we deduce that

$$|u^{[i]}(t)| \leq \zeta, \qquad \forall t \in [0,T]. \tag{8}$$

Step 2. Consider the mapping $\Psi_\zeta : \mathbb{R} \to \mathbb{R}$ defined by

$$\Psi_\zeta(z) = \begin{cases} z & \text{if } |z| \leq \zeta; \\ \dfrac{\zeta z}{|z|} & \text{if } |z| > \zeta. \end{cases}$$

Define the maximal monotone operator $M(t) : D\big(M(t)\big) \rightrightarrows [0,T]$ such that $D\big(M(t)\big) \subset [0,T]$ by

$$M(t)z = \mathcal{M}(t)\Psi_\zeta(z),$$

and the Carathéodory mapping $h_\Psi : [0,T]^{n+1} \to \mathbb{R}$ by

$$h_\Psi\big(t, x_1, x_2, \ldots, x_n\big) = h\big(t, \Psi_\zeta(x_1), \Psi_\zeta(x_2), \ldots, \Psi_\zeta(x_n)\big).$$

We have

$$\begin{aligned}|M(t)z| + |h_\Psi\big(t, x_1, \dots, x_n\big)| &= |\mathcal{M}(t)\Psi_\zeta(z)| + |h\big(t, \dots, \Psi_\zeta(x_n)\big)| \\ &\leq \beta(t)\big(1 + \sum_{i=1}^{n} |\Psi_\zeta(x_i)| + |\Psi_\zeta(y)|\big) \\ &\leq \beta(t)\big(1 + (n+1)\zeta\big) = \gamma(t).\end{aligned}$$

$\gamma(t) \in L^2_{\mathbb{R}}([0,T])$. By Theorem 3.2, the problem

$$\begin{cases} \dot{u}(t) \in -M(t)u(t) + h_\Psi\Big(t, u\Big(g_1\big(t, u(t)\big)\Big), \dots, u^{[n]}\Big(g_n\big(t, u(t)\big)\Big)\Big); \\ u(0) = u_0, \end{cases} \tag{9}$$

admits a solution $u \in \mathcal{AC}\,([0,T])$.

Let us show that u is solution of (9) if and only if u is solution of (1).
(i) We have $|\dot{u}(t)| \leq \gamma(t)$ if we suppose that u is a solution of (9). Then, a.e. $t \in [0,T]$

$$\begin{aligned}|u(t)| &\leq |u_0| + \int_0^t |\dot{u}(s)|\,\mathrm{d}s \\ &\leq |u_0| + \int_0^t |\gamma(s)|\,\mathrm{d}s \\ &= |u_0| + \|\beta\|_{L^1_{\mathbb{R}}}\,(1 + (n+1)\zeta) \\ &= \frac{|u_0| + \|\beta\|_{L^1_{\mathbb{R}}}}{1 - (n+1)\|\beta\|_{L^1_{\mathbb{R}}}} = \zeta\end{aligned}$$

and for $1 \leq i \leq n$, $|u^{[i]}(t)| \leq \zeta$. By definition of Ψ_ζ, we get

$$\Psi_\zeta\Big(u^{[i]}\Big(g_i\big(t, u(t)\big)\Big)\Big) = u^{[i]}\Big(g_i\big(t, u(t)\big)\Big).$$

Therefore, u is also a solution of (1).
(ii) If u is a solution of (1), by the relation (8),

$$\Psi_\zeta\Big(u^{[i]}\Big(g_i\big(t, u(t)\big)\Big)\Big) = u^{[i]}\Big(g_i\big(t, u(t)\big)\Big),$$

then,

$$h\Big(t, \dots, u^{[n]}\Big(g_n\big(t, u(t)\big)\Big)\Big) = h_\Psi\Big(t, \dots, u^{[n]}\Big(g_n\big(t, u(t)\big)\Big)\Big)$$

and $\mathcal{M}(t)u(t) = M(t)u(t)$, so u is a solution of (9). □

EXAMPLE. Consider the problem

$$(\mathcal{I})\begin{cases}\dot{u}(t)\in-\mathcal{M}u(t)+\frac{t}{16}u\left(\frac{t}{2}u(t)\right)-2u^{[2]}\left(\frac{u(t)+t}{2}\right) & \text{a.e. } t\in\left[0,\frac{1}{2}\right];\\ u\left(0\right)=0,\end{cases}$$

where the set-valued mapping

$$\mathcal{M}x=\begin{cases}0 & \text{if } x>0;\\ 1 & \text{if } x<0;\\ [0,1] & \text{if } x=0,\end{cases}$$

is a maximal monotone operator,

$$h\big(t,x_1,x_2\big)=\frac{t}{16}x_1-2x_2,\quad g_1(t,x)=\frac{t}{2}x\quad\text{and}\quad g_2(t,x)=\frac{x+t}{2}$$

are Carathéodory mappings. We have for all $(t,x,x_1,x_2)\in[0,\frac{1}{2}]^4$

$$|\mathcal{M}x|+|h\left(t,x_1,x_2\right)|,\leq\frac{t}{32}=\beta(t),$$

where $\beta\in L^2_{\mathbb{R}}\left(\left[0,\frac{1}{2}\right]\right)$ with $\|\beta\|_{L^1_{\mathbb{R}}}=\frac{1}{256}<\frac{1}{3}$. Hence,

$$|\mathcal{M}x|+|h(t,x_1,x_2)|\leq\beta(t)\left(1+|x|+|x_1|+|x_2|\right).$$

The hypotheses of Theorem 3.3 hold, then the problem $(\mathcal{I})$ admits a solution $u\in\mathcal{AC}\left(\left[0,\frac{1}{2}\right]\right)$.

