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CONTINUED FRACTION REPRESENTATIONS OF THE GENERALIZED OPERATOR ENTROPY

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ABSTRACT. The direct calculation of the generalized operator entropy proves to be difficult due to the appearance of rational exponents of matrices. The main motivation of this work is to overcome these difficulties and to present a practical and efficient method for this calculation using its representation by the matrix continued fraction. At the end of our work, we deduce a continued fraction expansion of the Bregman operator divergence. Some numerical examples illustrating the theoretical result are discussed.

1. Introduction

It is known that the real continued fraction expansions have an advantage that they converge faster than other numerical algorithms [6, 10]. So the extension of continued fractions theory from real numbers to the matrix case has seen several developments and interesting applications [4].

The theory of operator means for positive and bounded linear operators on Hilbert space was initiated by T. Ando [1] and established together with F. Kubo in connection with Löwner theory for the operator monotone functions [2, 16]. It is started from the presence of the notion of parallel sum as a tool for analyzing multiport electrical networks in engineering, see [3].

In 1850, Clausius introduced the notion of entropy in thermodynamics. Since then, several extensions and reformulations have been developed in various disciplines [5, 8]. Some mathematicians have investigated the so-called entropy inequalities, see [14] and references therein. For a real number $q \in \mathbb{R}$, $0 < q < 1$,

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we define the path

$$f_q(A,B) = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^q A^{1/2}.$$

The path $f_q(A,B)$ is the generalized geometric mean of A and B.

In the present paper, we also study the representation of the generalized operator entropy which is defined for two positive operators A and B on Hilbert space and any real number $q \in]0,1]$, by

$$S_q(A|B) = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^q \ln\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

The Bregman operator divergence introduced by Petz [14] is defined by

$$D(A|B) = B - A - S(A|B),$$

where

$$S(A|B) = A^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

In [6], Isa et al. have generalized $D(A|B)$ as follows

$$D_q(A|B) = f_{-q}(A|B) - f_q(A|B) - S_q(A|B).$$

At the end of our paper, we also express the continued fraction representation of the divergence operator $D_q(A|B)$.

For simplicity and clearness, we restrict ourselves to positive definite matrices, but our results can be, without special difficulties, projected to the case of positive definite operators from an infinite dimensional Hilbert space into itself.

The computation of $S_q(A|B)$ and $D_q(A|B)$ from the original definitions impose many difficulties by virtue of the appearance of the rational exponents of the matrices. One fundamental of this paper is to remove this difficulty and reveal a practical method, by matrix continued fraction.

2. Definitions and notations

Let $\mathcal{M}_m$ be the algebra of real square matrices, where m is a positive integer ≥ 2.

Let $A \in \mathcal{M}_m$, for a function f with a series representation on an open interval containing the eigenvalues of A, we are able to define the matrix function $f(A)$ via the Taylor series for f, se [12].

For any $A, B \in \mathcal{M}_m$ with B invertible, we write

$$\frac{A}{B} = B^{-1}A.$$

It is easy to verify that for all matrices C and D with C invertible we have

$$\frac{A}{B} = \frac{CA}{CB} \neq \frac{AC}{BC}.$$

Definition 2.1. Let $(A_n)_{n\geq 0}$, $(B_n)_{n\geq 0}$ be two nonzero sequences of $\mathcal{M}_m$. The continued fraction of (A_n) and (B_n) denoted by $K(B_n/A_n)$ is the quantity

$$A_0 + \cfrac{B_1}{A_1 + \cfrac{B_2}{A_2 + \cdots}} = \left[A_0; \frac{B_1}{A_1}, \frac{B_2}{A_2}, \cdots\right].$$

Sometimes, we use briefly the notation

$$\left[A_0; \frac{B_n}{A_n}\right]_{n=1}^{+\infty}.$$

The fractions

$$\frac{B_n}{A_n} \quad \text{and} \quad \frac{P_n}{Q_n} = \left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{n}$$

are called the nth partial quotient and the nth convergent of the continued fraction $K(B_n/A_n)$, respectively. When $B_n = I$ for all $n \geq 1$, then $K(I/A_n)$ is called a simple continued fraction. The continued fraction $K(B_n/A_n)$ converges in $\mathcal{M}$ if the sequence (F_n) converges in $\mathcal{M}$ in the sense that there exists a matrix $F \in \mathcal{M}_m$ such that $\lim_{n\to+\infty} ||F_n - F|| = 0$. In this case, we note that

$$F = \left[A_0; \frac{B_n}{A_n}\right]_{n=1}^{+\infty}.$$

The following proposition gives an appropriate method for calculating $K(B_n/A_n)$.

Proposition 2.2. *The elements $(P_n)_{n\geq -1}$ and $(Q_n)_{n\geq -1}$ of the n^{th} convergent of $K(B_n/A_n)$ are given by the relationships*

$$\begin{cases} P_{-1} = I, & P_0 = A_0, \\ Q_{-1} = 0, & Q_0 = I, \end{cases} \quad \text{and} \quad \begin{cases} P_n = A_n P_{n-1} + B_n P_{n-2}, \\ Q_n = A_n Q_{n-1} + B_n Q_{n-2}, \end{cases} \quad n \geq 1.$$

P r o o f. We prove it by induction. □

The proof of the next Proposition is elementary, and we leave it to the reader.

Proposition 2.3.

