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ON STRONGLY STAR g-COMPACTNESS OF TOPOLOGICAL SPACES

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ABSTRACT. In this paper, we introduce strongly star g-compactness as a topological covering property and compare its structure to other topological properties that have analogous structures. The characteristics of a strongly star g-compact subset and strongly star g-compact subspace are looked at. Finally, some finite intersection-like characteristics that will result in some situations akin to strongly star g-compactness are presented.

1. Introduction

N. Levine [22] introduced the idea of generalized open sets (g-open sets, for short) in topological spaces as the complement of generalized closed sets (g-closed sets, for short) for the first time in 1970. A set A is g-closed if and only if $\overline{A} \subseteq U$ whenever $A \subseteq U$ and U is open, where $\overline{A}$ represents the closer of A. Since then, a large number of mathematicians have tested various topological attribute ideas in novel g-open set contexts. Regardless of g-closed sets, an equivalent definition of g-open set is established by Bal et. al. in [8]. A subset A of a topological space X is called g-open set, if $V \subseteq \text{int}(A)$ whenever $V \subseteq A$ and for all closed set V, where $\text{int}(A)$ represents the interior of the set A (Theorem 4.2, [22]). Unlike an open set, which is always g-open, a g-open set can be a non-open set. The union of two g-closed sets is a g-closed set [22] which implies that intersection of two g-open set is always a g-open set. A collection $\mathcal{W}$ of open sets is called an open cover of a space X if $\bigcup \mathcal{W} = X$ and the space X is called compact space if each of its open cover has a finite sub-cover [15]. Balachandran et. al [9] first employed g-open sets in the definition of compact

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space to introduce g-compactness of a topological space. A collection $\mathcal{W}$ of g--open sets is called an g-open cover of a space X if $\bigcup \mathcal{W} = X$ and the space X is called g-compact space if each of its g-open cover has a finite sub-cover [9].

Assume that $\mathcal{W}$ is a family of X's subsets and P is a subset of the set X. Then, star of P in regards to $\mathcal{W}$ is indicated and characterized as

$$\mathrm{St}(P, \mathcal{W}) = \bigcup \{W \in \mathcal{W} : P \cap W \neq \emptyset\}.$$

For a singleton set $\{p\}$, we write $\mathrm{St}(p, \mathcal{W})$ instead of $\mathrm{St}(\{p\}, \mathcal{W})$ [14].

If for every open cover $\mathcal{W}$, there exists a finite set P such that $\mathrm{St}(P, \mathcal{W}) = X$, then the topological space X is said to be a strongly star-compact space [16] (also named as a star finite space / star compact space in some other literatures). Many generalizations of star compactness can be found in literature. Some of them are based on the variation of selection principles (for example, star Menger space, star Rothberger space [19,20] and star Hurewicz spaces [10]) and others are based on the generalization of open sets (for example, strongly star semi compact space [6]). Some other applications of star operator can be found in some recent papers in [1–5,7,11–13,17,20,23–26]. Now, in this paper, we employed g-open covers in the definition of strongly star compact space in order to define strongly star g-compactness of a topological space. The structure and properties of strongly star g-compactness are main objectives of this paper.

2. Preliminaries

In this section, we mention some prerequisite definitions, concepts and theorems for ready reference of the readers.

Through out the paper, a space X will always be a topological space (X, τ) where τ denotes the topology on X. In [22], it is proved that in every T_1-space g-open sets are open. All spaces considered are thus strictly not T_1. Specifically for this paper, following three symbols are adopted:

(1) $\mathcal{O}$ represents the collection of all open covers of a space X.
(2) $G\mathcal{O}(X)$ represents the collection of all g-open sets of a space X.
(3) $\mathcal{O}_g(X)$ represents the collection of all g-open covers of a space X.

For other symbols and notions we follow [15].

