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INTEGRABILITY OF SEQUENCES

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Abstract. This paper deals with density on the set of natural numbers and its connections to the distribution of sequences. Under the assumption of independence, some formulas are derived.

1. Introduction

The aim of this paper is to study a certain type of finitely additive measures of sets of natural numbers and their connections with the distribution of sequences on real line. This concept is firstly studied in [16] and [14]. This research was later developed by other authors. For the survey, we refer to monographs [8], [4], [13]. In papers [11], [12], [10], distribution of sequences with respect to the asymptotic density was studied. We apply some of the ideas from these papers for calculation of the distribution function with respect to the arbitrary "density".

We start with following notions.

Definition 1. Let $\mathcal{Y}$ be a system of subsets of $\mathbb{N}$ Then, $\mathcal{Y}$ will be called q-algebra if

i) $\emptyset, \mathbb{N} \in \mathcal{Y}$,

i) $A \in \mathcal{Y} \Rightarrow \mathbb{N} \setminus A \in \mathcal{Y}$,

iii) $A, B \in \mathcal{Y} \wedge A \cap B = \emptyset \Rightarrow A \cup B \in \mathcal{Y}$,

for arbitrary $A, B \subset \mathbb{N}$.

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Definition 2. A finitely additive probability measure ν defined on q-algebra $\mathcal{Y}$ will be called density if for each $A \subset \mathbb{N}$ we have

iv) $A \in \mathcal{Y} \Leftrightarrow \forall \varepsilon > 0 \exists S_1,\ \ S_2 \in \mathcal{Y};\ \ S_1 \subset A \subset S_2 \wedge \nu(S_2) - \nu(S_1) < \varepsilon.$

A sequence of real numbers $\{v(n)\}$ is called *ν-measurable* if for each real number x the set $v^{-1}((-\infty, x))$ belongs to $\mathcal{Y}$. The function

$$F(X) = \nu\big(v^{-1}((-\infty, x))\big)$$

is called *ν-distribution function* of $\{v(n)\}$.

Let $S \subset \mathbb{N}$. If the limit

$$\lim_{N\to\infty} \frac{|S \cap [1, N]|}{N} := \mathrm{d}(S)$$

does exist, we say that the set S has *asymptotic density*, and the value $\mathrm{d}(S)$ will be called *asymptotic density of S*. Let $\mathcal{D}$ denote the system of all subsets of $\mathbb{N}$ having asymptotic density. This set system is a q-algebra and d is a density on $\mathcal{D}$. Let us remark that properties i), ii), iii) are copied from the properties of the system $\mathcal{D}$. Condition iii) is weak.

Asymptotic density can be generalized in the following way. Let $\mathfrak{m} = \{\mathfrak{m}_k, k = 1, 2, 3, \dots\}$ be a sequence of probability measures on $\mathcal{P}(\mathbb{N})$. We say that a set S has *matrix density* with respect to this sequence if the proper limit $\mathfrak{m}(S) = \lim_{k\to\infty} \mathfrak{m}_k(S)$ exists (see [2]). If $\mathcal{D}_{\mathfrak{m}}$ is the system of all sets having the given matrix density, then $\mathcal{D}_{\mathfrak{m}}$ is a q-algebra and $\mathfrak{m}$ is a density on $\mathcal{D}_{\mathfrak{m}}$.

If for $S \subset \mathbb{N}$ the value $\frac{|S \cap [m, m+n]|}{n}$ converges uniformly for $n \to \infty$, $m \in \mathbb{N}$, then this value is called the *uniform density* of S denoted by $u(S)$, and we say that S has uniform density. The system of sets $\mathcal{U}$, all sets having uniform density, forms a q-algebra.

Unfortunately, for asymptotic density and many other concepts of density, the condition $A \cap B = \emptyset$ cannot be omitted. Because of this, we shall study some subsystems of q-algebra.