EXAMPLE. Let $C(t):[0,1]\to[0,t[$ be a set-valued mapping and consider the problem

$$(\mathcal{II})\begin{cases}\dot{u}(t)\in-\partial I_{C(t)}u(t)+t^5u(t)+\frac{1}{7\exp 1}u^{[2]}\left(\frac{1}{t^2u(t)+1}\right)-u^{[3]}\Big(\big(u(t)\big)^2\Big) & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{a.e. } t\in[0,1];\\ u\left(0\right)=0,\end{cases}$$

where the set-valued mapping

$$\partial I_{C(t)}x=\begin{cases}\mathbb{R}_- & \text{if } x=0;\\ 0 & \text{if } 0<x<t,\end{cases}$$

is the convex subdifferential (which is a maximal monotone operator) of the indicator function

$$I_{C(t)}x=\begin{cases}0 & \text{if } x\in C(t);\\ +\infty & \text{if } x\notin C(t).\end{cases}$$

Hence, for all $\xi>0$,

$$J_\xi(t)x=\begin{cases}0 & \text{if } x\leq 0;\\ x & \text{if } 0<x<t,\end{cases}$$

and $t\mapsto J_\xi(t)x$ is Lebesgue measurable.

For all $(t, x, x_1, x_2, x_3) \in [0,1]^5$:

$$\begin{aligned} h\left(t, x_1, x_2, x_3\right) &= t^5 x_1 + \frac{x_2}{7\exp 1} - x_3, \\ g_1\left(t, x\right) &= t, \\ g_2\left(t, x\right) &= \frac{1}{xt^2+1}, \\ g_3\left(t, x\right) &= x^2 \end{aligned}$$

are Carathéodory mappings and

$$\left|\partial I_{C(t)} x\right| + \left|h\left(t, x_1, x_2, x_3\right)\right| \leq t^5 + \frac{1}{7\exp 1} = \eta(t),$$

where

$$\beta \in L^2_{\mathbb{R}}([0,1]) \quad \text{with} \quad \|\beta\|_{L^1_{\mathbb{R}}} = \frac{1}{6} + \frac{1}{7\exp 1} < \frac{1}{4}.$$

Hence,

$$\left|\partial I_{C(t)} x\right| + \left|h\left(t, x_1, x_2, x_3\right)\right| \leq \beta(t)\left(1 + |x| + |x_1| + |x_2| + |x_3|\right).$$

The hypotheses of Theorem 3.3 hold, then the problem $(\mathcal{II})$ admits a solution $u \in \mathcal{AC}\left([0,1]\right)$.

4. An optimal control problem with no convexity assumptions

In this section, we try to provide a minimizer to the Lagrange optimal control (2)–(3) associated with differential inclusion (1).

It is well-known that we are always able to transform Lagrange optimal control problems into Lagrange calculus of variations problems. Therefore, (2)–(3) can be formed as the following problem: Minimize

$$\mathcal{I}(u) = \int_0^T L\Big(t, u(t), u\Big(g_1\big(t, u(t)\big)\Big), \ldots, u^{[n]}\Big(g_n\big(t, u(t)\big)\Big), \dot{u}(t)\Big)\, \mathrm{d}t, \tag{10}$$

on

$$\mathcal{X} = \left\{u \in \mathcal{C}([0,T]) : u(0) = u_0\right\}, \tag{11}$$

where, for $s = \left(s^0, s^1, \ldots, s^n\right) \in [0,T]^{n+1}$:

$$\begin{aligned} L : [0,T] \times\ & [0,T]^{n+1} \times \mathbb{R} \to [0, +\infty] \\ & (t, s, \mu) \mapsto L(t, s, \mu) = \inf\left\{\mu_0 : (\mu, \mu_0) \in Q(t, s)\right\}, \end{aligned}$$

is the Langrangian function associated to problem (2)–(3) (for more details

see [10], [12], [24]) and

$$Q(t,s) = \begin{cases} \Phi(t,s) & \text{if } (t,s) \in [0,T] \times [0,T]^{n+1}; \\ \emptyset & \text{otherwise,} \end{cases}$$

with

$$\Phi(t,s) = \left\{ \begin{array}{l} (\mu,\mu_0) \in \mathbb{R} \times \mathbb{R} : \mu_0 \geq h_0\left(t,s^1,\ldots,s^n,z\right), \\ \mu \in -\mathcal{M}(t)s^0 + h(t,s^1,\ldots,s^n,z) \text{ and } z \in Z(s^0) \end{array} \right\},$$

where $h_0 : [0,T] \times [0,T]^n \times \mathbb{R}^m \to \mathbb{R}$ is a mapping and $Z : [0,T] \rightrightarrows \mathbb{R}^m$ is a nonempty measurable multifunction.

Now, assume the existence of solutions of the associated convexified optimal control problem: Minimize

$$\mathcal{J}^{**}(u,z) = \int_0^T h_0^{**}\Big(t, u\Big(g_1\big(t,u(t)\big)\Big), \ldots, u^{[n]}\Big(g_n\big(t,u(t)\big)\Big), z(t)\Big)\,\mathrm{d}t; \tag{12}$$

on the class of admissible pairs

$$\mathcal{Y}^{**} = \big\{(u,z) \in \mathcal{C}([0,T]) \times \mathbb{R}^m : u \text{ is solution of (1) and } z(\cdot) \in \mathrm{co}Z\big(u(\cdot)\big)\big\}, \tag{13}$$

where

$$h_0^{**}(t,s^1,\ldots,s^n,\cdot)$$

is the bipolar of $h_0(t,s^1,\ldots,s^n,\cdot)$ and the control $z : [0,T] \to \mathbb{R}^m$ is measurable. Note that, the solution $(\overline{u}_c,\overline{z}_c)$ of (12)–(13) (resp. $(\overline{u},\overline{z})$ of (2)–(3)) is called a relaxed solution (resp. original solution).