Theorem 2.1 ([22]). *Suppose that $A \subset B \subset X$, A is a g-open set relative to B, and that B is a g-open set of X. Then, A is g-open relative to X.*

Theorem 2.2 ([22]). *A topological space X is said to be a $T_{\frac{1}{2}}$ space if and only if every g-closed set is closed (hence, every g-open set is open).*

Levine [22] proved that every T_1 space is $T_{\frac{1}{2}}$, and every $T_{\frac{1}{2}}$ space is T_0.

Definition 2.3 ([8]). A set $A \subseteq X$ is called a g-regular subset of a space X if A is both g-open and g-closed .

Definition 2.4 ([22]). A map $f : (X, \tau) \longrightarrow (Y, \sigma)$ is called:

(1) g-continuous if the inverse image of every closed set in Y is g-closed in X.
(2) gc-irresolute if the inverse image of every g-closed set in Y is g-closed in X.

Theorem 2.5 ([22]). *A map $f : (X, \tau) \longrightarrow (Y, \sigma)$ is g-continuous if the inverse image of every open set in Y is g-open in X.*

Theorem 2.6 ([22]). *A map $f : (X, \tau) \longrightarrow (Y, \sigma)$ is gc-irresolute if the inverse image of every g-open set in Y is g-open in X.*

Definition 2.7 ([15]). A non-empty family $\mathcal{F}$ of subsets of a space X is said to have the finite intersection property if every non-empty finite sub-family of $\mathcal{F}$ has non-empty intersection.

Theorem 2.8 ([15]). *A space X is compact if and only if every family of closed subsets having the finite intersection property has non-empty intersection.*

3. Strongly star g-compact spaces

Definition 3.1. A space X is called strongly star g-compact space if for every g-open cover $\mathcal{U}$ of X, there exists a finite set $F \subset X$ such that $\mathrm{St}(F,\mathcal{U}) = X$.

Proposition 3.2. *Every g-compact space is a strongly star g-compact space.*

P r o o f. Let $\mathcal{U} \in \mathcal{O}_g(X)$ be a cover of X. Since X is g-compact, there exists a finite subset $\mathcal{U}' = \{U_1, U_2, U_3, \ldots, U_m\} \subseteq \mathcal{U}$ such that $\bigcup \mathcal{U}' = X$. Now, if we collect $u_i \in U_i$ for each $i = 1, 2, 3, \ldots, m$, then $F = \{u_1, u_2, u_3, \ldots, u_m\}$ is a finite subset such that $\mathrm{St}(F,\mathcal{U}') = \bigcup \mathcal{U}' = X$. But, $X = \mathrm{St}(F,\mathcal{U}') \subseteq \mathrm{St}(F,\mathcal{U})$. Therefore, $\mathrm{St}(F,\mathcal{U}) = X$. Thus, X is a strongly star g-compact space. □

EXAMPLE. The converse of the above Proposition 3.2 may not be true, i.e., there exists a $T_{\frac{1}{2}}$ strongly star g-compact space which is not a g-compact space.

Let $X = \mathbb{R}$. Then, the collection

$$\tau = \{\emptyset, X\} \cup \big\{\{0\} \cup (X \setminus F) : F \subset X \text{ is countable}\big\}$$

is a topology on X.

Let $A \neq \emptyset$ be a set not containing '0' and $F \subseteq A$ be a non-empty closed subset. Since every open set contains '0', $\text{int}(A) = \emptyset$. Thus, $F \not\subseteq \text{int}(A)$. So, A cannot be an g-open set. By contra positivity we can conclude that every g-open set contains '0'. So for every g-open cover $\mathcal{U}$, the finite subset $E = \{0\} \subset X$ is such that $\text{St}(E, \mathcal{U}) = \bigcup \mathcal{U} = X$. Therefore, (X, τ) is a strongly star g-compact space.