(i) *For any two matrices C and D with C invertible, we have*

$$C\left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{n} D = \left[CA_0D\ \frac{B_1D}{A_1C^{-1}}, \frac{B_2C^{-1}}{A_2}, \frac{B_k}{A_k}\right]_{k=3}^{n}.$$

(ii) *If two matrices A and B are similar, with $A = XBX^{-1}$, then $f(A)$ and $f(B)$ are similar and we have $f(A) = Xf(B)X^{-1}$.*

In order to simplify the statement about some partial quotients of continued fractions with matrix arguments, we need the following proposition which is an example of equivalent continued fractions.

Proposition 2.4. *Let* $\left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{+\infty}$ *be a given continued fraction. Then, we have*

$$\frac{P_n}{Q_n} = \left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{n} = \left[A_0; \frac{X_k B_k X_{k-2}^{-1}}{X_k A_k X_{k-1}^{-1}}\right]_{k=1}^{n},$$

where $X_{-1} = X_0 = I$ *and* $X_1, X_2, \ldots X_n$ *are arbitrary invertible matrices.*

P r o o f. Let $\frac{P_n}{Q_n}$ and $\frac{\widetilde{P}_n}{\widetilde{Q}_n}$ be the nth convergent of the continued fractions

$$\left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{+\infty} \text{ and } \left[A_0; \frac{X_k B_k X_{k-2}^{-1}}{X_k A_k X_{k-1}^{-1}}\right]_{k=1}^{+\infty}, \text{ respectively.}$$

By proposition 2.1, for all $n \geq 1$, we can write

$$\widetilde{P}_n = X_n A_n X_{n-1}^{-1} \widetilde{P}_{n-1} + X_n B_n X_{n-2}^{-1} \widetilde{P}_{n-2},$$

which is equivalent to

$$X_n^{-1} \widetilde{P}_n = A_n \left(X_{n-1}^{-1} \widetilde{P}_{n-1}\right) + B_n \left(X_{n-2}^{-1} \widetilde{P}_{n-2}\right).$$

This last result together with the initial conditions proves that for all

$$n \geq 1, \quad X_{n-1}^{-1} \widetilde{P}_n = P_n.$$

A similar result can be obtained for Q_n. Consequently, both continued fractions have the same convergent, and Proposition 2.4 follows. □

Definition 2.5. Let A be a positive definite matrix in $\mathcal{M}_m, X \in \mathcal{M}_m$ and α a real number such that $0 < \alpha < 1$. We define the matrix A^α by the formulae

$$A^\alpha = \exp(\alpha \ln A),$$

where "exp" is the matrix exponential given by the series

$$\exp(Y) = \sum_{n=0}^{+\infty} \frac{Y^n}{n!}$$

and ln is the neperian logarithm defined by

$$\ln(A) = -2 \sum_{n=0}^{+\infty} \frac{1}{2n+1} \left(\frac{I-A}{I+A}\right)^{2n+1}.$$

3. Main results

3.1. Continued fraction representation of $A^q \ln(A)$

This section is devoted to give a continued fraction representation of $A^q \ln(A)$, where A is a positive definite matrix.

Theorem 3.1. *Let $A \in \mathcal{M}_m$ be a positive definite matrix and q a positive real number such that $0 < q < 1$. If we put*

$$A^q = \left[I; \frac{I}{A_k}\right]_{k=1}^{+\infty}, \quad \ln A = \left[I; \frac{I}{\widetilde{A_k}}\right]_{k=1}^{+\infty} \quad \text{and} \quad \varphi(A) = \frac{I-A}{I+A},$$

then the continued fraction expansion of $A^q \ln(A)$ is given by

$$A^q \ln(A) = \left[0; \frac{A_1+I}{A_1\widetilde{A}_1}, \frac{(A_1A_2+I)(A_1+I)+\widetilde{A}_1\widetilde{A}_2}{(A_1A_2+A_2+I)\widetilde{A}_2}, \frac{(A_1\widetilde{A}_1)^{-1}E_2F_3}{E_3-F_3}, \frac{E_{n-1}F_n}{E_n-F_n}\right]_{n=4}^{+\infty},$$

where

$$\begin{cases} A_1 &= \frac{-I-q\varphi(A)}{2q\varphi(A)}, \\ A_2 &= \frac{-6qI}{(q^2-1)\varphi(A)}, \\ A_{2k} &= \frac{-2q(q^2-2^2)\ldots(q^2-(2k-2)^2)}{(q^2-1)\ldots(q^2-(2k-1)^2)}(4k-1)I, \qquad k \geq 2, \\ A_{2k+1} &= \frac{-(q^2-1)\ldots(q^2-(2k-1)^2)}{2q(q^2-2^2)\ldots(q^2-4k^2)}(4k+1)\varphi^2(A), \quad k \geq 1. \end{cases}$$

For all $n \geq 1$, the expressions of $\widetilde{A}_n$ are given by the following relationships.

$$\begin{cases} \widetilde{A}_1 &= \big(-2\varphi(A)\big)^{-1}, \\ \widetilde{A}_2 &= \frac{6I}{\varphi(A)}, \\ \widetilde{A}_{2k} &= \frac{-2(2.4\ldots(2k-2))^2}{(3.5\ldots(2k-1))^2}(4k-1)I, \quad k \geq 2, \\ \widetilde{A}_{2k+1} &= \frac{(1.3.5\ldots(2k-1))^2}{(2.4\ldots(2k))^2\varphi(A)}(4k+1)I, \quad k \geq 1. \end{cases}$$

We also define

$$\begin{cases} E_n &= Q_n\widetilde{Q}_n(Q_{n-2}P_{n-1}+\widetilde{Q}_{n-1}\widetilde{P}_{n-2}), \\ F_n &= Q_{n-2}\widetilde{Q}_{n-2}(Q_nP_{n-1}+\widetilde{Q}_{n-1}\widetilde{P}_n). \end{cases}$$

The matrices P_n and Q_n (resp. $\widetilde{P}_n$ and $\widetilde{Q}_n$) are the numerator and denominator of the n^{th} convergent of A^q (resp. $\ln(A)$) which are defined for all $n \geq 1$ by

$$\begin{cases} P_n = A_n P_{n-1} + P_{n-2}\,, \\ Q_n = A_n Q_n + Q_{n-2}\,, \end{cases} \quad \text{and} \quad \begin{cases} \widetilde{P}_n = \widetilde{A}_n \widetilde{P}_{n-1} + \widetilde{P}_{n-2}\,, \\ \widetilde{Q}_n = \widetilde{A}_n \widetilde{Q}_{n-1} + \widetilde{Q}_{n-2}\,. \end{cases}$$

To prove Theorem 3.1, we begin by studying a real case.