Definition 3. A q-algebra $\mathcal{A}$ will be called *algebra* if

v) $\forall A, B \in \mathcal{A};\ \ A \cup B \in \mathcal{A}.$

If an algebra is a subsystem of a given q-algebra, we say that it is *subalgebra* of this q-algebra. The following well-known result will play an important role in the construction of algebra.

1. *If $\gamma^* : \mathcal{P}(\mathbb{N}) \to [0, 1]$ is such a function that*

vi) $\gamma^*(\emptyset) = 0, \gamma^*(\mathbb{N}) = 1$

vii) $A \subset B \Rightarrow \gamma^*(A) \le \gamma^*(B)$

viii) $\gamma^*(A \cap B) + \gamma^*(A \cup B) \le \gamma^*(A) + \gamma^*(B)$

for every $A, B \subset \mathbb{N}$*, then the system*

$$\mathcal{A} = \{A \subset \mathbb{N}; \gamma^*(A) + \gamma^*(\mathbb{N} \setminus A) = 1\}$$

is an algebra, and $\gamma = \gamma^*|_{\mathcal{A}}$ *is a finitely additive measure on this algebra.*

Example 1. A well-known subalgebra of $\mathcal{D}$ is the algebra of Buck measurable sets, see [1]. Denote

$$r + (m) = \{a \in \mathbb{N}; a \equiv r \pmod{m}\}$$

where $a \in \mathbb{Z}, m \in \mathbb{N}$, $0 + (m) := (m)$. Then, these sets belong to $\mathcal{D}$, and $d(r + (m)) = \frac{1}{m}$. Let $\mathcal{D}_0$ be the system of all sets in the form $\cup_{i=1}^{k} r_i + (m_i)$. For $S \subset \mathbb{N}$, define the function

$$\mu^*(S) = \inf\{d(A); S \subset A \wedge A \in \mathcal{D}_0\}.$$

A set $S \subset \mathbb{N}$ will be called μ^*-measurable if $\mu^*(S) + \mu^*(\mathbb{N} \setminus S) = 1$. If $\mathcal{D}_\mu$ denotes the system of all $\mu^m u*$-measurable sets, then this system is an algebra, and the restriction $\mu = \mu^*|_{\mathcal{D}_\mu}$ is a finitely additive probability measure on $\mathcal{D}_\mu$ which coincides with the asymptotic density. The elements of $\mathcal{D}_\mu$ are also called *Buck measurable sets.*

2. Integral of sequence

We recall the integral with respect to finitely additive probability measure (see [17]). For an overview of these topics, see [15].

Let $\mathcal{A} \subset \mathcal{P}(\mathbb{N})$ be an algebra of sets, and let $\lambda : \mathcal{A} \to [0, 1]$ be a density. Denote $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\{\mathcal{X}_A; A \in \mathcal{A}\})$-the linear hall of the indicator functions of the sets from $\mathcal{A}$.

If $A_1, \ldots, A_n$, $B_1, \ldots, B_m$ are two finite sequences of mutually disjoint sets from $\mathcal{A}$, then it can be proved in a standard way that from the equality

$$\sum_{i=1}^{n} a_i \mathcal{X}_{A_i} = \sum_{j=1}^{m} b_j \mathcal{X}_{B_j}$$

it follows that

$$\sum_{i=1}^{n} a_i \lambda(A_i) = \sum_{j=1}^{m} b_j \lambda(B_j).$$

This allows us to define the integral of a given $s \in \mathcal{L}(A)$ in a standard way. If $s = \sum_{i=1}^{n} a_i \mathcal{X}_{A_i}$, then

$$\int s \, \mathrm{d}\lambda = \sum_{i=1}^{n} a_i \lambda(A_i).$$

Let $\mathbf{I}(\mathcal{A})$ be a set of all bounded sequences from $\mathbf{cl}(\mathcal{L}(\mathcal{A}))$-the closure of $\mathcal{L}(\mathcal{A})$ with respect to the supremum metric. Each sequence $\{v(n)\} \in \mathbf{I}(\mathcal{A})$ can be assigned a value

$$\int v \,\mathrm{d}\lambda = \lim_{N\to\infty} \int s_N \,\mathrm{d}\lambda. \tag{1}$$

where $s_N \in \mathcal{L}(\mathcal{A}), N = 1, 2, 3, \ldots$, and $s_N \to v$ uniformly for $N \to \infty$.