On the other hand, consider the countable g-open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ with

$$U_n = \{0\} \cup \big\{X \setminus \{n, n+1, n+2, \dots\}\big\} \quad \text{for each } n \in \mathbb{N}.$$

Suppose $\mathcal{U}$ has a finite sub cover $\mathcal{U}' = \{U_{n_1}, U_{n_2}, U_{n_3}, \dots, U_{n_k}\}$. Since $\mathcal{U}'$ is finite, there exists a n_{max} such that $n_{max} = max\{n_1, n_2, n_3, \dots, n_k\}$ and so,

$$\bigcup \mathcal{U}' = \{0\} \cup \big\{X \setminus \{1, 2, 3, \dots, (n_{max} - 1)\}\big\}.$$

The portion $\{n_{max}, n_{max} + 1, n_{max} + 2, \dots\}$ of X remains uncovered. So, $\mathcal{U}$ cannot have a finite sub cover. Thus, (X, τ) is not a g-compact space.

In this space X, a set B is an open if and only if $0 \in B$ and $B = X \setminus F$ such that F is countable. Let A be a g-open and non-open set. Then, either $0 \notin A$ or $A = X \setminus F$ such that F is uncountable. If $0 \notin A$, then $\text{int}(A) = \emptyset$. Let $A = X \setminus F$, where F is uncountable and suppose there exists an open set $P = \{0\} \cup (X \setminus E)$ such that $P = \{0\} \cup (X \setminus E) \subseteq X \setminus F = A$. Then, E is a countable set and $F \subseteq E$, which contradicts the fact that F is uncountable. Thus, such an open set $P \subseteq A$ does not exist. Therefore, $\text{int}(A) = \emptyset$. In both cases we can find a non-empty finite subset $V \subseteq A$, which is obviously closed set and $V \not\subseteq \text{int}(A)$ which again contradicts the fact that A is g-open. So, in this space, non-empty g-open sets do not exist. In other words, we can say that every g-open set is open, i.e., X is a $T_{\frac{1}{2}}$ space.

PROPOSITION 3.3. *Every strongly star g-compact space is a strongly star compact space.*

P r o o f. The proof follows from the definition since every open set is g-open. □

EXAMPLE. The converse of the above Proposition 3.4 may not be true, i.e., there exists a T_0 strongly star compact space which is not strongly star g-compact.

Let us consider the topological space (X, τ) where

$$X = \mathbb{N} \quad \text{and} \quad \tau = \{\emptyset\} \cup \big\{\{1, 2, 3, \dots, n\} : n \in \mathbb{N}\big\}.$$

Therefore, every open set rather than $\emptyset$ contains the element $1 \in X$. So, $M = \{1\}$ is a finite set such that $\mathrm{St}(M,\mathcal{U}) = \bigcup\mathcal{U} = X$ for every open cover $\mathcal{U}$ of X. So, (X,τ) is a strongly star compact space.

On the other hand, for every singleton set A, there does not exist any closed set inside A rather than $\emptyset$. So, we can say that every singleton set is a g-open set in this topological space. So, for the open cover $\mathcal{W} = \{W_n = \{n\} : n \in \mathbb{N}\}$, if we consider any finite subset $M \subset X$, then $\mathrm{St}(M,\mathcal{W}) = M \neq X$. So, (X,τ) is not strongly star g-compact space.

Thus, we have the following arrow diagram (see [16] for examples of strongly star compact spaces which are not compact):

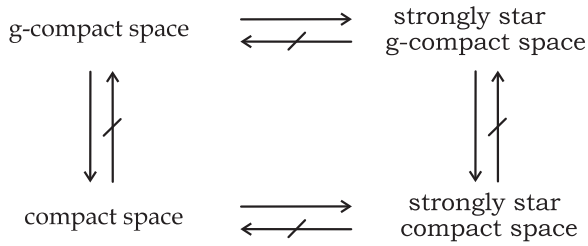


FIGURE 1. Relation diagram.

4. Strongly star g-compact subsets & subspaces

EXAMPLE. In general, a subspace of a strongly star g-compact space may not be a strongly star g-compact space.

Let $X = \{0\} \cup \mathbb{N}$, then $\mathcal{B} = \{\emptyset, \{0\}\} \cup \{\{0,n\} : n \in \mathbb{N}\}$ is a base for a suitable topology τ on X.