3.2. The real case

We begin by giving some lemmas concerning the real continued fraction which are important in the sequel. The following lemma characterizes equivalence of continued fractions.

Lemma 3.2 ([9]). *Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 1}$ be two non-zero sequences of real numbers. The continued fractions*

$$\left[a_0; \frac{b_1}{a_1}, \frac{b_2}{a_2}, \dots \frac{b_n}{a_n}, \dots\right] = \left[a_0 \frac{1}{a_1^*}, \frac{1}{a_2^*}, \dots, \frac{1}{a_n^*}, \dots\right]$$

are equal, where

$$\begin{cases} a_1^* &= \frac{a_1}{b_1}\,, \\ a_2^* &= \frac{b_1}{b_2} a_2\,, \\ a_{2k}^* &= \frac{b_1 b_3 \dots b_{2k-1}}{b_2 b_4 \dots b_{2k}} a_{2k}\,, & k \geq 2, \\ a_{2k+1}^* &= \dfrac{b_2 b_4 \dots b_{2k}}{b_1 b_3 \dots b_{2k+1}} a_{2k+1}\,, & k \geq 1. \end{cases}$$

We now give a lemma which expresses the n^{th} convergent for the product of two continued fractions.

Lemma 3.3 ([13]). *Let C and D be two real continued fractions which are defined by*

$$C = \left[c_0; \frac{1}{c_1}, \frac{1}{c_n}\right]_{n=2}^{+\infty}, \qquad D = \left[d_0; \frac{1}{d_1}, \frac{1}{d_n}\right]_{n=2}^{+\infty},$$

where c_k and d_k are non-zero real numbers. If we put

$$C_n = \left[c_0; \frac{1}{c_1}, \frac{1}{c_2}, \dots, \frac{1}{c_n}\right] = \frac{{}^c p_n}{{}^c q_n} \quad \text{and} \quad D_n = \left[d_0; \frac{1}{d_1}, \frac{1}{d_2}, \dots, \frac{1}{d_n}\right] = \frac{{}^d p_n}{{}^d q_n},$$

then, for all $n \geq 1$, we have

$$C_n D_n = \left[c_0 d_0; \frac{c_0 c_1 + d_0 d_1 + 1}{c_1 d_1}, \frac{c_1 d_1 f_2}{e_2 - f_2}, \frac{e_2 f_3}{e_3 - f_3}, \dots, \frac{e_{n-1} f_n}{e_n - f_n}\right],$$

where

$$\begin{cases} e_n &=^c \; q_n^d q_n \left({}^c q_{n-2}\; {}^c q_{n-1} +^d q_{n-2}\; {}^d q_{n-1}\right), \\ f_n &=^c \; q_{n-2}\; {}^d q_{n-2} \left({}^c q_{n-1}\; {}^c q_n +^d q_{n-1}\; {}^d q_n\right). \end{cases}$$

The following lemma gives two equivalent continued fraction expansions of λ^q, where λ and q are two strictly positive real numbers.

Lemma 3.4.

(i) *Let λ and q be two positive real numbers, $\lambda \neq 1$, $\varphi(\lambda) = \frac{1-\lambda}{1+\lambda}$. The continued fraction expansions of λ^q is*

$$\lambda^q = \left[1; \frac{2q\varphi(\lambda)}{-1-q\varphi(\lambda)}, \frac{(q^2-(k-1)^2)\varphi^2(\lambda)}{-(2k-1)}\right]_{k=2}^{+\infty}.$$

(ii) *The simple continued fraction of λ^q is given by*

$$\lambda^q = \left[1; \frac{1}{c_1^*}, \frac{1}{c_2^*} =, , \ldots, \frac{1}{c_n^*}, \ldots\right],$$

where

$$\begin{cases} c_1^* & = & \frac{-1-q\varphi(\lambda)}{2q\varphi(\lambda)}, \\ c_2^* & = & \frac{-6q}{(q^2-1)\varphi(\lambda)}, \\ c_{2k}^* & = & \frac{-2q\left(q^2-2^2\right)\ldots\left(q^2-(2k-2)^2\right)}{(q^2-1)\ldots\left(q^2-(2k-1)^2\right)}(4k-1), & k \geq 2, \\ c_{2k+1}^* & = & \frac{-\left(q^2-1\right)\ldots\left(q^2-(2k-1)^2\right)}{2q(q^2-2^2)\ldots(q^2-(2k)^2)}(4k+1)\varphi^2(\lambda), & k \geq 1. \end{cases}$$

P r o o f.

(i) See [15].

(ii) By appropriate iteration and by applying Lemma 3.2, we prove it. □

Lemma 3.5.

