sciendo

TATRA MOUNTAINS
Mathematical Publications

DOI: 10.2478/tmmp-2023-0017
Tatra Mt. Math. Publ. **84** (2023), 77–86

EXPONENTIAL STABILITY OF INTEGRO-DIFFERENTIAL VOLTERRA EQUATION ON TIME SCALES

Urszula Ostaszewska[1] — Ewa Schmeidel[2] — Małgorzata Zdanowicz[2]

[1]Faculty of Mathematics, University of Bialystok, Białystok, POLAND

[2]Institute of Computer Science, University of Bialystok, Białystok, POLAND

ABSTRACT. We study the Volterra integro-differential equation on time scales and provide sufficient conditions for boundness of all solutions of considered equation. Using that result, we present the conditions for exponential stability of considered equation. All the results proved on the general time scale include results for both integral and discrete Volterra equations.

1. Introduction

We consider a Volterra integro-dynamic equation of the following form

$$x^{\Delta}(t) = a(t)x(t) + \int_{t_0}^{t} A(t,s)x(s)\Delta s + f(t), \tag{1}$$

$t \in [t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \cap \mathbb{T}$, where $\mathbb{T}$ is a time scale. Functions $a, f : \mathbb{T} \to \mathbb{R}$ and $A : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ are given and we are interested to give sufficient conditions on these functions to obtain the exponential stability property of the above equation.

In the past, two types of mathematical equations were used to describe various dynamic processes: differential and integral equations, and difference and summation equations. The first one models phenomena in the continuous time, the second one in the discrete time. However, it is becoming evident that certain

© 2023 Mathematical Institute, Slovak Academy of Sciences.
2020 Mathematics Subject Classification: 34N05, 45D05.
Keywords: time scales, exponential stability, integro-differential Volterra equation.
Licensed under the Creative Commons BY-NC-ND 4.0 International Public License.

phenomena do not involve purely continuous or discrete aspects. Instead, they feature elements of both. An emerging area that has the potential to manage the above situations effectively is the field of dynamic equations on time scales. Providing a wide perspective, the time scales theory also leads to new results. Furthermore, since many time scales differ from reals and integers, studying dynamic equations on time scales leads to a more general and comprehensive theory.

The readers are referred to the books [4, 5] for the fundamentals of the time scales. Basic qualitative and quantitative results on Volterra equations on time scales with applications can be found in [10–12], and in the discrete case in [6, 13–15, 18], and the references cited therein. There is an interesting topic in mathematical modelling to study consensus on time scales. This problem concerns on stability, in particular exponential stability, of considered equation (see, for example, [8, 9, 16, 18]).

2. Preliminaries

Let us recall that a time scale $\mathbb{T}$ is a nonempty, closed subset of $\mathbb{R}$ with the topology inherited from the standard one in $\mathbb{R}$. The mapping $\sigma : \mathbb{T} \to \mathbb{T}$, defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, where $\inf \emptyset = \sup \mathbb{T}$, is called the forward jump operator. A point $t \in \mathbb{T}$ is called right-dense if $\sigma(t) = t$ and $t < \sup \mathbb{T}$. If $\sigma(t) > t$ then t is right-scattered. In a similar way, we define the backward jump operator (denoted by ρ), left-scattered and left-dense points. The function $\mu : \mathbb{T} \to [0, \infty)$ given by $\mu(t) = \sigma(t) - t$ is called the graininess function. The delta (or Hilger) derivative of $f : \mathbb{T} \to \mathbb{R}$ at a point $t \in \mathbb{T}^\kappa$, where

$$\mathbb{T}^\kappa := \begin{cases} \mathbb{T} \setminus \big(\rho(\sup \mathbb{T}), \sup \mathbb{T}\big] & \text{if} \quad \sup \mathbb{T} < \infty, \\ \mathbb{T} & \text{if} \quad \sup \mathbb{T} = \infty, \end{cases}$$

is defined in the following way.