Now, if $A \subseteq X$ is such that $0 \notin A$, then $\mathrm{int}(A) = \emptyset$ (since every open set contains 0). Thus, if there exists a closed set $\emptyset \neq B \subseteq A$, then $B \nsubseteq \mathrm{int}(A)$. So, any subset of X that does not contain 0 is not g-open. By the contra positivity, every g-open subset of X must contain 0. Now, suppose $\mathcal{U}$ is an arbitrary g-open cover of X, i.e., $\mathcal{U} \in \mathcal{O}_g(X)$. So, every element of $\mathcal{U}$ contains 0. Thus, we have a finite set $\{0\} = F$ such that $\mathrm{St}(F,\mathcal{U}) = X$. Therefore, X is a strongly star g-compact space.

Now, consider the subset $\mathbb{N} \subseteq X$ and the subspace $(\mathbb{N}, \tau_{\mathbb{N}})$. In this subspace, every singleton is an open set, hence is a g-open set. So, we consider the g--open cover $\mathcal{V} = \{V_n = \{n\} : n \in \mathbb{N}\}$. So, for every choice of finite set $F \subseteq \mathbb{N}$, we will have $\mathrm{St}(F,\mathcal{V}) = F \neq \mathbb{N}$. So, the subspace $(\mathbb{N}, \tau_{\mathbb{N}})$ is not a strongly star g-compact space although (X,τ) is a strongly star g-compact space.

Theorem 4.1. *Every g-regular subspace of a strongly star g-compact space is strongly star g-compact space.*

Proof. Let (A, τ_A) be a g-regular subspace of the space (X, τ) where X is a strongly star g-compact space. Let $\mathcal{U}_A$ be a cover of A such that $\mathcal{U}_A \in \mathcal{O}_g(A)$. By Proposition 2.1, every element of $\mathcal{U}_A$ belongs to $G\mathcal{O}(X)$. A is also g-closed, so $V = X \setminus A \in G\mathcal{O}(X)$. Finally, $\mathcal{U} = \{V\} \cup \mathcal{U}_A$ is a g-open cover of X. But X is a strongly star g-compact space. So, there exists a finite set F such that $\mathrm{St}(F,\mathcal{U}) = X$. Clearly, V, $\bigcup \mathcal{U}_A$ are disjoint sets; $V \cap F$, $F \setminus V$ are disjoint finite sets. So, we must have $\mathrm{St}\big((F \setminus V), \mathcal{U}_A\big) = A$. So, the subspace A is a strongly star g-compact subspace. Hence the theorem. □

Definition 4.2. Given a space X, a subset $A \subseteq X$ is called a strongly star g-compact subset relative to X if for every g-open cover $\mathcal{U}$ of A, where $\mathcal{U} \subseteq G\mathcal{O}(X)$, there exists a finite set $F \subseteq A$ such that $\mathrm{St}(F,\mathcal{U}) \supseteq A$.

Theorem 4.3. *Union of finite numbers of strongly star g-compact subsets relative to a space is strongly star g-compact subset relative to that space.*

Proof. The proof follows directly, hence omitted. □

Theorem 4.4. *An open set G of a space X is strongly star g-compact as subspace if and only if G is strongly star g-compact subset relative to X.*

Proof. Let (G, τ_G) be a strongly star g-compact subspace of the space (X, τ) where $G \in \tau$. $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ is a g-open cover of G where $\mathcal{U} \subseteq G\mathcal{O}(X)$. Clearly, $G \in G\mathcal{O}(X)$ and $G \cap U_\alpha \in G\mathcal{O}(X)$. $\therefore \mathcal{U}_G = \{G \cap U_\alpha : \alpha \in \Lambda\} \in \mathcal{O}_g(G)$. But (G, τ_G) is a strongly star g-compact subspace. So, there exists a finite subset $F \subseteq G$ such that $G = \mathrm{St}(F,\mathcal{U}_G)$. $\therefore G \subseteq \mathrm{St}(F,\mathcal{U})$. Therefore, G is strongly star g-compact subset relative to X.