(i) *Let λ be a real number such that $\lambda > 0$, $\lambda \neq 1$ and $\varphi(\lambda) = \frac{1-\lambda}{1+\lambda}$. A continued fraction expansion of $\ln(\lambda)$ is given by*

$$\ln(\lambda) = \left[0; -\frac{2\varphi(\lambda)}{1}, -\frac{\varphi(\lambda)^2}{3}, \frac{-2^2\varphi(\lambda)^2}{5}, \frac{-n^2\varphi(\lambda)^2}{2n+1}\right]_{n=3}^{+\infty}.$$

(ii) *The simple continued fraction of $\ln(\lambda)$ is*

$$\ln(\lambda) = \left[0; \frac{1}{d_1^*}, \frac{1}{d_2^*}, \ldots, \frac{1}{d_n^*}, \ldots\right]$$

where

$$\begin{cases} d_1^* &= \frac{1}{-2\varphi(\lambda)}, \\ d_2^* &= \dfrac{6}{\varphi(\lambda)}, \\ d_{2k}^* &= \frac{-2(2.4..(2k-2))^2}{(1.3.5...(2k-1))^2}(4k-1), \quad k \geq 2, \\ d_{2k+1}^* &= \frac{(1.3.5...(2k-1))^2}{(2.4...(2k))^2\varphi(\lambda)}(4k+1), \quad k \geq 1. \end{cases}$$

P r o o f.

(i) See [7].

(ii) We deduce it by applying Lemma 3.2. □

The next theorem is a real version of the previous Theorem 3.1.

Theorem 3.6. *With the same notations as bellow, let λ and q be two strictly positive real numbers such that $0 < q < 1$. A continued fraction representation of the real $\lambda^q \ln(\lambda)$ is given by:*

$$\lambda^q \ln(\lambda) = \left[0; \frac{c_1^*+1}{c_1^* d_1^*}, \frac{(c_1^* c_2^*+1)(c_1^*+1)+d_1^* d_2^*}{(c_1^* c_2^*+c_2^*+1)d_2^*}, \frac{(c_1^* d_1^*)^{-1} e_2 f_3}{e_3 - f_3}, \dots \right.$$
$$\left. \dots, \frac{e_{n-1} f_n}{e_n - f_n} \right]_{n=4}^{+\infty},$$

where

$$\begin{cases} e_n &= q_n \widetilde{q}_n \left(q_{n-2} p_{n-1} + \widetilde{q}_{n-1} \widetilde{p}_{n-2}\right), \\ f_n &= q_{n-2} \widetilde{q}_{n-2} \left(q_n p_{n-1} + \widetilde{q}_{n-1} \widetilde{p}_n\right), \end{cases}$$

p_n and q_n (resp. $\widetilde{p}_n$ and $\widetilde{q}_n$) are numerator and denominator of the nth convergent of λ^q (resp. $\ln(\lambda)$). They are defined by

$$\begin{cases} p_n &= c_n^* p_{n-1} + p_{n-2}, \\ q_n &= c_n^* q_{n-1} + q_{n-2}, \end{cases} \qquad \begin{cases} \widetilde{p}_n &= d_n^* \widetilde{p}_{n-1} + \widetilde{p}_{n-2}, \\ \widetilde{q}_n &= d_n^* \widetilde{q}_{n-1} + \widetilde{q}_{n-2}. \end{cases}$$

We apply Lemmas 3.2 3.3 and 3.4 by putting $C = \lambda^q$ and $D = \ln(\lambda)$.

P r o o f o f T h e o r e m 3.1. Let $A \in \mathcal{M}_m$ be a positive definite matrix. Then, there exists an invertible matrix X such that $A = XDX^{-1}$, where

$$D = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_m) \quad \text{and} \quad \lambda_i > 0, \quad \text{for } 1 \leq i < m.$$

As the functions $f(z) = z^q$ and $g(z) = \ln(z)$ are continuous in the open interval $\mathbb{R}_+^*$, then we get

$$A^q \ln(A) = XD^q \ln(D) X^{-1}.$$

Let us define the sequences (P_n) and (Q_n) the numerator and denominator of the nth convergent of $D^q \ln(D)$ by

$$\begin{cases} P_1 &= D_1+I, \\ P_2 &= (E_2'-F_2')(D_1+I), \\ P_3 &= (E_3'-F_3')P_2+(D_1\widetilde{D}_1)^{-1}E_2'F_3'P_1\,, \\ Q_1 &= D_1\widetilde{D}_1, \\ Q_2 &= (E_2'-F_2')Q_1+D_1\widetilde{D}_1F_2'\,, \\ Q_3 &= (E_3'-F_3')Q_2+(D_1\widetilde{D}_1)^{-1}E_2'F_3'Q_1\,, \end{cases}$$

and for $n \geq 4$,

$$\begin{cases} P_n &= (E_n'-F_n')\,P_{n-1}+E_{n-1}'F_n'P_{n-2}\,, \\ Q_n &= (E_n'-F_n')\,Q_{n-1}+E_{n-1}'F_n'Q_{n-2}\,, \end{cases}$$

$$\begin{cases} E_n' &= Q_n\widetilde{Q}_n(Q_{n-2}P_{n-1}+\widetilde{Q}_{n-1}\widetilde{P}_{n-2})\,, \\ F_n' &= Q_{n-2}\widetilde{Q}_{n-2}\left(Q_nP_{n-1}+\widetilde{Q}_{n-1}\widetilde{P}_n\right)\,. \end{cases}$$