Definition 2.1. The delta derivative of function f at a point t, denoted by $f^\Delta(t)$, is the number (provided it exists) such that given any $\varepsilon > 0$ there is a neighbourhood U of t with

$$\big|\big[f\big(\sigma(t)\big) - f(s)\big] - f^\Delta(t)[\sigma(t) - s]\big| \leq \varepsilon|\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

We say that a function f is delta differentiable on $\mathbb{T}^\kappa$ if $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is then called the (delta) derivative of f on $\mathbb{T}^\kappa$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at each right-dense point of $\mathbb{T}$ and it has finite left-sided limit at each left-dense point of $\mathbb{T}$ and the set of rd-continuous functions we denote by $C_{rd}(\mathbb{T}, \mathbb{R})$. We say that $F : \mathbb{T} \to \mathbb{R}$ is antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided for all $t \in \mathbb{T}^{\kappa}$ holds $F^{\Delta} = f$. Let us emphasize that each rd-continuous function has its antiderivative.

Definition 2.2. Let F be antiderivative of f and $s, t \in \mathbb{T}$. The delta integral of f is defined by

$$\int_s^t f(r)\Delta r = F(t) - F(s).$$

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ we will denote by $\mathcal{R}$. For $p, q \in \mathcal{R}$ we define circle addition $\oplus$ by

$$p \oplus q := p + q + \mu pq.$$

$p, q \in \mathcal{R}$. It is easy to check that $(\mathcal{R}, \oplus)$ is an Abelian group, and if $p \in \mathcal{R}$, then the inverse element is given by $\ominus p = \frac{-p}{1+\mu p}$. Moreover, if we define circle subtraction $\ominus$ by $p \ominus q := p \oplus (\ominus q)$ then $p \ominus q = \frac{p-q}{1+\mu q}$. The subset (in fact, the subgroup) of positively regressive functions can be distinguished in $\mathcal{R}$ as follows

$$\mathcal{R}^{+} = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \quad \text{for all} \quad t \in \mathbb{T}\}.$$

Let $p \in \mathcal{R}$. The exponential function $e_p(t, t_0), t \in \mathbb{T}$ is defined as the unique solution of the initial value problem of the form

$$x^{\Delta} = p(t)x, \qquad x\big(t_0\big) = 1.$$

Example. Let $p \in \mathcal{R}$ and $\alpha \in \mathbb{R}$. If $\mathbb{T} = \mathbb{R}$, then $e_p(t, t_0) = \exp\big(\int_{t_0}^t p(s)ds\big)$. In particular, if $p \equiv \alpha$, then $e_\alpha(t, t_0) = e^{\alpha(t-s)}$. If $\mathbb{T} = \mathbb{N}$, then $e_p(n, 1) = \prod_{k=1}^{n-1}\big(1 + p(k)\big)$. And for $p \equiv \alpha$ we have $e_\alpha(n, 1) = (1 + \alpha)^{n-1}$.

In the following theorem, we present some basic properties of the exponential function that will be needed later on in this paper.

Theorem 2.3. *Let $p, q \in \mathcal{R}$, then the following hold*

(i) $e_0(t, s) \equiv 0$ *and* $e_p(t, t) \equiv 0$;

(ii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$;

(iii) $e_p(t, s)e_p(s, r) = e_p(t, r)$;

(iv) $e_p(t, s)e_q(t, s) = e_{p\oplus q}(t, s)$;

(v) *if* $p \in \mathcal{R}^{+}$, *then* $e_p(t, t_0) > 0$ *for all* $t \in \mathbb{T}$;

(vi) $\int_a^b p(t)e_p\big(c, \sigma(t)\big)\Delta t = e_p(c, a) - e_p(c, b)$ *for all* $a, b, c \in \mathbb{T}$;

(vii) *if* $p \in \mathcal{R}^+$ *and* $p(t) \leq q(t)$ *for all* $t \in \mathbb{T}$, *then*

$$e_p(t,s) \leq e_q(t,s) \quad \text{for all} \quad t \geq s;$$

For the proof of the properties (i)–(vi) see [4, Theorems 2.36, 2.39, 2.48] and for the proof of (vii) see [2, Corollary 2.11]. The following Remark describes the monotonicity of the exponential function.