If $\{v(n)\} \in \mathbf{I}(\mathcal{A})$ is a sequence of elements of the interval $[a, b]$ and f is a real valued continuous function defined on this interval, then the sequence $\{f(v(n))\}$ belongs to $\mathbf{I}(\mathcal{A})$ as well. Therefore, each such a function can be assigned a value $\Phi(f) = \int f(v) \,\mathrm{d}\lambda$. This is a positive linear functional on the space of real continuous functions defined on the interval $[a, b]$. Riesz representation theorem yields

2. *Let $\{v(n)\} \in \mathbf{I}(\mathcal{A})$ be a sequence of elements of an interval $[a, b]$. There exists a non-decreasing function F, $F(a) = 0, F(b) = 1$ such that*

$$\int f(v) \,\mathrm{d}\lambda = \int_a^b f(x) \,\mathrm{d}F(x)$$

for each continuous real function f defined on the interval $[a, b]$.

If $\{v(n)\}$ is a λ-measurable sequence of elements of a certain interval $[a, b]$, $a < b$. Set

$$I_N^{(i)} = \left[a + i\frac{b-a}{N}, a + (i+1)\frac{b-a}{N}\right) \quad \text{for } i = 0, \ldots, N-2$$

and

$$I_N^{(N-1)} = \left[a + (N-1)\frac{b-a}{N}, b\right].$$

We define the sequence $\{s_N(n)\}$ where

$$s_N = \sum_{i=0}^{N-1} \left(a + i\frac{b-a}{N}\right) \mathcal{X}_{v^{-1}\left(I_N^{(i)}\right)}.$$

Clearly,

$$|v(n) - s_N(n)| \le \frac{b-a}{N}, n \in \mathbb{N}.$$

Thus the sequence of sequences $s_N, N=1, 2, 3, \ldots$, converges uniformly to $\{v(n)\}$. And so,

3. *Each λ-measurable sequence belongs to $\mathbf{I}(\mathcal{A})$.*

If s and s' are sequences from $\mathcal{L}(\mathcal{A})$ such that $s \le s'$, then

$$\int s \,\mathrm{d}\lambda \le \int s' \,\mathrm{d}\lambda$$

and so, from (1), we get

4. *If* $v_1, v_2 \in \mathcal{L}(\mathcal{A})$ *such that* $v_1 \leq v_2$, *then*

$$\int v_1 \, \mathrm{d}\lambda \leq \int v_2 \, \mathrm{d}\lambda.$$

From this, the method of Weyl's type criterion can be proved in a standard way (see [8, page 54]:

Theorem 1. *Let* $\{v(n)\}$ *be a* λ*-measurable sequence. A non-decreasing continuous function* F, $F(a) = 0, F(b) = 1$ *is* λ*-distribution function of this sequence if and only if*

$$\int f(v) \, \mathrm{d}\lambda = \int_a^b f(x) \, \mathrm{d}F(x)$$

for each monotone continuous function f *defined on the interval* $[a, b]$.

We say that bounded λ-measurable sequences $v_1, v_2, \dots, v_k$ are λ-*independent* if for all real numbers $x_1, \dots, x_j$ the set

$$v_{k_1}^{-1}((-\infty, x_1)) \cap \cdots \cap v_{k_j}^{-1}((-\infty, x_j))$$

belongs to $\mathcal{A}$ and

$$\begin{aligned} &\lambda\big(v_{k_1}^{-1}((-\infty, x_1)) \cap \cdots \cap v_{l_j}^{-1}((-\infty, x_j))\big) \\ &= \prod_{i=1}^{j} \lambda\big(v_{k_i}^{-1}((-\infty, x_i))\big). \end{aligned}$$