The matrices P_n, Q_n (resp. $\widetilde{P}_n, \widetilde{Q}_n$) are the numerator and denominator of the nth convergent of D^q (resp. $\ln(D)$) which are defined by

$$\begin{cases} P_n = D_nP_{n-1}+P_{n-2}\,, \\ Q_n = D_nQ_{n-1}+Q_{n-2}\,, \end{cases} \quad \text{and} \quad \begin{cases} \widetilde{P}_n = \widetilde{D}_n\widetilde{P}_{n-1}+\widetilde{P}_{n-2}\,, \\ \widetilde{Q}_n = \widetilde{d}_n\widetilde{Q}_{n-1}+\widetilde{Q}_{n-2}\,. \end{cases}$$

We recall that

$$\begin{cases} D_1 &= \frac{-I-q\varphi(D)}{2q\varphi(D)}, \\ D_2 &= \frac{6qI}{(q^2-1)\varphi(D)} \\ D_{2k} &= \frac{-2q(q^2-2^2)\ldots(q^2-(2k-2)^2)}{(q^2-1)\ldots(q^2-(2k-1)^2)}(4k-1)I, \qquad k\geq 1 \\ D_{2k+1} &= \frac{-(q^2-1)\ldots(q^2-(2k-1)^2)}{2q(q^2-2^2)\ldots(q^2-4k^2)}(4k+1)\varphi^2(D), \quad k\geq 1 \end{cases}$$

and

$$\begin{cases} \widetilde{D}_1 &= (-2\varphi(D))^{-1}, \\ \widetilde{D}_2 &= \frac{6I}{\varphi(D)}, \\ \widetilde{D}_{2k} &= \frac{-2(2.4\ldots(2k-2))^2}{(1.3.5\ldots(2k-1))^2}(4k-1)I, \quad k\geq 2, \\ \widetilde{D}_{2k+1} &= \frac{(1.3.5\ldots(2k-1))^2}{(2.4\ldots(2k))^2\varphi(D)}(4k+1)I, \quad k\geq 1. \end{cases}$$

We see that P_n, Q_n, E'_n and F'_n are diagonal matrices, so we put

$$\begin{cases} P_n &= \operatorname{diag}(p_n^1, \quad p_n^2, \ldots, p_n^m), \\ Q_n &= \operatorname{diag}(q_n^1, \quad q_n^2, \ldots, q_n^m), \end{cases} \qquad \begin{cases} E'_n &= \operatorname{diag}(e_n^1, \quad e_n^2, \ldots, e_n^m), \\ F'_n &= \operatorname{diag}(f_n^1, \quad f_n^2, \ldots, f_n^m). \end{cases}$$

For each $1 \leq i \leq m$, we obtain

$$\begin{cases} p_1^i = c_{i1}^* + 1, \quad p_2^i = \quad (e_2^i - f_2^i)(c_{i1}^* + 1), \qquad p_3^i = (e_3^i - f_3^i)p_2^i + (c_{i1}^* d_{i1}^*)^{-1} p_1^i \,, \\ q_1^i = c_{i1}^* d_{i1}^* \,, \quad q_2^i = (e_2^i - f_2^i)q_1^i + c_{i1}^* d_{i1}^* f_2^i \,, \quad q_3^i = (e_3^i - f_3^i)q_2^i + (c_{i1}^* d_{i1}^*)^{-1} q_1^i \end{cases}$$

and for $n \geq 4$, we have

$$\begin{cases} p_n^i &= (e_n^i - f_n^i)p_{n-1}^i + e_{n-1}^i f_n^i p_{n-2}^i \,, \\ q_n^i &= (e_n^i - f_n^i)q_{n-1}^i + e_{n-1}^i f_n^i q_{n-2}^i \,, \end{cases}$$

$$\begin{cases} e_n^i &= q_n^i \widetilde{q}_n^i (q_{n-2}^i p_{n-1}^i + \widetilde{q}_{n-1}^i \widetilde{p}_{n-2}^i) \,, \\ f_n^i &= q_{n-2}^i \widetilde{q}_{n-2}^i (q_n^i p_{n-1}^i + \widetilde{q}_{n-1}^i \widetilde{p}_n^i) \,. \end{cases}$$

By Lemma 3.2, we deduce that $\dfrac{p_n^i}{q_n^i}$ converges to $\lambda_i^q \ln(\lambda_i)$ for $1 \leq i \leq m$. It follows that the matrix $\dfrac{P_n}{Q_n}$ converges to $D^q \ln(D)$. So, we get

$$D^q \ln(D) = \left[0; \frac{D_1 + I}{D_1 \widetilde{D}_1}, \frac{(D_1 \widetilde{D}_1)^{-1} E'_2 F'_3}{E'_3 - F'_3}, \frac{E'_{n-1} F'_n}{E'_n - F'_n} \right]_{n=4}^{+\infty}.$$

By Proposition 3.2, we have

$$\begin{aligned} A^q \ln(A) &= X \big(D^q \ln(D) \big) X^{-1} \\ &= \left[0; \frac{(D_1 + I)X^{-1}}{D_1 \widetilde{D}_1 X^{-1}}, \frac{(D_1 \widetilde{D}_1) F'_2 X^{-1}}{E'_2 - F'_2}, \frac{(D_1 \widetilde{D}_1)^{-1} E'_2 F'_3}{E'_3 - F'_3}, \frac{E'_{n-1} F'_n}{E'_n - F'_n} \right]_{n=4}^{+\infty}. \end{aligned}$$

