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NEW CRITERIA ON THE OSCILLATORY BEHAVIOUR OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH MIXED NEUTRAL TERMS

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ABSTRACT. The authors present some new criteria for the oscillation of second order nonlinear differential equations with mixed nonlinear neutral terms and mixed deviating arguments. The approach used is to linearize the equation under consideration and then to deduce the oscillation from that of the linear form. In so doing, the authors obtain new oscillation criteria via a comparison with the first order equations whose oscillatory behaviors are known. They illustrate their results with some examples.

1. Introduction

This paper is devoted to obtaining some new criteria on the oscillatory behavior of all solutions of the nonlinear second order differential equation with mixed neutral terms

$$\Big(a(t)\big(y'(t)\big)^{\alpha}\Big)' + q(t)x^{\alpha}\big(\tau(t)\big) + p(t)x^{\alpha}\big(\omega(t)\big) = 0, \tag{1}$$

where $y(t) = x(t) + p_1(t)x^{\beta}\big(\sigma(t)\big) - p_2(t)x^{\delta}\big(\sigma(t)\big)$. We will assume that:

(i) α, β, and δ are the ratios of odd positive integers with $\alpha \geq 1$;

(ii) p_1, p_2, a, p, q, $c : [t_0, \infty) \to \mathbb{R}^+$ are continuous functions;

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(iii) τ, σ, $\omega : [t_0, \infty) \to \mathbb{R}$ are continuous functions with $\tau(t) \le t$, $\sigma(t) \le t$, $\omega(t) \ge t$, $\tau(t) \to \infty$, and $\sigma(t) \to \infty$ as $t \to \infty$;

(iv) $h(t) = \sigma^{-1}\big(\tau(t)\big) \le t$, $h(t) \to \infty$ as $t \to \infty$, and $h^*(t) = \sigma^{-1}\big(\omega(t)\big) \ge t$.

For $t \ge t_1 \ge t_0$, we let $A(v,u) = \displaystyle\int_u^v \frac{1}{a^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s$ and ask that

$$A(t,t_1) = \int_{t_1}^{t} \frac{1}{a^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s \to \infty \text{ as } t \to \infty. \tag{2}$$

By a *solution* of equation (1), we mean a function $x(t)$ such that the quasi-derivatives $a(t)\big(y'(t)\big)^{\alpha}$ and $\big(a(t)\big(y'(t)\big)^{\alpha}\big)'$ are continuous on $[T_x, \infty)$, $T_x \ge t_0$, and which satisfy Eq. (1) on $[T_x, \infty)$. We consider only those solutions $x(t)$ of (1) that are nontrivial and continuable, i.e.,

$$\sup\{|x(t)| : t \ge T\} > 0 \quad \text{for all } T \ge T_x.$$

A solution of (1) is said to be *oscillatory* if it has arbitrarily large zeros, and *nonoscillatory* otherwise. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

In the last twenty years, there has been a great deal of research activity on the oscillation and nonoscillation of solutions of various forms of neutral differential equations, see for example [1, 4, 5, 11–14, 19, 20, 22–24, 26]. For general oscillation results, we refer the reader to the monographs [2, 3] and the papers [7–10, 25]. In addition to their theoretical interest, the qualitative study of neutral equations has significant practical importance due to fact that they arise in various phenomena including problems involving electric networks containing lossless transmission lines (such as in interconnect switching circuits in high-speed computers), in the study of vibrating masses attached to an elastic bar, and in the solution of vibrational problems with time delays. We refer the reader to Hale's monograph [15] for further applications in science and technology.

Early results on the oscillatory behavior of second-order differential equations with a single sublinear neutral term can be found, for example, in [2, 3] where some oscillation results for (1) appear in the linear case $\alpha = \gamma = 1$ and $p_2(t) = 0$ by using Riccati transformations and some inequalities. Grace [5] established some new results for the case where $p_1(t) = 0$ and $\delta = 1$.

The approach to be used in this paper is to linearize equation (1) and then deduce its oscillatory behavior from that of the linear form via a comparison with first order equations and inequalities whose oscillatory behaviors are known. The obtained results are new and improve the known oscillation criteria that appeared in the literature.