Theorem 2. *If* v_1, v_2 *are* λ*-measurable sequences of elements of a given interval* $[a, b]$ *with continuous* λ*-distribution functions, then the sequences are* λ*-independent if and only if*

$$\int f_1(v_1) f_2(v_2) \, \mathrm{d}\lambda = \int f_1(v_1) \, \mathrm{d}\nu \int f_2(v_2) \, \mathrm{d}\lambda$$

for all continuous functions f_1, f_2 *defined on the interval* $[a, b]$.

Since every continuous function can be uniformly approximated by polynomial function on a closed interval, from Theorem 7, we get

Proposition 1. *Under the assumptions of Theorem 7, we have that the mentioned sequences are* λ*-independent if and only if*

$$\int v_1^{m_1} v_2^{m_2} \, \mathrm{d}\lambda = \int v_1^{m_1} \, \mathrm{d}\lambda \int v_2^{m_2} \, \mathrm{d}\lambda.$$

3. Subalgebra of $\mathcal{Y}$

We repeat the constructions from [7] and [12].

Let $\{v_1(n)\}, \{v_2(n)\}$ be two ν-measurable sequences of elements of a given interval $[a,b]$, where ν is a density defined on q-algebra $\mathcal{Y}$. Suppose that these sequences are ν-independent. Consider the intervals $I_N^{(i)}, i = 0, \dots, N-1$ mentioned above. Let $E_N^{(k)}$ denote all the sets

$$v_1^{-1}\Big(I_N^{(i)}\Big) \cap v_2^{-1}\Big(I_N^{(j)}\Big), \quad i,j = 0, \dots N-1.$$

Then, we have a system of decompositions of $\mathbb{N}$ in the form

$$\mathcal{E}_N = \Big\{E_N^{(k)}, k = 1, \dots, N^2\Big\}, \quad N = 1, 2, 3, \dots$$

Let $\mathcal{A}_0$ denote the algebra of all subsets of $\mathbb{N}$ in the form $\cup_{i=1}^r E_{N_i}^{(k_i)}$. Let us denote

$$\lambda^*(S) = \inf\{\nu(E); E \in \mathcal{A}_0\}.$$

We have $\lambda^*(\emptyset) = 0$, $\lambda(\mathbb{N}) = 1$ and

$$\lambda^*(S_1 \cup S_2) + \lambda^*(S_1 \cap S_2) \le \lambda^*(S_1) + \lambda^*(S_2).$$

This yields:

5. *The system of sets*

$$\mathcal{A} = \{S \subset \mathbb{N}; \lambda^*(S) + \lambda^*(\mathbb{N} \setminus S) = 1\}$$

is an algebra of sets, and $\lambda = \lambda^*|_{\mathcal{A}}$ is a density on $\mathcal{A}$. Moreover, $\mathcal{A} \subset \mathcal{Y}$ and $\lambda(S) = \nu(S)$ for $S \in \mathcal{A}$.

Proposition 2. *If the sequence $\{v_1(n)\}$ has a continuous ν- distribution function, then it is λ-measurable and*

$$\lambda\big(v_1^{-1}([a,x))\big) = F(x)$$

where F is a ν-distribution function of $\{v_1(n)\}$.

Proof. For each $\frac{k(b-a)}{n} \in [a,b]$, the set

$$v_1^{-1}\left(\left[a, a + \tfrac{k(b-a)}{n}\right)\right)$$

is a union of the sets of the form $E_j^{(n)}$ for a suitable finite set of indices j. Thus, it belongs to $\mathcal{A}_0$. If k_1, k_2 are such naturals numbers that

$$\frac{a + k_1(b-a)}{n} \le x < \frac{a + k_2(b-a)}{n}$$

for a given $x \in [a,b]$, then

$$v_1^{-1}\left(\left[a, a + \frac{k_1(b-a)}{n}\right)\right) \subset v_1^{-1}([a,x)) \subset v_1^{-1}\left(\left[a, a + \frac{k_2(b-a)}{n}\right)\right).$$