Remark 1. Let $p(t) < 0$ for all $t \in \mathbb{T}$ and $p \in \mathcal{R}^+$. Then the following statements hold:

(i) The function $e_p(t,s)$ is increasing with respect to the variable s.

(ii) The function $e_p(t,s)$ is decreasing with respect to the variable t.

We finish this section by recalling the theorem related to the properties of the delta integral.

Theorem 2.4. *If* $a, b \in \mathbb{T}$ *and* $f : \mathbb{T} \to \mathbb{R}$ *is a rd-continuous function such that* $f(t) \geq 0$ *for all* $a \leq t < b$, *then*

$$\int_a^b f(t)\Delta t \geq 0.$$

3. Exponential stability of the Volterra equation

Let $t_0 \in \mathbb{T}$, where $\mathbb{T}$ is unbounded above (i.e., $\sup \mathbb{T} = \infty$), in this case $\mathbb{T}^\kappa = \mathbb{T}$.

We consider the equation (1) with the initial condition

$$x(t_0) = x_0 \tag{2}$$

and assume that

$$a \in \mathcal{R}, A \in C_{rd}(\mathbb{T} \times \mathbb{T}, \mathbb{R}) \quad \text{and} \quad f \in C_{rd}(\mathbb{T}, \mathbb{R}). \tag{A}$$

Definition 3.1. Equation (1) is said to be exponentially stable if there exist positive constants M and d such that for any solution of the corresponding homogeneous equation with the initial condition $x(t_0) = x_0$ the following inequality holds

$$|x(t)| \leq M|x_0|e_{-d}(t,t_0).$$

If assumption (A) holds, then the first order nonhomogenous dynamic linear equation corresponding to equation (1) of the form

$$x^\Delta(t) = a(t)x(t) + f(t). \tag{3}$$

is called regressive and by variation of constants formula (see [4, Theorem 2.77]) the unique solution of this equation with the initial condition (2) is given by

$$x(t) = e_a(t,t_0)x_0 + \int_{t_0}^{t} e_a\big(t,\sigma(r)\big)f(r)\Delta r. \tag{4}$$

One can check that the solution of equation (1) satisfies the following

$$\begin{aligned} x(t) = e_a(t,t_0)x_0 &+ \int_{t_0}^{t} e_a\big(t,\sigma(r)\big)\int_{t_0}^{r} A(r,s)x(s)\Delta s\Delta r \\ &+ \int_{t_0}^{t} e_a\big(t,\sigma(r)\big)f(r)\Delta r, \end{aligned} \tag{5}$$

where $x_0 = x(t_0)$.

Lemma 3.2. *Assume that $a(t) \le 0$ for all $t \in \mathbb{T}$. If $a \in \mathcal{R}^+$ then the following inequalities hold*

$$-1 \le \int_{t_0}^{t} e_a\big(t,\sigma(r)\big)a(r)\Delta r \le 0 \tag{6}$$

and $e_a(t,t_0) \in [0,1]$.

Proof. Consider equation (3) with $f(t) = -a(t)$ and $x_0 = 1$. Note that in this case the function $x(t) \equiv 1$ is a solution of the initial value problem given by (3) and (2). Hence, by (4), we immediately obtain that

$$1 = e_a(t,t_0) - \int_{t_0}^{t} e_a\big(t,\sigma(r)\big)a(r)\Delta r.$$

Due to Theorem 2.3, we can write

$$e_a(t,t_0) = e_a\big(t,\sigma(r)\big)e_a\big(\sigma(r),t_0\big).$$

Since $a \in \mathcal{R}^+$ we have that $e_a(t,t_0) > 0$ for all $t \in \mathbb{T}$ and $e_a\big(\sigma(r),t_0\big) > 0$ for all $r \in \mathbb{T}$. It follows that, in particular, $e_a\big(t,\sigma(r)\big) > 0$ for all $r \in [t_0,t)_{\mathbb{T}}$. This finishes the proof since function a takes nonpositive values. □