Conversely, let $G \in \tau$ be strongly star g-compact subset relative to X. Suppose $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ is a cover of G, where $\mathcal{U} \in \mathcal{O}_g(G)$. By Theorem 2.1, $U_\lambda \in G\mathcal{O}(X)$ for all $\lambda \in \Lambda$. $\therefore \mathcal{U}$ is a cover of G such that $\mathcal{U} \subseteq G\mathcal{O}(X)$. But G is strongly star g-compact subset relative to X. So, there exists a finite subset $F \subseteq G$ such that $G = \mathrm{St}(F,\mathcal{U})$. So, G is a strongly star g-compact subspace. □

Theorem 4.5. *A g-continuous image of a strongly star g-compact space is a strongly star compact space.*

Proof. Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a g-continuous map from a strongly star g-compact space (X, τ) onto a space (Y, σ). Let $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$ be an open cover of Y. Then, $\mathcal{U} = \{f^{-1}(V_\alpha) : \alpha \in \Lambda\}$ is a g-open cover of X. Since X is strongly star g-compact, there exists a finite set $F = \{x_1, x_2, x_3, \ldots x_k\}$ such that $\mathrm{St}(F,\mathcal{U}) = X$. This implies that $\bigcup\{f^{-1}(V_\alpha) \in \mathcal{U} : F \cap f^{-1}(V_\alpha) \neq \emptyset\} = X$.

We take $\mathcal{U}' = \{f^{-1}(V_\alpha) \in \mathcal{U} : F \cap f^{-1}(V_\alpha) \neq \emptyset\}$. So, $\bigcup \mathcal{U}' = X$. Clearly,

$$\bigcup \{f(f^{-1}(V_\alpha)) : f^{-1}(V_\alpha) \in \mathcal{U}'\} = f(X)$$
$$\implies \quad \bigcup \{V_\alpha : f^{-1}(V_\alpha) \in \mathcal{U}'\} = Y.$$

Suppose $\mathcal{W} = \{V_\alpha : f^{-1}(V_\alpha) \in \mathcal{U}'\}$. Thus, $\bigcup \mathcal{W} = Y$ and $\mathcal{W} \subseteq \mathcal{V}$.

Also, for every $x_i \in F \subseteq X$, $i = 1, 2, 3, \ldots k$, there exists a $y_i \in Y$ such that $f(x_i) = y_i$. $\therefore$ $E = \{f(x_i) : i = 1, 2, 3, \ldots, k\}$ is a finite set in Y. We want to show that $\mathrm{St}(E, \mathcal{W}) = Y$.

Suppose $\mathrm{St}(E, \mathcal{W}) \neq Y$. But $\bigcup \mathcal{W} = Y$. $\therefore$ there exists a $W \in \mathcal{W}$ such that $E \cap W = \emptyset$. This implies that $F \cap f^{-1}(V_\beta) = \emptyset$ where $W = f^{-1}(V_\beta) \in \mathcal{U}'$, which is a contradiction to the fact that

$$F \cap f^{-1}(V_\alpha) \neq \emptyset \ \ \forall f^{-1}(V_\alpha) \in \mathcal{U}'.$$

Therefore, $\mathrm{St}(E, \mathcal{W}) = Y$. But $\mathcal{W} \subseteq \mathcal{V}$. So, $\mathrm{St}(E, \mathcal{V}) = Y$. Hence the theorem. □

Theorem 4.6. *A gc-irresolute image of a strongly star g-compact space is a strongly star g-compact space.*

Proof. The proof is similar to the proof of Theorem 4.6, hence omitted. □

5. Finite intersection like properties for strongly star g-compactness

Although compactness of a topological space is defined by using open covers, the finite intersection property helps us to represents the compactness in terms of the family of closed subsets. In the similar way, we want to explain the strongly star g-compactness in terms of g-closed subsets. So, in this section, we have investigated for some finite intersection-like properties for the strongly star g-compactness.