Moreover,

$$\lambda\left(v_1^{-1}\left(\left[a, a+\frac{k_2(b-a)}{n}\right)\right)\right) - \lambda\left(v_1^{-1}\left(\left[a, a+\frac{k_1(b-a)}{n}\right)\right)\right) = F\left(a+\frac{k_2(b-a)}{n}\right) - F\left(a+\frac{k_1(b-a)}{n}\right). \quad (2)$$

Let $\varepsilon > 0$. Since F is uniformly continuous on $[a, b]$, such n, k_1, k_2 can be chosen that the difference in (2) is smaller than the given ε. □

4. Compact space

The metric given by this system of decomposition can be defined as usual: Set $\psi_N(a, b) = 0$ if a, b belong to the same set $E_N^{(k)}$ for suitable $k \in \{1, \ldots, N^2\}$, and $\psi_N(a, b) = 1$ for the other case. Then, the function

$$\mathfrak{d}(a, b) = \sum_{N=1}^{\infty} \frac{\psi_N(a, b)}{2^N}, \quad a, b \in \mathbb{N},$$

is metric on $\mathbb{N}$. Denote by $(\mathbb{M}, \mathfrak{d})$ the completion of the metric space $(\mathbb{N}, \mathfrak{d})$. Then the sets

$$H_N^{(k)} = \mathrm{cl}\left(E_N^{(k)}\right), \quad N = 1, 2, 3, \ldots, k = 1, \ldots, N^2$$

form an open base of the metric space $(\mathbb{M}, \mathfrak{d})$. Let $\mathcal{B}_0$ denote the algebra of all sets in the form $\cup_{i=1}^{r} H_{N_i}^{(k_i)}$. The function $\Delta : \mathcal{B}_0 \to [0, 1]$, where

$$\Delta\left(\bigcup_{i=1}^{r} H_{N_i}^{(k_i)}\right) = \nu\left(\bigcup_{i=1}^{r} E_{N_i}^{(k_i)}\right)$$

is a probability measure on $\mathcal{B}_0$. This yields that the set function P^* given by

$$P^*(A) = \inf\left\{\sum_{j=1}^{\infty} \Delta(H_j),\ A \subset \bigcup_{j=1}^{\infty} H_j,\ H_j \in \mathcal{B}_0\right\}$$

is an outer measure defined on the system of all subset of $\mathbb{M}$. Thus, the system of sets

$$\mathcal{B} = \{S \subset \mathbb{M}; P^*(S) + P^*(\mathbb{M} \setminus S) = 1\}$$

is a σ-algebra containing all Borel subsets of $\mathbb{M}$, and the restriction $P = P^*|_{\mathcal{B}}$ is a borel probability measure.

Proposition 3. *For each $S \subset \mathbb{N}$ we have*

$$\lambda^*(S) = P\big(\mathrm{cl}(S)\big).$$

If $\{v(n)\}$ is a sequence uniformly continuous with respect to metric $\mathfrak{d}$, then it can be extended to a uniformly continuous function $\tilde{v} : \mathbb{M} \to \mathbb{R}$ as

$$\tilde{v}(\alpha) = \lim_{k\to\infty} v(n_k),$$

where $n_k \to \alpha$ for $k \to \infty$, $n_k \in \mathbb{N}$. From the continuity of $\tilde{v}$, we get that this function is measurable. Thus it can be considered as random variable. Let $F(x) = P(\tilde{v} < x)$ be its distribution function. From Proposition 3, we get (see [10], Theorem 3).