2. Main Results

In this section, we obtain some oscillation criteria for equation (1) in the case where $\beta < 1$ and $\delta > 1$. To obtain our result, we need the following two lemmas.

Lemma 2.1. *Let $q : [t_0, \infty) \to \mathbb{R}^+$, $g : [t_0, \infty) \to \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ be continuous functions with f nondecreasing, $xf(x) > 0$ for $x \neq 0$, and $g(t) \to \infty$ as $t \to \infty$.*

(1) *If the first order delay differential inequality (i.e., $g(t) \leq t$)*

$$y'(t) + q(t)f\Big(y\big(g(t)\big)\Big) \leq 0$$

has an eventually positive solution, then so does the delay equation

$$y'(t) + q(t)f\Big(y\big(g(t)\big)\Big) = 0.$$

(2) *If the first order advanced differential inequality (i.e., $g(t) \geq t$)*

$$y'(t) - q(t)f\Big(y\big(g(t)\big)\Big) \geq 0$$

has an eventually positive solution, then so does the advanced equation

$$y'(t) - q(t)f\Big(y\big(g(t)\big)\Big) = 0.$$

Lemma 2.1 is an immediate extension of known results in [17, 18, 21].

Lemma 2.2 ([16])**.** *If X and Y are nonnegative, then:*

$$X^\lambda + (\lambda - 1)Y^\lambda - \lambda XY^{\lambda-1} \geq 0 \quad \text{for } \lambda > 1, \tag{3}$$

$$X^\lambda - (1 - \lambda)Y^\lambda - \lambda XY^{\lambda-1} \leq 0 \quad \text{for } 0 < \lambda < 1. \tag{4}$$

Equality holds in each case if and only if $X = Y$.

In what follows, for any continuous function $b : [t_0, \infty) \to (0, \infty)$, we let

$$\begin{aligned} g_1(t) &:= (1-\beta)\beta^{\frac{\beta}{1-\beta}} b^{\frac{\beta}{\beta-1}}(t) p_1^{\frac{1}{1-\beta}}(t), \\ g_2(t) &:= (\delta - 1)\, \delta^{\frac{\delta}{1-\delta}} p_2^{\frac{1}{1-\delta}}(t)\, b^{\frac{\delta}{\delta-1}}(t), \end{aligned}$$

$$P(t) = \frac{p(t)}{(p_2(h^*(t)))^{\frac{\alpha}{\delta}}}, \quad \text{and} \quad Q(t) = \frac{q(t)}{(p_2(h(t)))^{\frac{\alpha}{\delta}}}.$$

We now present our main oscillation result.

Theorem 2.3. *Let $\beta < 1$, $\delta > 1$, conditions* (i)–(iv) *and* (2) *hold, and assume there are continuous functions $b : [t_0, \infty) \to (0, \infty)$ and $\varphi : [t_0, \infty) \to R$ such that*

$$\lim_{t\to\infty} [g_1(t) + g_2(t)] = 0, \tag{5}$$

and

$$\varphi(t) \leq t, \quad \text{and} \quad \rho(t) := \varphi\big(h^*(t)\big) > t. \tag{6}$$

If the first order advanced equation

$$Z'(t) - \frac{1}{\alpha} P(t) A^{\frac{\alpha}{\delta}}\big(h^*(t), \rho(t)\big) Z^{1-\alpha+\frac{\alpha}{\delta}}\big(\rho(t)\big) = 0 \tag{7}$$

is oscillatory and for every $k \in (0,1)$ *and nondecreasing function* $\xi(t) \in [t_0, \infty)$ *with* $h(t) \le \xi(t) \le t$ *for* $t \ge t_0$*, both of the delay equations*

$$w'(t) + kA^{\alpha}\big(\tau(t), t_1\big) q(t) w\big(\tau(t)\big) = 0 \tag{8}$$

and

$$W'(t) - \frac{1}{\alpha} Q(t) A^{\frac{\alpha}{\delta}}\big(\xi(t), h(t)\big) W^{\frac{\alpha}{\delta}-\alpha+1}\big(\xi(t)\big) = 0 \tag{9}$$

are oscillatory for every $t_1 \ge t_0$*, then equation* (1) *is oscillatory.*

P r o o f. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0$, $x\big(\sigma(t)\big) > 0$ and $x\big(\tau(t)\big) > 0$, for $t \ge t_1$ for some $t_1 \ge t_0$. It follows from equation (1) that

$$\Big(a(t)\big(y'(t)\big)^{\alpha}\Big)' = -q(t)x^{\alpha}\big(\tau(t)\big) - p(t)x^{\alpha}\big(\omega(t)\big) \le 0. \tag{10}$$