Let us define the sequence $(X_n)_{n \geq -1}$ by $X_{-1} = X_0 = I$, and for all $n \geq 1$, $X_n = X$. Then, we have

$$\begin{cases} \frac{X_1 (D_1 + I) X^{-1} X_{-1}^{-1}}{X_1 D_1 \widetilde{D}_1 X^{-1} X_0^{-1}} &= \frac{A_n + I}{A_1 \widetilde{A}_1}, \\ \frac{X_2 (D_1 \widetilde{D}_1 F'_2 X^{-1}) X_0^{-1}}{X_2 (E'_2 - F'_2) X_1^{-1}} &= \frac{A_1 \widetilde{A}_1 F_2}{E_2 - F_2}, \\ \frac{X_n (E'_{n-1} F'_n) X_{n-2}^{-1}}{X_n (E'_n - F'_n) X_{n-1}^{-1}} &= \frac{E_{n-1} F_n}{E_n - F_n} \end{cases}$$

with

$$X E'_n X^{-1} = E_n \quad \text{and} \quad X F'_n X^{-1} = F_n \quad \text{for all } \ n \geq 2.$$

By applying the result of Proposition 2.3 to the sequence $(X_n)_{n \geq -1}$, we finish the proof of Theorem 3.1 □

3.3. Representation of the generalized operator entropy

Theorem 3.7. *Let A and B be two positive definite matrices in $\mathcal{M}_m$, q a positive real number such that $0 < q < 1$. A continued fraction representation of the generalized operator entropy $S_q(A|B)$ is given by*

$$S_q(A|B) = \left[0; \frac{(A'_1+I)A^{1/2}}{A'_1\widetilde{A'}_1A^{-1/2}}, \frac{((A'_1A_2+I)(A'_1+I)+\widetilde{A'}_1\widetilde{A'}_2)A^{-1/2}}{(A'_1A'_2+A'_2+I)\widetilde{A'}_2}, \right.$$
$$\left. \frac{(A'_1\widetilde{A'}_1)^{-1}E_2F_3}{E_3-F_3}, \frac{E_{n-1}F_n}{E_n-F_n}\right]_{n=4}^{+\infty}$$

where

$$A'_k = A_k, \qquad \widetilde{A'}_k = \widetilde{A}_k,$$

which are defined in Theorem 3.1*, and by replacing $\varphi(A)$ with*

$$\varphi(A,B) = \varphi\big(A^{-1/2}BA^{-1/2}\big) = A^{1/2}\left(\frac{A-B}{A+B}\right)A^{-1/2}.$$

P r o o f. We have

$$\big(A^{-1/2}BA^{-1/2}\big)^q \ln\big(A^{-1/2}BA^{-1/2}\big) = A^{-1/2}S_q(A|B)A^{-1/2}.$$

By applying Theorem 3.1, the continued fraction representation of $S_q(A|B)$ is

$$S_q(A|B) = A^{1/2}\left[0; \frac{A'_1+I}{A'_1\widetilde{A'}_1}, \frac{(A'_1A_2+I)(A'_1+I)+\widetilde{A'}_1\widetilde{A'}_2}{(A'_1A'_2+A'_2+I)\widetilde{A'}_2}, \right.$$
$$\left. \frac{(A'_1\widetilde{A'}_1)^{-1}E_2F_3}{E_3-F_3}, \frac{E_{n-1}F_n}{E_n-F_n}\right]_{n=4}^{+\infty} A^{1/2}.$$

That is

$$S_q(A|B) = \left[0; \frac{(A'_1+I)A^{1/2}}{A'_1\widetilde{A'}_1A^{-1/2}}, \frac{((A'_1A_2+I)(A'_1+I)+\widetilde{A'}_1\widetilde{A'}_2)A^{-1/2}}{(A'_1A'_2+A'_2+I)\widetilde{A'}_2}, \right.$$
$$\left. \frac{(A'_1\widetilde{A'}_1)^{-1}E_2F_3}{E_3-F_3}, \frac{E_{n-1}F_n}{E_n-F_n}\right]_{n=4}^{+\infty}$$

which completes the proof of Theorem 3.7. □

Let $q \in \mathbb{R}$, such that $0 < q < 1$, and let n be an integer. We gave some properties of $S_n(A|B)$, which are shown in [5].

Lemma 3.8. *Let A and B be two positive definite matrices in $\mathcal{M}_m$, $n \in \mathbb{N}^*$, then we have*

$$\begin{aligned} &\text{(i)} && S_n(A|B) &&= (BA^{-1})^n S(A|B) = S(A|B)(A^{-1}B)^n. \\ &\text{(ii)} && S_{2n}(A|B) &&= (BA^{-1})^n S(A|B)(A^{-1}B)^n. \\ &\text{(iii)} && S_{2n+1}(A|B) &&= (BA^{-1})^n S_1(A|B)(A^{-1}B)^n. \end{aligned}$$

The next result gives a continued fraction expansions of the relative operator entropy $S_n(A|B)$.

Corollary 3.9. *Let A and B be two positive definite matrices in $\mathcal{M}_m$, $n \in \mathbb{N}^*$. A continued fraction representation of the generalized operator entropy $S_n(A|B)$ is given by*

$$S_n(A|B) = \left[0; \frac{2(BA^{-1})^n A\left(\frac{A-B}{A+B}\right)}{I}, \frac{-k^2 A\left(\frac{A-B}{A+B}\right)^2 A^{-1}}{(2k+1)I} \right]_{k=1}^{+\infty}.$$

P r o o f. In order to prove Corollary 3.9, we recall a continued fraction expansion of the relative operator entropy $S(A|B)$ (see [7]).

$$S(A|B) = \left[0; \frac{2A\left(\frac{A-B}{A+B}\right)}{I}, \frac{-k^2 A\left(\frac{A-B}{A+B}\right)^2 A^{-1}}{(2k+1)I} \right]_{k=1}^{+\infty}. \qquad \square$$

By applying Proposition 2.2, we complete the proof of Corollary 3.9.