6. *For each point of continuity of F, the set $v^{-1}((a,x])$ belongs to $\mathcal{Y}$ and*

$$\nu\big(v^{-1}((a,x])\big) = F(x).$$

From **18** in [10], we get

7. *If the sequences $\{v_1(n)\}, \{v_2(n)\}$ have continuous ν-distribution functions, then the random variables $\tilde{v}_1, \tilde{v}_2$ are independent.*

Let us consider a rectangle $R = [x_1, x_2] \times [y_1, y_2]$. Under the conditions of **7**, we get

$$\nu\big(\{n \in \mathbb{N}; (v_1, v_2) \in R\}\big) = \big(F_1(x_2) - F_1(x_1)\big)\big(F_2(y_2) - F_2(y_1)\big).$$

This leads to

Theorem 3. *Let $\{v_1(n)\}, \{v_2(n)\}$ be bounded ν-measurable and ν-independent sequences with continuous ν-distribution functions F_1, F_2. If $S \subset (-\infty,\infty) \times (-\infty,\infty)$ is such set that its indicator function $\mathcal{X}_S$ is Riemann Steltjes integrable with respect to F_1, F_2, then the set $\big\{n \in \mathbb{N}; \big(v_1(n), v_2(n)\big) \in S\big\}$ is ν-measurable and*

$$\nu\Big(\big\{n \in \mathbb{N}; \big(v_1(n), v_2(n)\big) \in S\big\}\Big) = \iint\limits_S \mathrm{d}F_1 \, \mathrm{d}F_2.$$

A sequence $\{v(n)\}$ of elements of the interval $[0,1]$ is called *uniformly distributed* mod 1 if for each $x \in [0,1]$ the set $v^{-1}([0,x))$ belongs to $\mathcal{Y}$ and $\nu\big(v^{-1}([0,x))\big) = x$.

Example 2. If the sequences $\{v_1(n)\}, \{v_2(n)\}$ are ν-uniformly distributed mod 1, then the set

$$K = \big\{n \in \mathbb{N}; v_1^2(n) + v_2^2(n) \le 1\big\}$$

belongs to $\mathcal{Y}$ and $\nu(K) = \frac{\pi}{4}$.

Similarly, the set

$$D = \{n \in \mathbb{N}; v_1(n) \le v_2(n)\}$$

belongs to $\mathcal{Y}$ and $\nu(D) = \frac{1}{2}$.

5. Product of uniformly distributed sequences

In [11], the distribution of sum of independent sequences is studied. In this section, we observe the product of such sequences.

8. *Let $\{v_1(n)\}, \{v_2(n)\}$ be two ν-independent sequences of elements of the interval $[0,1]$ uniformly continuous with respect to the metric $\mathfrak{d}$ having continuous ν-distribution functions H_1, H_2. Then, the sequence $\{v_1(n)v_2(n)\}$ has a continuous ν-distribution function F given by*

$$F(x) = H_1(x) + \int_x^1 H_2\left(\frac{x}{t}\right) \mathrm{d}H_1(t), \quad \text{for } x \in (0,1]. \tag{3}$$

P r o o f. It suffices to prove that F given by (3) is the distribution function of the random variable $\tilde{v_1}\tilde{v_2}$. Let $x \in (0,1]$. Then, $\tilde{v_1}\tilde{v_2} \le x$ if and only if the random vector $(\tilde{v_1}, \tilde{v_2})$ belongs to the set

$$S = \{(s,t); st \le x \wedge 0 < s \le 1, 0 < t \le 1\}.$$

This yields

$$P(\tilde{v_1}\tilde{v_2} \le x) = \iint_S \mathrm{d}H_2(s)\,\mathrm{d}H_1(t).$$

We have

$$\begin{aligned}\iint_S \mathrm{d}H_2(s)\,\mathrm{d}H_1(t) &= \int_0^x\int_0^1 \mathrm{d}H_2(s)\,\mathrm{d}H_1(t)\\ &\quad+ \int_x^1\int_0^{x/t} \mathrm{d}H_2(s)\,\mathrm{d}H_1(t)\\ &= H_1(x) + \int_x^1 H_2\left(\frac{x}{t}\right)\mathrm{d}H_1(t).\end{aligned}$$