Hence, $\big(a(t)(y'(t))\big)^{\alpha}$ is nonincreasing and eventually of one sign. That is, there exists $t_2 \ge t_1$ such that $y'(t) > 0$ or $y'(t) < 0$ for $t \ge t_2$.

We see that

$$\Big(a(t)\big(y'(t)\big)^{\alpha}\Big)' = \Big(\big(a^{\frac{1}{\alpha}}(t)y'(t)\big)^{\alpha}\Big)' = \alpha\big(a^{\frac{1}{\alpha}}(t)y'(t)\big)^{\alpha-1}\big(a^{\frac{1}{\alpha}}(t)y'(t)\big)', \tag{11}$$

and so from (1),

$$\big(a^{\frac{1}{\alpha}}(t)y'(t)\big)' + \frac{1}{\alpha}\big(a^{\frac{1}{\alpha}}(t)y'(t)\big)^{1-\alpha}\big[q(t)x^{\alpha}\big(\tau(t)\big) + p(t)x^{\alpha}\big(\omega(t)\big)\big] = 0. \tag{12}$$

Also, from (1)

$$\Big(a(t)\big(y'(t)\big)^{\alpha}\Big)' + q(t)x^{\alpha}\big(\tau(t)\big) = -p(t)x^{\alpha}\big(\omega(t)\big) \le 0, \tag{13}$$

which implies

$$\Big(a^{\frac{1}{\alpha}}(t)y'(t)\Big)' + \frac{1}{\alpha}\Big(a^{\frac{1}{\alpha}}(t)y'(t)\Big)^{1-\alpha} q(t)x^{\alpha}\big(\tau(t)\big) \le 0. \tag{14}$$

We distinguish the following four cases for $t \ge t_2$:

(I)	$y(t) > 0$	and	$y'(t) < 0$,
(II)	$y(t) > 0$	and	$y'(t) > 0$,
(III)	$y(t) < 0$	and	$y'(t) > 0$,
(IV)	$y(t) < 0$	and	$y'(t) < 0$.

First, we consider Case (I). Since $y'(t) < 0$ for $t \geq t_2$, by (14)

$$a^{\frac{1}{\alpha}}(t)y'(t) \leq -c < 0, \quad \text{so} \quad y'(t) \leq -ca^{-\frac{1}{\alpha}}(t)$$

for $t \geq t_2$ and some $c > 0$. Condition (2) implies $\lim_{t\to\infty} y(t) = -\infty$, contradicting the fact that $y(t) > 0$.

For Case (II), from the definition of $y(t)$, we have

$$y(t) = x(t) + \left[b(t)x\big(\sigma(t)\big) - p_2(t)x^{\delta}\big(\sigma(t)\big)\right] + \left[p_1(t)x^{\beta}\big(\sigma(t)\big) - b(t)x\big(\sigma(t)\big)\right],$$

or

$$x(t) = y(t) - \left[b(t)x\big(\sigma(t)\big) - p_2(t)x^{\delta}\big(\sigma(t)\big)\right] - \left[p_1(t)x^{\beta}\big(\sigma(t)\big) - b(t)x\big(\sigma(t)\big)\right]. \quad (15)$$

If we apply (3) with $\lambda = \delta > 1$, $X = p_2^{\frac{1}{\delta}}(t)x(t)$ and $Y = \left(\frac{1}{\delta}b(t)p_2^{-\frac{1}{\delta}}(t)\right)^{\frac{1}{\delta-1}}$, we obtain

$$\Big(b(t)x\big(\sigma(t)\big) - p_2(t)x^{\delta}\big(\sigma(t)\big)\Big) \leq (\delta-1)\delta^{\frac{\delta}{1-\delta}}p_2^{\frac{1}{1-\delta}}(t)b^{\frac{\delta}{\delta-1}}(t) := g_2(t),$$