4. The operator divergence

The Bregman operator divergence is $D(A|B) = B - A - S(A|B)$.
We see that its positivity is assured by

$$S(A|B) = A^{1/2}\ln\big(A^{-1/2}BA^{-1/2}\big)A^{1/2} \le A^{1/2}(A^{-1/2}BA^{-1/2} - I)A^{1/2} = B - A.$$

In Isa et al. [5], $D(A|B)$ has been generalized as follows

$$D_q(A|B) = f_{-q}(A|B) - f_q(A|B) - S_q(A|B),$$

where

$$f_q(A|B) = A^{1/2}\big(A^{-1/2}BA^{-1/2}\big)^q A^{1/2}.$$

Theorem 4.1. *Let A and B be two positive definite matrices in $\mathcal{M}_m$, q a positive real number such that $0 < q < 1$. A continued fraction representation of the operator divergence $D_q(A|B)$ is given by*

$$\begin{aligned} D_q(A|B) \;=\; \Big[& A^{-1/2}BA^{-1/2} - I; \\ & \frac{(A'_1 + I + \widetilde{A'}_0\widetilde{A'}_1)A^{1/2}}{A'_1\widetilde{A'}_1A^{-1/2}}, \frac{((A'_1A'_2 + I)(A'_1 + I) + \widetilde{A'}_1\widetilde{A'}_2)A^{-1/2}}{(A'_1A'_2 + A'_2 + I)\widetilde{A'}_2}, \\ & \frac{(A'_1\widetilde{A'}_1)^{-1}E_2F_3}{E_3 - F_3}, \frac{E_{n-1}F_n}{E_n - F_n}\Big]_{n=4}^{+\infty}. \end{aligned}$$

For $k \geq 2$, $\widetilde{A'}_k$ are the same as in Theorem 3.7. The matrices A'_k, E_k and F_k for all $k \geq 1$ are also the same as in Theorem 3.7.

Proof. Since $f_{-q}(A|B) = f_{1+q}(A|B)$, then we have

$$\begin{aligned} D_q(A|B) \;&=\; A^{1/2}\Big(\big(A^{-1/2}BA^{-1/2}\big)^{q+1} - \big(A^{-1/2}BA^{-1/2}\big)^q \\ &\quad -\big(A^{-1/2}BA^{-1/2}\big)^q \ln\big(A^{-1/2}BA^{-1/2}\big)\Big)A^{1/2} \\ &=\; A^{1/2}\big(A^{-1/2}BA^{-1/2}\big)^q \\ &\quad \times\Big(-\ln\big(A^{-1/2}BA^{-1/2}\big) + A^{-1/2}BA^{-1/2} - I\Big)A^{1/2}. \end{aligned}$$

So, the proof of Theorem 4.1 is shown in a similar way as to that of Theorem 3.7 by replacing

$$\ln(A^{-1/2}BA^{-1/2}) \quad \text{with} \quad \Big(-\ln\big(A^{-1/2}BA^{-1/2}\big) + A^{-1/2}BA^{-1/2} - I\Big). \quad \square$$

5. Numerical applications

This paragraph will provide some numerical data to illustrate the preceding results. The focus will be on the results obtained for the generalized operator entropy and the operator divergence. We notice that the software used for numerical calculations is maple as it was the case in [11].

5.1. Numerical example of $A^q \ln(A)$,

We start this section by giving an example to illustrate the theoretical results obtained in Theorems 3.1 and 3.6

Example 1. Table 1 clarifies the differences between $\lambda^q \ln \lambda$ and its firsts convergents F_n when applying Theorem 3.6

TABLE 1.

λ	q	$\lambda^q \ln \lambda - F_1$	$\lambda^q \ln \lambda - F_2$	$\lambda^q \ln \lambda - F_3$	$\lambda^q \ln \lambda - F_4$	$\lambda^q \ln \lambda - F_5$
2	1/2	4.6924×10^{-2}	1.4780×10^{-3}	4.4621×10^{-5}	1.3310×10^{-6}	3.9410×10^{-8}
2	1/3	3.9977×10^{-2}	1.2693×10^{-3}	3.8450×10^{-5}	1.1501×10^{-6}	3.4323×10^{-8}
3	1/2	2.3618×10^{-1}	1.8554×10^{-2}	1.3690×10^{-3}	9.9731×10^{-5}	7.2241×10^{-6}

Now, we pass to a more general case than the previous one, the matrix case.

EXAMPLE 2. Let $A \in \mathcal{M}_m$ be a positive definite matrix such that

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

We calculate the difference between the exact value of $A^q lnA$ and its first convergents. We will take for example $q = \frac{1}{3}$. We get

$$A^q lnA - F_1 = \begin{pmatrix} 8.867877 \times 10^{-2} & 1.379741 \times 10^{-1} \\ 1.379741 \times 10^{-1} & 2.266528 \times 10^{-1} \end{pmatrix},$$

$$A^q lnA - F_2 = \begin{pmatrix} 9.348031 \times 10^{-2} & 1.508712 \times 10^{-2} \\ 1.508712 \times 10^{-2} & 2.443515 \times 10^{-2} \end{pmatrix},$$

$$A^q lnA - F_3 = \begin{pmatrix} 9.361897 \times 10^{-4} & 1.514529 \times 10^{-3} \\ 1.514529 \times 10^{-3} & 2.450718 \times 10^{-3} \end{pmatrix},$$

$$A^q lnA - F_4 = \begin{pmatrix} 9.212614 \times 10^{-5} & 1.490607 \times 10^{-4} \\ 1.490605 \times 10^{-4} & 2.411865 \times 10^{-4} \end{pmatrix},$$

$$A^q lnA - F_5 = \begin{pmatrix} 9.002541 \times 10^{-6} & 1.456551 \times 10^{-5} \\ 1.456532 \times 10^{-5} & 2.356712 \times 10^{-5} \end{pmatrix}.$$

5.2. Numerical example of the generalized operator entropy

Now, we turn to the first main goal of this work, the approximation of the generalized operator entropy by using continued fractions.