Theorem 5.1. *For a topological space (X, τ), following statements are equivalent:*

(1) *X is a strongly star g-compact space.*

(2) *For a given family $\mathcal{U}$ of g-open sets, if for every finite subset $F \subseteq X$, $\mathrm{St}(F, \mathcal{U}) \neq X$, then $\mathcal{U}$ cannot cover X.*

(3) *For every family $\mathcal{F}$ of g-closed sets having the property $\bigcap_{F \in \mathcal{F}} F = \emptyset$, we can find a finite subset $P \subseteq X$ and a sub-family $\mathcal{E} \subseteq \mathcal{F}$ such that $P \cap (X \setminus F) \neq \emptyset$ for all $F \in \mathcal{E}$ and $\bigcap_{F \in \mathcal{E}} F = \emptyset$.*

(4) *If $\mathcal{F}$ is a family of g-closed sets such that for every finite set $P \subseteq X$ and for every sub-family $\mathcal{E} \subseteq \mathcal{F}$ having the property $\bigcap_{F\in\mathcal{E}} F \neq \emptyset$ gives $P \cap (X \setminus F) = \emptyset$, for every $F \in \mathcal{E}$, then $\bigcap_{F\in\mathcal{F}} F \neq \emptyset$.*

P r o o f.

(1) $\Longrightarrow$ (2): Let X be strongly star g-compact space. If possible, suppose that $\mathcal{U}$ is a family of g-open sets which covers X. But by condition (2), for every finite subset $F \subseteq X$, $\mathrm{St}(F,\mathcal{U}) \neq X$, which contradicts the fact that X is strongly star g-compact. So, $\mathcal{U}$ cannot be a cover of X.

(2) $\Longrightarrow$ (1): Let $\mathcal{U}$ be an arbitrary g-open cover of X and let condition (2) hold. Clearly, $\mathcal{U}$ is a collection of g-open sets. If possible, suppose that for every finite set $F \subseteq X$, $\mathrm{St}(F,\mathcal{U}) \neq X$. So, by condition (2), $\mathcal{U}$ is not a cover of X, which is a contradiction. So, there must exist a finite subset $F \subseteq X$ such that $\mathrm{St}(F,\mathcal{U}) = X$.

$\therefore X$ is a strongly star g-compact space.

(1) $\Longrightarrow$ (3): Let X be a strongly star g-compact space. Suppose $\mathcal{F}$ is a family of g-closed sets such that $\bigcap_{F\in\mathcal{F}} F = \emptyset$.

$$\therefore X \setminus \left(\bigcap_{F\in\mathcal{F}} F\right) = X \setminus \emptyset$$

$$\Longrightarrow \bigcup_{F\in\mathcal{F}} (X \setminus F) = X$$

$\therefore \mathcal{G} = \{G = (X \setminus F) : F \in \mathcal{F}\}$ is a g-open cover of X. But X is strongly star $G\mathcal{O}$-compact. So, there exists a finite set $P \subseteq X$ such that $\mathrm{St}(P,\mathcal{G}) = X$.

$$\begin{aligned}
\Longrightarrow \quad & \bigcup\{(X \setminus F) \in \mathcal{G} : P \cap (X \setminus F) \neq \emptyset\} && = X \\
\Longrightarrow \quad & X \setminus \textstyle\bigcap_{\{X\setminus F\in\mathcal{G}:P\cap(X\setminus F)\neq\emptyset\}} F && = X \\
\Longrightarrow \quad & \textstyle\bigcap_{\{X\setminus F\in\mathcal{G}:P\cap(X\setminus F)\neq\emptyset\}} F && = \emptyset \\
\Longrightarrow \quad & \textstyle\bigcap\{F : X \setminus F \in \mathcal{G} \quad \text{and} \quad P \cap (X \setminus F) \neq \emptyset\} && = \emptyset.
\end{aligned}$$