and applying (4) with $\lambda = \beta < 1$, $X = p_1^{\frac{1}{\beta}}(t)x(t)$, and $Y = \left(\frac{1}{\beta}b(t)p_1^{-\frac{1}{\beta}}(t)\right)^{\frac{1}{\beta-1}}$ gives

$$\Big(p_1(t)x^{\beta}\big(\sigma(t)\big) - b(t)x\big(\sigma(t)\big)\Big) \leq (1-\beta)\beta^{\frac{\beta}{1-\beta}}b^{\frac{\beta}{\beta-1}}(t)p_1^{\frac{1}{1-\beta}}(t) := g_1(t).$$

Thus, from (15),

$$x(t) \geq \left[1 - \frac{g_1(t) + g_2(t)}{y(t)}\right] y(t) \quad (16)$$

for $t \geq t_2$. Since $y(t)$ is nondecreasing, there exists a constant $C > 0$ such that $y(t) \geq C$ for $t \geq t_2$, and so

$$x(t) \geq \left[1 - \frac{g_1(t) + g_2(t)}{C}\right] y(t), \quad t \geq t_2.$$

In view of (5), there exists a constant $c_1 \in (0,1)$ such that

$$x(t) \geq c_1 y(t) \quad (17)$$

for $t \geq t_3$ for some $t_3 \geq t_2$. Thus, from (17),

$$\Big(a^{\frac{1}{\alpha}}(t)y'(t)\Big)' + \frac{c_1^{\alpha}}{\alpha}\Big(a^{\frac{1}{\alpha}}(t)y'(t)\Big)^{1-\alpha} q(t)y^{\alpha}\big(\tau(t)\big) \leq 0.$$

Clearly, we see that

$$y(t) \geq \int_{t_3}^{t} a^{-\frac{1}{\alpha}}(s)\left(a^{\frac{1}{\alpha}}(s)y'(s)\right) \mathrm{d}s \geq A(t,t_3)\left(a^{\frac{1}{\alpha}}(t)y'(t)\right), \quad (18)$$

so

$$\Big(a^{\frac{1}{\alpha}}(t)y'(t)\Big)' + \frac{c_1^{\alpha}}{\alpha}\left(\frac{y(t)}{A(t,t_3)}\right)^{1-\alpha} q(t)y^{\alpha}\big(\tau(t)\big) \leq 0 \quad (19)$$

for $t \geq t_3$.

Since $\frac{y(t)}{A(t,t_3)}$ is nonincreasing,

$$\Big(a^{\frac{1}{\alpha}}(t)y'(t)\Big)' + \frac{c_1^\alpha}{\alpha}A^{\alpha-1}\big(\tau(t),t_3\big)q(t)y\big(\tau(t)\big) \le 0. \tag{20}$$

Letting $w(t) = a^{\frac{1}{\alpha}}(t)y'(t)$ and using (18) in (20), we obtain

$$w'(t) + \frac{c_1^\alpha}{\alpha}A^{\alpha}\big(\tau(t),t_3\big)q(t)w\big(\tau(t)\big) \le 0. \tag{21}$$

It follows from Lemma 2.1(1) that the corresponding differential equation also has a positive solution, which is a contradiction.

Next, we consider the cases (III) and (IV) in which $y(t) < 0$ for $t \ge t_2$. Let $z(t) = -y(t) = -x(t) - p_1(t)x^\beta\big(\sigma(t)\big) + p_2(t)x^\delta\big(\sigma(t)\big) \le p_2(t)x^\delta\big(\sigma(t)\big)$, so that

$$x\big(\sigma(t)\big) \ge \left(\frac{z(t)}{p_2(t)}\right)^{\frac{1}{\delta}} \quad \text{or} \quad x(t) \ge \left(\frac{z(\sigma^{-1}(t))}{p_2(\sigma^{-1}(t))}\right)^{\frac{1}{\delta}}.$$