We prove the existence of solutions of (2)–(3) in 3 steps:

i) Transform the convexified problem (12)–(13) into a convex calculus of variations problem:

$$\mathcal{I}^{**}(u) = \int_0^T L^{**}\Big(t,u(t),u\Big(g_1\big(t,u(t)\big)\Big),\ldots,u^{[n]}\big(g_n\big(t,u(t)\big)\Big),\dot{u}(t)\Big)\,\mathrm{d}t; \tag{14}$$

so that the minimizer to (14) on $\mathcal{X}$ will be $\overline{u}_c(\cdot)$.

ii) Prove the existence of $\overline{u}(\cdot)$ solution of the non-convex calculus of variations problem (10)–(11).

iii) Use $\overline{u}(\cdot)$ to obtain a solution $\big(\overline{u}(\cdot),\overline{z}(\cdot)\big)$ of the non-convex optimal control problem (2)–(3).

We start by demonstrating the following propositions.

Proposition 4.1. *Let $\mathcal{M}(t) : D\big(\mathcal{M}(t)\big) \subset \mathbb{R} \rightrightarrows \mathbb{R}$ be a maximal monotone operator, $Z : [0,T] \rightrightarrows \mathbb{R}^m$, $\mathrm{co}Z : [0,T] \rightrightarrows \mathbb{R}^m$ be measurable multifunctions, $h : [0,T] \times [0,T]^n \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory mapping, $h_0 : [0,T] \times [0,T]^n \times \mathbb{R}^m \to \mathbb{R}$ be a mapping and h_0^{**} be the bipolar of h_0.*

Assume that:

$(\mathcal{H}_1^{\mathrm{c}})$ $\mathrm{gph}(Z)$ *and* $\mathrm{gph}(\mathrm{co}Z)$ *are closed;*

$(\mathcal{H}_2^{\mathrm{c}})$ *the functions* h_0 *and* h_0^{**} *are* lsc *on* $[0,T]^{n+1}$*;*

$(\mathcal{H}_3^{\mathrm{c}})$ *for all* $z \in \mathbb{R}^m$*,* $z \mapsto h(t,x,z)$ *is affine;*

$(\mathcal{H}_4^{\mathrm{c}})$ *for any sequence* $(t_\rho)_\rho$ *converging to* t *in* $[0,T]$*,*

$$\lim_{\rho\to+\infty} \mathrm{dis}\big(\mathcal{M}(t_\rho), \mathcal{M}(t)\big) = 0;$$

$(\mathcal{H}_5^{\mathrm{c}})$ *for all* $(t,s) \in [0,T]\times[0,T]^{n+1}$*, the sets* $Q(t,s)$ *and* $co\big(Q(t,s)\big)$ *are closed.*

Then,

1. *the Lagrangian* L *is a well defined Borel mapping a.e.* $t \in [0,T]$ *such that* $L(t,s,\cdot)$ *is* lsc *and*

$$L\big(t,s,\mu\big) = \inf_{z\in H(t,s,\mu)} h_0(t,s^1,\ldots,s^n,z); \tag{15}$$

where

$$H(t,s,\mu) = \begin{cases} \Lambda\,(t,s,\mu) & \text{if } (t,s)\in[0,T]\times[0,T]^{n+1};\\ \emptyset, & \text{otherwise,}\end{cases} \tag{16}$$

$$\Lambda\,(t,s,\mu) = \left\{\begin{array}{l} y\in\mathbb{R}^m : y\in Z\left(s^0\right) \quad \text{and}\\ \mu\in -\mathcal{M}(t)s^0 + h\left(t,s^1,\ldots,s^n,y\right)\end{array}\right\};$$

2. *the bipolar* L^{**} *is Borel a.e.* $t\in[0,T]$*, such that*

$$L^{**}(t,s,\mu) = \inf_{z\in H_0(t,s,\mu)} h_0^{**}\left(t,s^1,\ldots,s^n,z\right); \tag{17}$$

where

$$H_0(t,s,\mu) = \begin{cases} \Lambda_{co}(t,s,\mu) & \text{if } (t,s)\in[0,T]\times[0,T]^{n+1};\\ \emptyset, & \text{otherwise,}\end{cases} \tag{18}$$

$$\Lambda_{co}(t,s,\mu) = \left\{\begin{array}{l} y\in\mathbb{R}^m : y\in \mathrm{co}Z(s^0) \quad \text{and}\\ \mu\in -\mathcal{M}(t)s^0 + h(t,s^1,\ldots,s^n,y)\end{array}\right\}.$$

P r o o f.

1. Let $\big((t_\rho,s_\rho,\mu_\rho)\big)_\rho$ be a sequence in $[0,T]\times[0,T]^{n+1}\times\mathbb{R}$ converging to (t,s,μ); for a.e. $t\in[0,T]$ and all $s\in[0,T]^{n+1}$, $Q(t_\rho,s_\rho)\subset Q(t,s)$.