Suppose,

$$\mathcal{E} = \{F : X \setminus F \in \mathcal{G} \quad \text{and} \quad P \cap (X \setminus F) \neq \emptyset\}.$$

Therefore,

$$\mathcal{E} = \{F \in \mathcal{F} : P \cap (X \setminus F) \neq \emptyset\}.$$

Clearly, $\mathcal{E}$ is a sub-family of $\mathcal{F}$, $P \subseteq X$ is finite with

$$\bigcap_{F\in\mathcal{E}} F = \emptyset \quad \text{and} \quad P \cap (X \setminus F) \neq \emptyset \quad \text{for all} \ \ F \in \mathcal{E}.$$

(3) $\Longrightarrow$ (1): Let $\mathcal{U}$ be an arbitrary g-open cover of X and let condition (3) hold. So, $\mathcal{F} = \{F = X \setminus U : U \in \mathcal{U}\}$ is a family of g-closed sets with

$$\bigcap_{F\in\mathcal{F}} F = \bigcap_{U\in\mathcal{U}} (X \setminus U) = X \setminus \bigcup_{U\in\mathcal{U}} U = X \setminus X = \emptyset.$$

So, by condition (3), there exists a finite subset $P \subseteq X$ and a sub-family $\mathcal{E} \subseteq \mathcal{F}$ such that

$$\begin{aligned} P \cap (X \setminus F) \neq \emptyset \quad \text{for all} \quad F \in \mathcal{E} \quad \text{and} \quad \bigcap_{F\in\mathcal{E}} F = \emptyset. \\ \Longrightarrow \quad \textstyle\bigcap_{X\setminus U\in\mathcal{E}} (X \setminus U) &= \emptyset \\ \Longrightarrow \quad X \setminus \textstyle\bigcup_{(X\setminus U)\in\mathcal{E}} U &= \emptyset \\ \Longrightarrow \quad \textstyle\bigcup_{(X\setminus U)\in\mathcal{E}} U &= X. \end{aligned}$$

Suppose $\{U : X \setminus U \in \mathcal{E}\} = \mathcal{U}'$, clearly $\mathcal{U}'$ is a sub-family of $\mathcal{U}$ and $\bigcup_{U\in\mathcal{U}'} U = X$. Moreover, $P \cap U \neq \emptyset$, $\forall X \setminus U \in \mathcal{E}$

$$\begin{aligned} &\Longrightarrow \quad P \cap U \neq \emptyset, \ \forall U \in \mathcal{U}' \\ &\Longrightarrow \quad \mathrm{St}(P,\mathcal{U}') = \textstyle\bigcup_{U\in\mathcal{U}'} U = X \\ &\Longrightarrow \quad \mathrm{St}(P,\mathcal{U}) = X. \end{aligned}$$

Therefore, X is a strongly star g-compact space.

(1) $\Longrightarrow$ (4): Let (X,τ) be a strongly star g-compact space. Suppose $\mathcal{F}$ is a collection of g-closed sets such that for every finite set $P \subseteq X$ and for every sub-family $\mathcal{E} \subseteq \mathcal{F}$ with the properties $\bigcap_{F\in\mathcal{E}} F \neq \emptyset$ and the set

$$P \cap (X \setminus F) = \emptyset \forall F \in \mathcal{E}.$$

If possible, suppose $\bigcap_{F\in\mathcal{F}} F = \emptyset$

$$\begin{aligned} &\Longrightarrow \quad X \setminus \textstyle\bigcap_{F\in\mathcal{F}} F &= X \setminus \emptyset \\ &\Longrightarrow \quad \textstyle\bigcup_{F\in\mathcal{F}} (X \setminus F) &= X. \end{aligned}$$

Therefore, $\mathcal{G} = \{G = X \setminus F : F \in \mathcal{F}\}$ is a g-open cover of X. Since (X,τ) is strongly star g-compact, there exists a finite set $P \subseteq X$ such that $\mathrm{St}(P,\mathcal{G}) = X$.