Using this inequality in (1), we have

$$\begin{aligned}\Big(a(t)\big(z'(t)\big)^\alpha\Big)' &= q(t)x^\alpha\big(\tau(t)\big) + p(t)x^\alpha\big(\omega(t)\big)\\ &\ge \frac{q(t)}{\big(p_2(\sigma^{-1}(\tau(t)))\big)^{\frac{\alpha}{\delta}}}z^{\frac{\alpha}{\delta}}\Big(\sigma^{-1}\big(\tau(t)\big)\Big)\\ &\quad + \frac{p(t)}{\big(p_2(\sigma^{-1}(\omega(t)))\big)^{\frac{\alpha}{\delta}}}z^{\frac{\alpha}{\delta}}\Big(\sigma^{-1}\big(\omega(t)\big)\Big)\\ &:= Q(t)z^{\frac{\alpha}{\delta}}\big(h(t)\big) + P(t)z^{\frac{\alpha}{\delta}}\big(h^*(t)\big).\end{aligned} \tag{22}$$

Analogous to (11), (22) can be written as

$$\Big(a^{\frac{1}{\alpha}}(t)z'(t)\Big)' \ge \frac{1}{\alpha}\Big(a^{\frac{1}{\alpha}}(t)z'(t)\Big)^{1-\alpha}\left[Q(t)z^{\frac{\alpha}{\delta}}\big(h(t)\big) + P(t)z^{\frac{\alpha}{\delta}}\big(h^*(t)\big)\right], \tag{23}$$

from which we can easily see that

$$\Big(a^{\frac{1}{\alpha}}(t)z'(t)\Big)' \ge \frac{1}{\alpha}\Big(a^{\frac{1}{\alpha}}(t)z'(t)\Big)^{1-\alpha}Q(t)z^{\frac{\alpha}{\delta}}\big(h\,(t)\big) \tag{24}$$

and

$$\Big(a^{\frac{1}{\alpha}}(t)z'(t)\Big)' \ge \frac{1}{\alpha}\Big(a^{\frac{1}{\alpha}}(t)z'(t)\Big)^{1-\alpha}P(t)z^{\frac{\alpha}{\delta}}\big(h^*(t)\big). \tag{25}$$

In Case (III) where we have $y(t) < 0$ and $y'(t) > 0$, notice that $z'(t) = -y'(t) < 0$ for $t \ge t_2$. For $t_2 \le u \le v$,

$$z(u) - z(v) = -\int_u^v a^{-\frac{1}{\alpha}}(s)\Big(a(s)\big(z'(s)\big)^\alpha\Big)^{\frac{1}{\alpha}}\,\mathrm{d}s \ge A(v,u)\Big(-a^{\frac{1}{\alpha}}(v)\big(z'(v)\big)\Big).$$

Letting $u = h(t)$ and $v = \xi(t)$ in the above inequality gives

$$z\big(h(t)\big) \geq A\big(\xi(t), h(t)\big)\Big(-a^{\frac{1}{\alpha}}\big(\xi(t)\big)\Big)\Big(z'\big(\xi(t)\big)\Big), \tag{26}$$

and using this in (24) yields

$$-\Big(a^{\frac{1}{\alpha}}(t)z'(t)\Big)' + \frac{1}{\alpha}\Big(a^{\frac{1}{\alpha}}(t)z'(t)\Big)^{1-\alpha}Q(t)\Big(A\big(\xi(t), h(t)\big)\Big(-a^{\frac{1}{\alpha}}\big(\xi(t)\big)\Big)\Big(z'\big(\xi(t)\big)\Big)\Big)^{\frac{\alpha}{\delta}} \leq 0,$$

which has the form

$$W'(t) - \frac{1}{\alpha}Q(t)A^{\frac{\alpha}{\delta}}\big(\xi(t), h(t)\big)W^{\frac{\alpha}{\delta}-\alpha+1}\big(\xi(t)\big) \leq 0,$$

where $W(t) = -a^{\frac{1}{\alpha}}(t)z'(t)$. The remainder of the proof in this case is similar to the proof of Case (I) and hence is omitted.