Indeed, for $\mu, \mu_0) \in Q(t_\rho, s_\rho)$, we have

$$\begin{cases} \mu_0 & \geq & h_0\left(t_\rho, s_\rho^1, \ldots, s_\rho^n, z\right); \\ \mu & \in & -\mathcal{M}\left(t_\rho\right) s_\rho^0 + h\left(t_\rho, s_\rho^1, \ldots, s_\rho^n, z\right); \\ s_\rho^0 & \in & D\big(\mathcal{M}(t_\rho)\big); \\ z & \in & Z\left(s_\rho^0\right). \end{cases}$$

As h is a Carathéodory mapping, Using Lusin's theorem, h is continuous a.e. $t \in [0,T]$ and

$$\lim_{\rho\to+\infty} \mu - h\left(t_\rho, s_\rho^1, \ldots, s_\rho^n, z\right) = \mu - h\left(t, s^1, \ldots, s^n, z\right) \quad \text{a.e. } t \in [0,T].$$

Using $(\mathcal{H}_4^c)$ and Lemma 3 in [21], we obtain

$$\mu \in -\mathcal{M}(t)s^0 + h\left(t, s^1, \ldots, s^n, z\right) \quad \text{a.e. } t \in [0,T]. \tag{19}$$

We get by $(\mathcal{H}_1^c)$, $(\mathcal{H}_2^c)$ and relation (19),

$$\begin{cases} \mu_0 & \geq & \liminf\limits_{\rho\to+\infty} h_0\left(t_\rho, s_\rho^1, \ldots, s_\rho^n, z\right) \geq h_0\left(t, s^1, \ldots, s^n, z\right); \\ \mu & \in & -\mathcal{M}(t)s^0 + h\left(t, s^1, \ldots, s^n, z\right); \\ z & \in & Z\left(s^0\right). \end{cases}$$

On the other hand,

$$\begin{aligned} \lim_{\rho\to+\infty} |L(t,s,\mu) - L(t_\rho, s_\rho, \mu_\rho)| &= \lim_{\rho\to+\infty} \left| \inf_{(\mu,\mu_0)\in Q(t,s)} \mu_0 - \inf_{(\mu_\rho,\mu_0)\in Q(t_\rho,s_\rho)} \mu_0 \right| \\ &\leq \lim_{\rho\to+\infty} \left| \inf_{(\mu,\mu_0)\in Q(t,s)} \mu_0 - \inf_{(\mu_\rho,\mu_0)\in Q(t,s)} \mu_0 \right|. \end{aligned}$$

That yields the continuity of L a.e. $t \in [0,T]$ so, L is well-defined Borel a.e. $t \in [0,T]$. For every $(t,s) \in [0,T] \times [0,T]^{n+1}$;

$$\begin{aligned} \mathrm{epi}L(t,s,\cdot) &= \{(\mu,\mu_0) \in \mathbb{R}\times\mathbb{R} : \mu_0 \geq L(t,s,\mu)\} \\ &= \left\{(\mu,\mu_0) \in \mathbb{R}\times\mathbb{R} : \mu_0 \geq \inf_{(\mu,\mu_0)\in Q(t,s)} \mu_0\right\} \\ &= Q\left(t,s\right), \end{aligned}$$

by $(\mathcal{H}_5^c)$, $L\left(t,s,\cdot\right)$ is lsc and we can rewrite $Q(t,s)$ as follows

$$Q(t,s) = \left\{ \begin{array}{l} (\mu,\mu_0) \in \mathbb{R}\times\mathbb{R} : \mu \in -\mathcal{M}(t)s^0 + h(t,s^1,\ldots,s^n,z), \\ \mu_0 \;=\; h_0(t,s^1,\ldots,s^n,z) + \delta, z \in Z(s^0) \text{ and } \delta \geq 0 \end{array} \right\},$$

Then, (15)–(16) hold.

2. We have

$$L^{**}(t,s,\mu)=\inf\left\{\mu_0:(\mu,\mu_0)\in\overline{co}\big(\mathrm{epi}L(t,s,\cdot)\big)\right\},$$

by the closure of $Q(t,s)$;

$$\overline{co}\big(\mathrm{epi}L(t,s,\cdot)\big)=co\big(Q(t,s)\big)$$

with

$$co\big(Q(t,s)\big)=\begin{cases}\Phi_{co}(t,s) & \text{if } (t,s)\in[0,T]\times[0,T]^{n+1};\\ \emptyset, & \text{otherwise,}\end{cases}$$

and

$$\Phi_{co}\,(t,s)=\left\{\begin{array}{l}(\mu,\mu_0)\in\mathbb{R}\times\mathbb{R}:\mu_0\geq h_0^{**}(t,s^1,\ldots,s^n,z),\\ \mu\in-\mathcal{M}(t)s^0+h(t,s^1,\ldots,s^n,z)\text{ and }z\in\mathrm{co}Z(s^0)\end{array}\right\}.$$

Using the same reasoning as in 1, L^{**} is also Borel a.e. $t\in[0,T]$. Moreover,

$$co\,(Q\,(t,s))=\left\{\begin{array}{l}(\mu,\mu_0)\in\mathbb{R}\times\mathbb{R}:\mu\in-\mathcal{M}(t)s^0+h(t,s^1,\ldots,s^n,z),\\ \mu_0=h_0^{**}(t,s^1,\ldots,s^n,z)+\delta,z\in\mathrm{co}Z(s^0)\text{ and }\delta\geq0\end{array}\right\},$$

so (17)–(18) hold. □

Proposition 4.2. *If the hypotheses $(\mathcal{H}_1^c)$–$(\mathcal{H}_5^c)$ hold true. In addition to the following condition:*

(AC) *for $\omega\in\mathbb{R}$ fixed and all $(t,s)\in[0,T]\times[0,T]^{n+1}$, $z\in\mathrm{co}Z\left(s^0\right)$, there exist two functions $\lambda_1:\mathbb{R}\to[0,1[$ and $\lambda_2:\mathbb{R}\to]1,+\infty[$, such that;*

$$\begin{aligned}\inf_{z\in H_0(t,s,\omega)}h_0^{**}(t,s^1,\ldots,s^n,z)\geq\big(1-\alpha(\omega)\big)&\inf_{z\in H(t,s,\omega\lambda_1(\omega))}h_0(t,s^1,\ldots,s^n,z)\\ &+\ \alpha(\omega)\inf_{z\in H(t,s,\omega\lambda_2(\omega))}h_0(t,s^1,\ldots,s^n,z),\end{aligned}$$

where $\alpha(\omega)=\frac{1-\lambda_1(\omega)}{\lambda_2(\omega)-\lambda_1(\omega)}$ with the convention $0.\infty=0$. Then, $L\,(t,s,\cdot)$ is almost convex.