EXAMPLE 3. Let A and B be two positive definite matrices in $\mathcal{M}_m$ such that

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 5 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

We calculate the difference between $S_q(A \mid B)$ and its first five convergents.

Taking $q = \frac{1}{3}$ for example, we get the following results

$$F_1 - S_q(A \mid B) = \begin{pmatrix} 5.419098 \times 10^{-2} & -4.186083 \times 10^{-2} & 6.136209 \times 10^{-3} \\ -4.186083 \times 10^{-2} & -9.860205 \times 10^{-2} & -8.974242 \times 10^{-3} \\ 6.136209 \times 10^{-3} & -8.974242 \times 10^{-3} & 4.130494 \times 10^{-2} \end{pmatrix},$$

$$F_2 - S_q(A \mid B) = \begin{pmatrix} 1.377942 \times 10^{-3} & -1.519802 \times 10^{-3} & 2.817516 \times 10^{-4} \\ -1.519802 \times 10^{-3} & -3.845145 \times 10^{-3} & -4.732079 \times 10^{-4} \\ 2.817514 \times 10^{-4} & -4.732083 \times 10^{-4} & 7.862637 \times 10^{-4} \end{pmatrix},$$

$$F_3 - S_q(A \mid B) = \begin{pmatrix} 3.277531 \times 10^{-5} & -5.387752 \times 10^{-5} & 8.625401 \times 10^{-6} \\ -5.387778 \times 10^{-5} & -1.420605 \times 10^{-5} & -1.930841 \times 10^{-5} \\ 8.625333 \times 10^{-6} & -1.930885 \times 10^{-5} & 1.466103 \times 10^{-5} \end{pmatrix},$$

$$F_4 - S_q(A \mid B) = \begin{pmatrix} 7.181323 \times 10^{-6} & -1.90962 \times 10^{-6} & 0.2162 \times 10^{-7} \\ -1.909831 \times 10^{-6} & -5.167555 \times 10^{-6} & -0.727901 \times 10^{-7} \\ 2.161531 \times 10^{-7} & -7.283501 \times 10^{-7} & 2.634102 \times 10^{-7} \end{pmatrix},$$

$$F_5 - S_q(A \mid B) = \begin{pmatrix} 1.312031 \times 10^{-8} & -6.722313 \times 10^{-8} & 6.513041 \times 10^{-9} \\ -6.741832 \times 10^{-8} & -1.865521 \times 10^{-7} & -2.721352 \times 10^{-8} \\ 6.431258 \times 10^{-9} & -2.761852 \times 10^{-8} & -2.135401 \times 10^{-9} \end{pmatrix}.$$

5.3. Numerical example of the operator divergence

We finish this paragraph by illustrating the theoretical results concerning the divergence operator.

We keep the same data as we have in the previous example. Calculating the difference between $D_q(A \mid B)$ and its first five convergents, we get the following results.

Example 4.

$$F_1 - D_q(A \mid B) = \begin{pmatrix} -6.926496 \times 10^{-2} & 3.318154 \times 10^{-2} & -9.988851 \times 10^{-3} \\ 3.318154 \times 10^{-2} & 7.019678 \times 10^{-2} & 4.414054 \times 10^{-3} \\ -9.988850 \times 10^{-3} & 4.414055 \times 10^{-3} & -4.828836 \times 10^{-3} \end{pmatrix},$$

$$F_2 - D_q(A \mid B) = \begin{pmatrix} -1.821578 \times 10^{-3} & 1.179261 \times 10^{-3} & -4.227273 \times 10^{-4} \\ 1.179261 \times 10^{-3} & 2.784671 \times 10^{-3} & 3.187543 \times 10^{-4} \\ -4.227271 \times 10^{-4} & 3.187548 \times 10^{-4} & -9.338488 \times 10^{-4} \end{pmatrix},$$

$$F_3 - D_q(A \mid B) = \begin{pmatrix} -4.588545 \times 10^{-5} & 4.116862 \times 10^{-5} & -1.326901 \times 10^{-6} \\ 4.116972 \times 10^{-4} & 1.035625 \times 10^{-4} & 1.392721 \times 10^{-5} \\ -1.326882 \times 10^{-5} & 1.392771 \times 10^{-5} & -1.801721 \times 10^{-5} \end{pmatrix},$$

$$F_4 - D_q(A \mid B) = \begin{pmatrix} -0.111545 \times 10^{-6} & 1.4436231 \times 10^{-6} & -3.629356 \times 10^{-7} \\ 1.4447147 \times 10^{-6} & 3.785555 \times 10^{-6} & 5.380134 \times 10^{-7} \\ -3.627251 \times 10^{-7} & 5.385451 \times 10^{-7} & -3.501232 \times 10^{-7} \end{pmatrix},$$

$$F_5 - D_q(A \mid B) = \begin{pmatrix} -2.701243 \times 10^{-8} & 0.5062 \times 10^{-8} & -1.002133 \times 10^{-8} \\ 5.072145 \times 10^{-7} & 1.4255 \times 10^{-7} & 2.170121 \times 10^{-8} \\ -9.821384 \times 10^{-8} & 2.222199 \times 10^{-8} & -3.210055 \times 10^{-8} \end{pmatrix}.$$

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