Finally, we consider Case (IV), where $z(t) > 0$ and $z'(t) > 0$. Now,

$$\begin{aligned} z(t) \geq z(t) - z\big(\varphi(t)\big) &= \int_{\varphi(t)}^{t} a^{-\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)z'(s)\,\mathrm{d}s \\ &\geq A(t, \varphi(t))a^{\frac{1}{\alpha}}\big(\varphi(t)\big)z'\big(\varphi(t)\big) \end{aligned}$$

or

$$z(h^*(t)) \geq A\big(h^*(t), \rho(t)\big)a^{\frac{1}{\alpha}}\big(\rho(t)\big)z'\big(\rho(t)\big).$$

Using this inequality in (25), we obtain

$$\big(a^{\frac{1}{\alpha}}(t)z'(t)\big)' \geq \frac{1}{\alpha}\Big(a^{\frac{1}{\alpha}}(t)z'(t)\Big)^{1-\alpha}P(t)\Big(A\big(h^*(t), \rho(t)\big)a^{\frac{1}{\alpha}}\big(\rho(t)\big)z'\big(\rho(t)\big)\Big)^{\frac{\alpha}{\delta}},$$

or

$$\big(a^{\frac{1}{\alpha}}(t)z'(t)\big)' \geq \frac{1}{\alpha}P(t)A^{\frac{\alpha}{\delta}}\big(h^*(t), \rho(t)\big)\Big(a^{\frac{1}{\alpha}}\big(\rho(t)\big)z'\big(\rho(t)\big)\Big)^{1-\alpha+\frac{\alpha}{\delta}}.$$

Setting $Z(t) = a^{\frac{1}{\alpha}}(t)z'(t)$, this becomes

$$Z'(t) \geq \frac{1}{\alpha}P(t)A^{\frac{\alpha}{\delta}}\big(h^*(t), \rho(t)\big)Z^{1-\alpha+\frac{\alpha}{\delta}}\big(\rho(t)\big).$$

By Lemma 2.1(2), the corresponding differential equation also has a positive solution, which again is a contradiction. This completes the proof of the theorem. □

Now we consider some special cases of equation (1). First, we consider (1) with $p_2(t) \equiv 0 \equiv p(t)$, namely,

$$\Big(a(t)\Big(x(t) + p_1(t)x^{\beta}\big(\sigma(t)\big)'\Big)^{\alpha}\Big)' + q(t)x^{\alpha}\big(\tau(t)\big) = 0. \tag{27}$$

From the proof of Theorem 2.3, we conclude that equation (27) is oscillatory provided one of the following conditions holds:

(A) The second order equation $\big(a^{\frac{1}{\alpha}}(t)y'(t)\big)' + dA^{\alpha-1}\big(\tau(t),t_1\big)q(t)y\big(\tau(t)\big) = 0$ is oscillatory for any positive constant d.

(B) Equation (8) is oscillatory.

(C) $\liminf_{t\to\infty}\int_{\tau(t)}^{t} q(s)A^{\alpha}(\tau(s),t_1)\,\mathrm{d}s = \infty$ if $1 \le \alpha$.

For the special case

$$\left(\left(a(t)\Big(x(t) - p_2(t)x^{\delta}\big(\sigma(t)\big)\Big)'\right)^{\alpha}\right)' + q(t)x^{\alpha}\big(\tau(t)\big) + p(t)x^{\alpha}\big(\omega(t)\big) = 0, \quad (28)$$

which is equation (1) with $p_1(t) \equiv 0$, Theorem 2.3 implies equation (28) is oscillatory if one of the following conditions is satisfied:

(C_1) The second order equation $\big(a^{\frac{1}{\alpha}}(t)y'(t)\big)' + \frac{1}{\alpha}A^{\alpha-1}\big(\tau(t),t_1\big)q(t)y\big(\tau(t)\big) = 0$ and the first order equations (7) and (9) are oscillatory.

(C_2) The first order equation $w'(t) + A^{\alpha}\big(\tau(t),t_1\big)q(t)w\big(\tau(t)\big) = 0$ is oscillatory for $t \ge t_1$ for every $t_1 \ge t_0$, and the first order equations (7) and (9) are oscillatory.

For the special case of (1) with $\alpha = \delta$

$$\left(\left(a(t)\Big(x(t) + p_1(t)x^{\beta}\big(\sigma(t)\big) - p_2(t)x^{\alpha}\big(\sigma(t)\big)\Big)'\right)^{\alpha}\right)' + q(t)x^{\alpha}\big(\tau(t)\big) + p(t)x^{\alpha}\big(\omega(t)\big) = 0, \quad (29)$$

we have the following interesting result.