Proof. If $(t,s)\notin[0,T]\times[0,T]^{n+1}$, then $L^{**}(t,s,\cdot)=L(t,s,\cdot)\equiv+\infty$. Hence, $L\,(t,s,\cdot)$ is almost convex. Fix $t\in[0,T]$, $s\in[0,T]^{n+1}$ and $\omega\in\mathbb{R}$, by definition of L and L^{**}

$$L^{**}\,(t,s,\omega)<L\,(t,s,\omega)$$

by (AC), there exist $\lambda_1 : \mathbb{R} \to [0,1[$, $\lambda_2 : \mathbb{R} \to]1, +\infty[$ and $\alpha(\omega) = \frac{1-\lambda_1(\omega)}{\lambda_2(\omega)-\lambda_1(\omega)} \in]0,1[$, such that

$$\begin{aligned} L^{**}(t,s,\omega) &= \inf_{z \in H_0(t,s,\omega)} h_0^{**}\left(t, s^1, \ldots, s^n, z\right) \\ &\geq \big(1-\alpha(\omega)\big) \inf_{z \in H(t,s,\omega\lambda_1(\omega))} h_0(t, s^1, \ldots, s^n, z) \\ &\quad + \alpha(\omega) \inf_{z \in H(t,s,\omega\lambda_2(\omega))} h_0(t, s^1, \ldots, s^n, zt) \\ &= \big(1-\alpha(\omega)\big) L\big(t, s, \omega\lambda_1(\omega)\big) + \alpha(\omega) L\big(t, s, \omega\lambda_2(\omega)\big). \end{aligned}$$

On the other hand,

$$\begin{aligned} L^{**}(t,s,\omega) &\leq \big(1-\alpha(\omega)\big) L^{**}\big(t, s, \omega\lambda_1(\omega)\big) + \alpha(\omega) L^{**}\big(t, s, \omega\lambda_2(\omega)\big) \\ &< \big(1-\alpha(\omega)\big)\ L\ \big(t, s, \omega\lambda_1(\omega)\big) + \alpha(\omega)\ L\ \big(t, s, \omega\lambda_2(\omega)\big). \end{aligned}$$

Hence,

$$L^{**}(t,s,\omega) = \big(1-\alpha(\omega)\big)\ L\ \big(t, s, \omega\lambda_1(\omega)\big) + \alpha(\omega)\ L\ \big(t, s, \omega\lambda_2(\omega)\big).$$

So, L is almost convex. □

Before announcing our main theorem, we need to state the below proposition.

Proposition 4.3 ([8]). *Let $L : [0,T] \times [0,T]^{n+1} \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a Borel function such that $L(t,s,\cdot)$ is almost convex and* lsc, *$L(\cdot,\cdot,0)$ is* lsc *and $L^{**}(\cdot,\cdot,\cdot)$ is Borel.*

If there exists a minimizer $\overline{u}_c(\cdot)$ to the convexified calculus of variations problem (14)*, then there exists a minimizer $\overline{u}(\cdot)$ to the non-convex calculus of variations problem* (10)–(11)*, such that*

$$\mathcal{I}(\overline{u}) = \mathcal{I}^{**}(\overline{u}_c). \tag{20}$$

Now, we present our main theorem in this section:

Theorem 4.4. *Under the hypotheses of Theorem* (3.3)*, Proposition* (4.1) *and* (AC)*. Let*

$$D_0 = \left\{ \begin{array}{l} (t,s,\mu_0) \in [0,T] \times [0,T]^{n+1} \times \mathbb{R} : \mu_0 = h_0\left(t, s^1, \ldots, s^n, z\right) + \delta, \\ 0 \in -\mathcal{M}(t)s^0 + h\left(t, s^1, \ldots, s^n, z\right), \ \ z \in Z\left(s^0\right) \ \ \textit{and} \ \ \delta \geq 0 \end{array} \right\}$$

be a closed set. If the relaxed solution $(\overline{u}_c, \overline{z}_c)$ exists then the original solution $(\overline{u}, \overline{z})$ exists.

Proof. Assume that $\inf_{(u,z)\in\mathcal{Y}} \mathcal{J}(u,z) < +\infty$. If $(\overline{u}_c, \overline{z}_c) \in \mathcal{Y}^{**}$ is a relaxed solution, so

$$\mathcal{I}^{**}(\overline{u}_c) \leq \mathcal{J}^{**}(\overline{u}_c, \overline{z}_c) < +\infty. \tag{21}$$

i) Let us prove that $\overline{u}_c$ is a minimizer to $\mathcal{I}^{**}(\cdot)$.