□

9. *If we suppose in* **8** *that H_1 has a continuous derivation, then*

$$F(x) = H_1(x) + \int_x^1 H_2\Big(\frac{x}{t}\Big) H_1'(t)\,\mathrm{d}t.$$

This yields

10. *If we suppose in* **8** *that the sequence $\{v_2(n)\}$ is ν-uniformly distributed modulo 1, and H_1 has a continuous derivation, we get*

$$F(x) = H_1(x) + x\int_x^1 \frac{H_1'(t)}{t}\,\mathrm{d}t.$$

And so

11. *If both sequences* $\{v_1(n)\}, \{v_2(n)\}$ *are* ν*-uniformly distributed modulo* 1*, then the* ν*-distribution function of* $\{v_1(n)v_2(n)\}$ *is*

$$F(x) = x - x\ln x, \quad \textit{for } x \in (0,1] \quad \textit{and} \quad F(0) = 0.$$

From Proposition 1, we get

12. *If* $\{w_1(n)\}, \{w_2(n)\}, \{w_3(n)\}$ *are bounded* ν*-measurable and* ν*-independent sequences with continuous* ν*-distribution functions such that the sequence*

$$\{w_1(n)w_2(n)\}$$

has a continuous distribution function, then the sequences

$$\{w_1(n)w_2(n)\}, \quad \{w_3(n)\}$$

are ν*-independent.*

Suppose now that $\{v_1(n)\}, \dots, \{v_k(n)\}$ are ν-uniformly distributed modulo 1 sequences, continuous with respect to the metric $\mathfrak{d}$, of elements of the interval $(0,1]$, which are ν-independent. Denote $u_k(n) = v_1(n)\dots v_k(n)$ for $n \in \mathbb{N}$, and k a given natural number. We derive the ν-distribution function for this sequence.

Let $g(y)$ be a polynomial. Let $\widehat{g}(y)$ denote a primitive polynomial corresponding to $g(y)$ (i.e., $\widehat{g}'(y) = g(y)$) such that $\widehat{g}(0) = 0$. Define the polynomials $g_k(y), k \in \mathbb{N}$ as follows:

$$g_1(y) = 1, \quad g_k(y) = 1 + g_{k-1}(0) - g_{k-1}(y) - \widehat{g}_{k-1}(y).$$

Put

$$F_k(x) = xg_k(\ln x), \quad x \in (0,1], \quad k \in \mathbb{N}.$$

Theorem 4. *The function* $F_k(x)$ *is the* ν*-distribution function of the sequence* $\{u_k(n)\}$*.*

Proof. For $k = 1$, the assertion holds.

Suppose that the sequence $\{u_{k-1}(n)\}$ has a continuous ν-distribution function F_{k-1} where

$$F_{k-1}(x) = xg_{k-1}(\ln x),$$

where $g_{k-1}(t)$ is a polynomial of degree $k-2$. Then

$$\begin{aligned} F_k(x) &= x + x\int_x^1 \frac{F'_{k-1}(t)}{t}\,\mathrm{d}t \\ &= x + x\int_x^1 \frac{g_{k-1}(\ln t) + g'_{k-1}(\ln t)}{t}\,\mathrm{d}t. \end{aligned}$$

Substituting $y = \ln t$, we get

$$\begin{aligned} F_k(x) &= x + x \int\limits_{\ln x}^{0} \big(g'_{k-1}(y) + g_{k-1}(y)\big)\,\mathrm{d}y \\ &= x + x\big(g_{k-1}(0) - g_{k-1}(\ln x) - \widehat{g}_{k-1}(\ln x)\big) \\ &= x g_k(\ln x). \end{aligned}$$

□

EXAMPLE 3. We have

$$\begin{aligned} F_3(x) &= x(1 + \frac{1}{2}\ln^2 x), \\ F_4(x) &= x\left(1 - \ln x - \frac{1}{2}\ln^2 x - \frac{1}{6}\ln^3 x\right). \end{aligned}$$

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