Corollary 2.4. *Let the hypotheses of Theorem 2.3 hold with equations* (7) *and* (9) *be replaced by*

$$Z'(t) - \frac{1}{\alpha}P(t)A\big(h^*(t),\rho(t)\big)Z^{2-\alpha}\big(\rho(t)\big) = 0 \quad (30)$$

and

$$W'(t) + \frac{1}{\alpha}Q(t)A\big(\xi(t),h(t)\big)W^{2-\alpha}\big(\xi(t)\big) = 0. \quad (31)$$

Then equation (29) *is oscillatory.*

The following corollary employs integral conditions rather than requiring the oscillatory behavior of some first order equations (see, for example, Grace *et al.* [6, Theorem A and Theorem 3.2]).

Corollary 2.5. *Let the covering assumptions of Theorem 2.3 hold. If*

$$\liminf_{t\to\infty}\int_t^{\rho(t)} P(s)A\big(h^*(s),\rho(s)\big)\,\mathrm{d}s \begin{cases} =\infty, & \text{if} \quad 1<\alpha<2,\\ >\frac{1}{e}, & \text{if} \qquad \alpha=1, \end{cases} \tag{32}$$

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} q(s)A^{\alpha}\big(\tau(s),t_1\big)\,\mathrm{d}s=\infty \quad \text{if } 1\le\alpha, \tag{33}$$

and

$$\liminf_{t\to\infty}\int_{\xi(t)}^{t} Q(s)A\big(\xi(s),h(s)\big)\,\mathrm{d}s \begin{cases} =\infty, & \text{if} \quad 1<\alpha<2,\\ >\frac{1}{e}, & \text{if} \qquad \alpha=1, \end{cases} \tag{34}$$

then equation (29) *is oscillatory.*

It is possible to formulate other results similar to Corollary 2.5.

3. Examples and Discussion

The following examples illustrate our main result.

Example. Consider the second order neutral differential equation

$$\left[\frac{1}{t^5}\left(\left(x(t)+\frac{1}{t}x^{\frac{3}{4}}\left(\frac{t}{2}\right)-tx^{\frac{5}{4}}\left(\frac{t}{2}\right)\right)'\right)^{\frac{5}{4}}\right]'+q(t)x^{\frac{5}{4}}\left(\frac{t}{3}\right)+p(t)x^{\frac{5}{4}}(2t)=0, \tag{35}$$

where $q(t)$ and $p(t)$ are positive continuous functions to be chosen below. Here,

$$\alpha=\tfrac{5}{4}=\delta>1,\quad \beta=\tfrac{3}{4}<1,\quad a(t)=\tfrac{1}{t^5},\qquad p_1(t)=\tfrac{1}{t},$$
$$p_2(t)=t,\qquad \sigma(t)=\tfrac{t}{2},\quad \tau(t)=\tfrac{t}{3},\quad \text{and}\quad \omega(t)=2t.$$

Also, $h(t)=\sigma^{-1}\big(\tau(t)\big)=\frac{2t}{3}$ and $h^*(t)=\sigma^{-1}\big(\omega(t)\big)=4t$, so conditions (i)–(iv) and (2) hold.

We take $b(t)\equiv 1$ so that

$$\lim_{t\to\infty}\left[g_1(t)+g_2(t)\right]=\lim_{t\to\infty}\left[\frac{1}{4}\left(\frac{3}{4}\right)^3\frac{1}{t}+\frac{1}{4}\left(\frac{5}{4}\right)^{-5}\frac{1}{t^4}\right]=0,$$

which means (5) holds. Taking $\varphi(t)=\frac{t}{3}$, we see that (6) holds since $\rho(t)=\varphi(h^*(t))=\frac{4t}{3}>t$. By choosing the functions p and q appropriately all the conditions of Theorem 2.3 will be satisfied, and hence equation (35) will be oscillatory. This conclusion is not attainable from existing results in the literature.