Let $u \in \mathcal{X}$ such that; $\mathcal{I}^{**}(u) < +\infty$. Set

$$\mathcal{U} = \left\{ \begin{array}{c} u \in \mathcal{X} : L^{**}\Big(t, u(t), u\Big(g_1\big(t, u(t)\big)\Big), \ldots, u^{[n]}\Big(g_n\big(t, u(t)\big)\Big), \dot{u}(t)\Big) \\ = h_0^{**}\Big(t, u\Big(g_1\big(t, u(t)\big)\Big), \ldots, u^{[n]}\Big(g_n\big(t, u(t)\big)\Big), z\Big) \end{array} \right\}$$

and let Γ be a measurable multifunction with closed values

$$\Gamma(t) = \big\{ z(t) \in \mathbb{R}^m : z(t) \in \mathrm{co} Z\big(u(t)\big), u \in \mathcal{U} \quad \text{and verifies (1)} \big\}.$$

There exists a measurable selection $z(\cdot) \in \Gamma(\cdot)$ (Theorem 1.14.1 in [2]) such that

$$(u, z) \in \mathcal{Y}^{**}.$$

We have by (17),

$$\begin{aligned} \mathcal{I}^{**}(\overline{u}_c) &= \int_0^T L^{**}\Big(t, \overline{u}_c(t), \overline{u}_c\Big(g_1\big(t, \overline{u}_c(t)\big)\Big), \ldots, \overline{u}_c^{[n]}\Big(g_n\big(t, \overline{u}_c(t)\big)\Big), \dot{\overline{u}}_c(t)\Big)\, \mathrm{d}t \\ &\leq \int_0^T h_0^{**}\Big(t, \overline{u}_c\Big(g_1\big(t, \overline{u}_c(t)\big)\Big), \ldots, \overline{u}_c^{[n]}\Big(g_n\big(t, \overline{u}_c(t)\big)\Big), \overline{z}_c(t)\Big)\, \mathrm{d}t \\ &= \mathcal{J}^{**}\left(\overline{u}_c, \overline{z}_c\right). \end{aligned}$$

$(\overline{u}_c, \overline{z}_c)$ is a minimizer to $\mathcal{J}^{**}(\cdot, \cdot)$, then $\mathcal{J}^{**}\left(\overline{u}_c, \overline{z}_c\right) \leq \mathcal{J}^{**}\left(u, z\right)$ and since $z(t) \in \Gamma(t)$;

$$\begin{aligned} \mathcal{J}^{**}\left(u, z\right) &= \int_0^T h_0^{**}\Big(t, u\Big(g_1\big(t, u(t)\big)\Big), \ldots, u^{[n]}\Big(g_n\big(t, u(t)\big)\Big), z(t)\Big)\, \mathrm{d}t \\ &= \int_0^T L^{**}\Big(t, u(t), u\Big(g_1\big(t, u(t)\big)\Big), \ldots, u^{[n]}\Big(g_n\big(t, u(t)\big)\Big), \dot{u}(t)\Big)\, \mathrm{d}t \\ &= \mathcal{I}^{**}\left(u\right). \end{aligned}$$

Hence, for all $u \in \mathcal{X}$, $\mathcal{I}^{**}\left(\overline{u}_c\right) \leq \mathcal{I}^{**}\left(u\right)$, i.e., $\overline{u}_c$ is a minimizer to $\mathcal{I}^{**}\left(\cdot\right)$.

ii) We have

$$\begin{aligned}
\mathrm{epi}L\left(\cdot,\cdot,0\right) &= \left\{\left((t,s),\mu_0\right) \in [0,T]^{n+2}\times\mathbb{R} : \mu_0 \geq L\left(t,s,0\right)\right\}\\
&= \left\{\begin{array}{l} \left((t,s),\mu_0\right) \in [0,T]^{n+2}\times\mathbb{R} : \\ \mu_0 \geq \inf_{z\in H\left(t,s,0\right)} h_0\left(t,s^1,\ldots,s^n,z\right) nt \end{array}\right\}\\
&= \left\{\begin{array}{l} \left((t,s),\mu_0\right) \in [0,T]^{n+2}\times\mathbb{R} : \\ \mu_0 = h_0\left(t,s^1,\ldots,s^n,z\right)+\delta, \\ 0 \in -\mathcal{M}(t)s^0 + h\left(t,s^1,\ldots,s^n,z\right), \\ z\in Z\left(s^0\right) \text{ and } \delta\geq 0 \end{array}\right\}\\
&= D_0,
\end{aligned}$$

which is closed, so $L\left(\cdot,\cdot,0\right)$ is lsc. By Proposition 4.3 and since $\overline{u}_c$ is a minimizer of $\mathcal{I}^{**}(\cdot)$, then there exists a minimizer $\overline{u}$ to $\mathcal{I}(\cdot)$ such that (20) is verified and by (21), $\mathcal{I}\left(\overline{u}\right) < +\infty$.

iii) Similarly as in Step 2, it follows that there exists a measurable $\overline{z}(\cdot)\in\overline{\Gamma}(\cdot)$ where

$$\overline{\Gamma}(t) = \left\{z(t)\in\mathbb{R}^m : z(t)\in Z\big(u(t)\big), u\in\overline{\mathcal{U}} \text{ and verifies } \ (1)\right\},$$

and

$$\overline{\mathcal{U}} = \left\{\begin{array}{l} u\in\mathcal{X} : L\Big(t,u(t),u\Big(g_1\big(t,u(t)\big)\Big),\ldots,u^{[n]}\Big(g_n\big(t,u(t)\big)\Big),\dot{u}(t)\Big) \\ \quad = h_0\Big(t,u\Big(g_1\big(t,u(t)\big)\Big),\ldots,u^{[n]}\Big(g_n\big(t,u(t)\big)\Big),z\Big) \end{array}\right\},$$

such that $(\overline{u},\overline{z})\in\mathcal{Y}$, and for all $(u,z)\in\mathcal{Y}$,

$$\begin{aligned}
\mathcal{J}\left(\overline{u},\overline{z}\right) &= \int_0^T h_0\Big(t,\overline{u}\Big(g_1\big(t,\overline{u}(t)\big)\Big),\ldots,\overline{u}^{[n]}\Big(g_n\big(t,\overline{u}(t)\big)\Big),\overline{z}(t)\Big)\,\mathrm{d}t\\
&= \int_0^T L\Big(t,\overline{u}(t),\overline{u}\Big(g_1\big(t,\overline{u}(t)\big)\Big),\ldots,\overline{u}^{[n]}\Big(g_n\big(t,\overline{u}(t)\big)\Big),\dot{\overline{u}}(t)\Big)\,\mathrm{d}t\\
&= \mathcal{I}\left(\overline{u}\right).
\end{aligned}$$