$$\begin{aligned} &\Longrightarrow \quad \textstyle\bigcup\{G \in \mathcal{G} : P \cap G \neq \emptyset\} &= X \\ &\Longrightarrow \quad \textstyle\bigcup\{X \setminus F \in \mathcal{G} : P \cap (X \setminus F) \neq \emptyset\} &= X \\ &\Longrightarrow \quad X \setminus \left(\textstyle\bigcap_{\{X\setminus F\in G : P\cap(X\setminus F)\neq\emptyset\}} F\right) &= X \\ &\Longrightarrow \quad \textstyle\bigcap_{\{F\in\mathcal{F} : P\cap(X\setminus F)\neq\emptyset\}} F &= \emptyset. \end{aligned}$$

Suppose $\mathcal{E} = \{F \in \mathcal{F} : P \cap (X \setminus F) \neq \emptyset\}$, then $P \subseteq X$ is finite and $\mathcal{E}$ is a sub-family of $\mathcal{F}$ such that

$$P \cap (X \setminus F) \neq \emptyset.$$

But $\bigcap_{F\in\mathcal{E}} F = \emptyset$, which is a contradiction. Therefore,

$$\bigcap_{F\in\mathcal{F}} F \neq \emptyset.$$

(4) $\Longrightarrow$ (1): Suppose condition (4) holds. If possible, suppose that X is not a strongly star g-compact space, i.e., there exists an open cover $\mathcal{U}$ such that for every finite set $P \subseteq X, \mathrm{St}(P,\mathcal{U}) \neq X$.

$$\Longrightarrow \quad \bigcup\{U \in \mathcal{U} : P \cap U \neq \emptyset\} \neq X \quad \text{for every finite subset } P \subseteq X$$

$$\Longrightarrow \quad X \setminus \bigcup\{U \in \mathcal{U} : P \cap U \neq \emptyset\} \neq \emptyset \quad \text{for every finite subset } P \subseteq X$$

$$\Longrightarrow \quad \bigcap\{(X \setminus U) : U \in \mathcal{U} \text{ and } P \cap U \neq \emptyset\} \neq \emptyset \quad \text{for every finite subset } P \subseteq X.$$

Suppose $\mathcal{F} = \{F = X \setminus U : U \in \mathcal{U}\}$, then $\mathcal{F}$ is a family of g-closed subsets of X and

$$\mathcal{E} = \{F = X \setminus U : U \in \mathcal{U}$$

and

$$P \cap U \neq \emptyset\} = \{F : F \in \mathcal{F}$$

and

$$P \cap (X \setminus F) \neq \emptyset\}$$

is a sub-family of $\mathcal{F}$ for all finite set $P \subseteq X$.

Now, for the finite subset $P \subseteq X$, the sub-family $\mathcal{E} \subseteq \mathcal{F}$ has the properties $P \cap (X \setminus F) \neq \emptyset \quad \forall F \in \mathcal{E}$ and $\bigcap_{F\in\mathcal{E}} F \neq \emptyset$.

Now, by condition (4), $\bigcap_{F\in\mathcal{F}} F \neq \emptyset$

$$\begin{aligned}
\Longrightarrow \quad & X \setminus \textstyle\bigcap_{F\in\mathcal{F}} F && \neq \quad X.\\
\Longrightarrow \quad & \textstyle\bigcup_{F\in\mathcal{F}} (X \setminus F) && \neq \quad X.\\
\Longrightarrow \quad & \textstyle\bigcup\{U = X \setminus F : F \in \mathcal{F}\} && \neq \quad X.\\
\Longrightarrow \quad & \textstyle\bigcup\{U : U \in \mathcal{U}\} && \neq \quad X.
\end{aligned}$$

That is, $\mathcal{U}$ is not a cover of X. Which is a contradiction.

So, there must exist a finite set

$$P \subseteq X \quad \text{such that} \quad \mathrm{St}(P,\mathcal{U}) = X.$$

Therefore, (X, τ) is a strongly star g-compact space. Hence the theorem.

□

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