EXAMPLE. Consider the equation

$$\left[\frac{1}{t^2}\left(\left(x(t)+\frac{1}{t}x^{\frac{1}{2}}\left(\frac{t}{2}\right)-(t+1)x^{\frac{3}{2}}\left(\frac{t}{2}\right)\right)'\right)^2\right]'+\ q(t)x^2\left(\frac{t}{4}\right)+p(t)x^2(3t)=0,\ (36)$$

where $q(t)$ and $p(t)$ are positive continuous functions to be chosen. Since

$$\begin{aligned} &a(t)=\tfrac{1}{t^2},\quad p_1(t)=\tfrac{1}{t},\quad p_2(t)=(t+1),\quad \sigma(t)=\tfrac{t}{2},\\ &\tau(t)=\tfrac{t}{4},\quad \omega(t)=3t,\qquad \beta=\tfrac{1}{2},\qquad \delta=\tfrac{3}{2},\quad \textit{and}\quad \alpha=2, \end{aligned}$$

we have $h(t)=\frac{t}{2}$ and $h^*(t)=6t$, so it is easy to see that (i)–(iv) and (2) hold. Choosing $b(t)=t^{\frac{1}{2}}$ and $\varphi(t)=\frac{t}{2}$, we have $g_1(t)=\frac{1}{4}(t^{-\frac{3}{2}})$, $g_2(t)=t^{\frac{3}{2}}(t+1)^{-2}$, $\rho(t)=3t$, $P(t)=\frac{p(t)}{(1+6t)^{\frac{4}{3}}}$, and $Q(t)=\frac{3q(t)}{(3+2t)^{\frac{4}{3}}}$, so (5) and (6) hold. We also see that $A\big(h^*(t),\rho(t)\big)=3t$ and $A^\alpha\big(\tau(t),t_1\big)=\big(\frac{t}{4}-t_1\big)^2$. Again, we see that choosing the functions p and q appropriately, all the conditions of Theorem 2.3 will be satisfied, and so equation (36) will be oscillatory. It is not possible to deduce this from previously known theorems.

EXAMPLE.

$$\left[\left(\left(x(t)+\frac{1}{t}x^{\frac{1}{2}}\left(\frac{t}{2}\right)-t^2x^2\left(\frac{t}{2}\right)\right)'\right)^2\right]'+\ q(t)x^2\left(\frac{t}{3}\right)+p(t)x^2(t+1)=0,\ (37)$$

which means

$$\begin{aligned} &a(t)\equiv 1,\quad p_1(t)=\tfrac{1}{t},\quad p_2(t)=t^2,\qquad \alpha=2,\\ &\beta=\tfrac{1}{2},\qquad \delta=2,\qquad \sigma(t)=\tfrac{t}{2},\quad \tau(t)=\tfrac{t}{3},\quad \text{and}\quad \omega(t)=t+1. \end{aligned}$$

It is easy to see that $h(t)=\frac{2t}{3}$ and $h^*(t)=2(t+1)$, so (i)–(iv) and (2) hold. With $b\equiv 1$, we have $g_1(t)=\frac{1}{4t^2}$, $g_2(t)=\frac{1}{4t}$, and taking $\varphi(t)=\frac{t}{2}$, we see that $\rho(t)=t+1$ and (5) and (6) hold. With $\xi(t)=t$, we have $A\big(h^*(t),\rho(t)\big)=t+1$ and $A\big(\xi(t),h(t)\big)=\frac{t}{3}$. Now with appropriate choices of $p(t)$ and $q(t)$, equations (30) and (31) will be oscillatory, so by Corollary 2.4, equation (37) is oscillatory. Once again, we can not determine this from other known results.

Remark 1. It would be of interest to obtain similar results for the second order equation

$$\Big(a(t)\big(y'(t)\big)^\alpha\Big)'=\ q(t)x^\alpha\big(\tau(t)\big)+p(t)x^\alpha\big(\omega(t)\big),\tag{38}$$

or for high order equations of the form

$$\Big(a(t)\big(y^{(n-1)}(t)\big)^\alpha\Big)'+\ q(t)x^\gamma\big(\tau(t)\big)+p(t)x^\mu\big(\omega(t)\big)=0\tag{39}$$

and

$$\Big(a(t)\big(y^{(n-1)}(t)\big)^\alpha\Big)'=\ q(t)x^\gamma\big(\tau(t)\big)+p(t)x^\mu\big(\omega(t)\big)\tag{40}$$

Also, results for equations (1), (38), (39), or (40) with $\beta \geq 1$ and/or $\delta \leq 1$ would be of interest.

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