Since $\overline{u}$ is a minimizer to $\mathcal{I}(\cdot)$, we have for all $u \in \mathcal{X}$, $\mathcal{I}(\overline{u}) \leq \mathcal{I}(u)$ and by (15),

$$\begin{aligned}\mathcal{J}(\overline{u},\overline{z}) &\leq \mathcal{I}(u)\\ &= \int_0^T L\Big(t, u(t), u\Big(g_1\big(t,u(t)\big)\Big), \dots, u^{[n]}\Big(g_n\big(t,u(t)\big)\Big), \dot{u}(t)\Big)\,\mathrm{d}t\\ &\leq \int_0^T h_0\Big(t, u\Big(g_1\big(t,u(t)\big)\Big), \dots, u^{[n]}\Big(g_n\big(t,u(t)\big)\Big), z(t)\Big)\,\mathrm{d}t\\ &= \mathcal{J}\left(u,z\right),\end{aligned}$$

i.e. $(\overline{u},\overline{z})$ is an original solution. The proof is complete. □

Example. Let the Lagrange optimal control problem be defined as:
Minimize

$$\mathcal{J}(u,z) = \int_0^1 h_0\left(t, u(t), u^{[2]}\left(\frac{1}{u(t)t^2+1}\right), u^{[3]}\big(u(t)^2\big), z_1(t), z_2(t)\right)\mathrm{d}t,$$

on

$$\mathcal{Y} = \left\{ \begin{array}{l} (u,z) \in \mathcal{C}([0,1]) \times \mathbb{R}^2 : u \text{ is a solution of } (\mathcal{II}) \\ \text{and } z(\cdot) = \big(z_1(\cdot), z_2(\cdot)\big) \in Z\big(u(\cdot)\big) \end{array} \right\},$$

where $Z : [0,1] \rightrightarrows \mathbb{R}^2$ is a measurable multifunction defined by $Z(\cdot) \equiv \mathbb{R}^2$, $h_0 : [0,1] \times [0,1]^3 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$h_0(t, x_1, x_2, x_3, z_1, z_2) = g(t, x_1, x_2, x_3) + f(z_1, z_2),$$

with

$$g(t, x_1, x_2, x_3) = |t-1| + |x_1 - 2| + |x_2 - 3| + |x_3 - 4|$$

and

$$f\left(z_1, z_2\right) = \begin{cases} |z_1 + z_2|^2 & \textit{if } z_1 \neq 0 \textit{ or } z_2 \neq 0; \\ 0 & \textit{if } z_1 = z_2 = 0. \end{cases}$$

$(\mathcal{II})$ is the problem defined in the previous example where the mapping $h : [0,1] \times [0,1]^3 \times \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$h\left(t, x_1, x_2, x_3, z_1, z_2\right) = t^5 x_1 + \frac{x_2}{7\exp 1} - x_3 + z_1 + z_2.$$

Let's show that the hypotheses in Theorem 4.4 are satisfied:

$(\mathcal{H}_c^1)$ The sets $\mathrm{gph}\,(Z)$ and $\mathrm{gph}\,(\mathrm{co}Z)$ are closed since

$$\mathrm{gph}\,(Z) = \mathrm{gph}\,(\mathrm{co}Z) = [0,1] \times \mathbb{R}^2.$$

$(\mathcal{H}_c^2)$ The mapping h_0 is continuous, so it is lsc, we have

$$h_0^{**}\,(t, x_1, x_2, x_3, \cdot, \cdot) = 0, \quad \text{hence } \ h_0^{**} \quad \text{is lsc.}$$

$(\mathcal{H}_c^3)$ By definition of h, it is clear that h is an affine mapping on $\mathbb{R}^2$.

$(\mathcal{H}_c^4)$ For a sequence $(t_\rho)_\rho \in [0,1]$ converging to t, we have

$$\lim_{\rho \to 0} \mathrm{dis}\big(\partial I_{C(t_\rho)}, \partial I_{C(t)}\big) = \lim_{\rho \to 0} d_H\big(C\,(t_\rho)\,, C(t)\big) = 0,$$

d_H indicates the distance of Hausdorff.

$(\mathcal{H}_c^5)$ The sets

$$Q(t, x, x_1, x_2, x_3) = \mathrm{co}Q(t, x, x_1, x_2, x_3) = \mathbb{R} \times [0, +\infty[$$

are closed.

(AC) We have

$$\begin{aligned} &\inf_{z \in H_0\left(t, x, x_1, x_2, x_3, \omega\right)} h_0^{**}\big(t, x_1, x_2, x_3, z_1, z_2\big) \\ &= \inf_{z \in H\left(t, x, x_1, x_2, x_3, \omega\right)} h_0\big(t, x_1, x_2, x_3, z_1, z_2\big) \\ &= 0. \end{aligned}$$

Then (AC) holds.

In addition, $D_0 = [0,1] \times [0,1]^4 \times [0, +\infty[$ is a closed set. Since $\mathcal{J}^{**}\,(u, z) = 0$ admits a minimizer, then by Theorem 4.4, $\mathcal{J}\,(u, z)$ admits a minimizer.

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REFERENCES

[1] AFFANE, D.—GHALIA, S.: *First-order iterative differential inclusion*, Electron. J. Math. Anal. Appl. **10** (2022), 1–10.

[2] AUBIN, J.-P.—CELLINA, A.: *Differential Inclusions. Set-valued Maps and Viability Theory*. Grundlehren Math. Wiss. [Fundamental Principles of Mathematical Sciences], Vol. 264 Springer-Verlag, Berlin, 1984.

[3] BERINDE, V.: *Existence and approximation of solutions of some first order iterative differential equations*, Miskolc Math. Notes **11** (2010), 13–26.

[4] BERTSEKAS, D. P.: *Convex Optimization Theory*. Athena Scientific, Nashua, NH, 2009.

[5] BRÉZIS, H.: *Opérateurs Maximaux Monotones et Semi-groupes de Contractions Dans les Espaces de Hilbert*. North-Holland Math. Stud. Vol. 5, Elsevier, Amsterdam, 1973.

[6] BUICĂ, A.: *Existence and continuous dependence of solutions of some functional--differential equations*, Babeș-Bolyai Univ., Fac. Math. Comput. Sci., Res. Semin., Preprint **1995** (1995), 1–13.

[7] CARLOTA, C.—CHÁ, S.: *An existence result for non-convex optimal control problems*, WSEAS Trans. Syst. Control. **9** (2014), 687–697.

[8] CARLOTA, C.—ORNELAS, A.: *Existence of vector minimizers for nonconvex 1-dim integrals with almost convex Lagrangian*, J. Differ. Equations **243** (2007), 414–426.

[9] CELLINA, A.—ORNELAS, A.: *Existence of solutions to differential inclusions and to time optimal control problems in the autonomous case*, SIAM J. Control Optim. **42** (2003), 260–265.

[10] CESARI, L.: *Optimization—Theory and Applications. Problems with Ordinary Differential Equations*. Appl. Math. (New York). Vol. 17, Springer, New York, 1983.

[11] CHENG, S. S.: *Smooth solutions of iterative functional differential equations*, In: Proceedings of the international conference: Dynamical systems and applications 2024, Antalya, Turkey, July 5–10, 2004.

[12] CLARKE, F. H.: *Optimization and Nonsmooth Analysis*. Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley, New York, NY, 1983.

[13] EDER, E.: *The functional differential equation* $x'(t) = x(x(t))$, J. Differ. Equations **54** (1984), 390–400.

[14] EL-SAYED, A. M. A.—EBEAD, H.: *On an initial value problem of delay-refereed differential equation*, Int. J. Math. Trends Technol. **66** (2020), 32–37.

[15] EL-SAYED, A. M. A.—HAMDALLAH, E.—EBEAD, H.: *Positive solutions of an initial value problem of a delay-self-reference nonlinear differential equation*, Malaya J. Matematik. **8** (2020), 1001–1006.

[16] FEČKAN, M.: *On a certain type of functional differential equations*, Math. Slovaca **43** (1993), 39–43.

[17] GHALIA, S.—AFFANE, D.: *Control problem governed by an iterative differential inclusion*, Rend. Circ. Mat. Palermo (2) **72** (2023), no. 4, 2621–2642.

[18] GHALIA, S.—AFFANE, D.: *On the attainable set of iterative differential inclusions*, Math. Slovaca **73** (2023), 1479–1498.

[19] KAUFMANN, E. R.: *Existence and uniqueness of solutions for a second-order iterative boundary-value problem*, Electron. J. Differ. Equ. **2018** (2018), 1–6.

[20] KUMLIN, P.: *A note on fixed point theory*, Functional analysis lecture (2004).

[21] KUNZE, M.—MONTEIRO MARQUES, M. D. P.: *BV solutions to evolution problems with time-dependent domains*, Set-Valued Anal. **5** (1997), 57–72.

[22] LAURAN, M.: *Existence results for some differential equations with deviating argument*, Filomat **25** (2011), 21–31.

[23] OPREA, N.: *Numerical solutions of first order iterative functional-differential equations by spline functions of even degree*, Acta Marisiensis Ser. Technol. **6** (2009), 34–37.

[24] PEDREGAL, P.: *On the generality of variational principles*, Milan J. Math. **71** (2003), 319–356.

[25] PELCZAR, A.: *On some iterative-differential equations. I*, Zesz. Nauk. Uniw. Jagielloń. Pr. Mat. **167** (1968), 53–56.

[26] PODISUK, M.: *On simple iterative ordinary differential equations*, Science Asia. **28** (2002), 191–198.

[27] SI, J.-G.—WANG, X.-P.—CHENG, S. S.: *Nondecreasing and convex C^2-solutions of an iterative functional-differential equation*, Aequationes Math. **60** (2000), 38–56.

[28] WANG, K.: *On the equation $x'(t) = f(x(x(t)))$*, Funkcial. Ekvacioj, Ser. Int. **33** (1990), no. 3, 405–425.

[29] YANG, D.—ZHANG, W.: *Solutions of equivariance for iterative differential equations*, Appl. Math. Lett. **17** (2004), 759–765.

[30] ZHANG, P.: *Analytic solutions for iterative functional differential equations*, Electron. J. Differ. Equ. **2012** (2012), 1–7.

[31] ZHANG, P.—GONG, X.: *Existence of solutions for iterative differential equations*, Electron. J. Differ. Equ. **2014** (2014), 1–10.

[32] ZHOU, J.—SHEN, J.: *Positive solutions of iterative functional differential equations and application to mixed-type functional differential equations*, Discrete Contin. Dyn. Syst., Ser. B **27** (2022), 3605–3624.

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