Remark 2. If $\mathbb{T} = \mathbb{R}$ then every continuous function $a : \mathbb{R} \to \mathbb{R}$ is contained in $\mathcal{R}^+$. If $\mathbb{T} = \mathbb{N}$ then the function $a : \mathbb{N} \to \mathbb{R}$ satisfies the assumption of the above Lemma provided

$$-1 < a(n) \le 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

Now we are in a position to present sufficient conditions on the boundness of solutions of equation (1).

Theorem 3.3. *Let assumption* (A) *holds. Assume* $a \in \mathcal{R}^+$ *such that* $a(t) \leq -\alpha < 0$ *for all* $t \in \mathbb{T}$, *where* $\alpha > 0$. *Moreover, assume that there exists constant* $c \in (0,1)$ *such that*

$$\sup_{t\in\mathbb{T}} \frac{1}{|a(t)|} \int_{t_0}^{t} |A(t,s)|\Delta s \leq c. \tag{7}$$

If the function f *is bounded, then all solutions of* (1) *are bounded.*

Proof. For bounded function f exists positive constant F such that $|f(t)| \leq F$ for all $t \in \mathbb{T}$. Let $x : \mathbb{T} \to \mathbb{R}$ be a solution of the initial value problem (1), (2). Since x satisfies (5) then for any $t > t_0$, by Lemma 3.2, we have

$$\begin{aligned} |x(t)| &\leq |x_0| + \int_{t_0}^{t} e_a\big(t,\sigma(r)\big) \int_{t_0}^{r} |A(r,s)x(s)|\Delta s\Delta r \\ &\quad + \int_{t_0}^{t} e_a\big(t,\sigma(r)\big)|f(r)|\Delta r \end{aligned}$$

and in consequence

$$\begin{aligned} |x(t)| &\leq |x_0| + \int_{t_0}^{t} e_a\big(t,\sigma(r)\big)|a(r)| \int_{t_0}^{r} \frac{|A(r,s)|}{|a(r)|} \sup_{s\in[t_0,t]_{\mathbb{T}}} |x(s)|\Delta s\Delta r \\ &\quad + F\int_{t_0}^{t} \frac{1}{|a(r)|}\, e_a\big(t,\sigma(r)\big)|a(r)|\Delta r \\ &\leq |x_0| + c \sup_{s\in[t_0,t]_{\mathbb{T}}} |x(s)| \int_{t_0}^{t} e_a\big(t,\sigma(r)\big)|a(r)|\Delta r \\ &\quad + \frac{F}{\alpha}\int_{t_0}^{t} e_a\big(t,\sigma(r)\big)|a(r)|\Delta r. \end{aligned}$$

Applying Lemma 3.2 we get

$$|x(t)| \leq |x_0| + c \sup_{s\in[t_0,t]_{\mathbb{T}}} |x(s)| + \frac{F}{\alpha}.$$

Now, we can conclude that

$$\sup_{s\in[t_0,t]_{\mathbb{T}}} |x(s)| \leq |x_0| + c \sup_{s\in[t_0,t]_{\mathbb{T}}} |x(s)| + \frac{F}{\alpha},$$

what gives

$$\sup_{s\in[t_0,t]_{\mathbb{T}}} |x(s)| \leq \frac{|x_0| + \frac{F}{\alpha}}{1-c}.$$

Since it holds for an arbitrary $t > t_0$, the proof is completed. □

EXAMPLE. Let $\mathbb{T}$ be the time scale such that $0 \leq \mu(t) \leq 1$ for all $t \in \mathbb{T}$. We consider the following equation

$$x^{\Delta}(t) = -\frac{x(t)}{\mu^2(t)+1} + \frac{1}{t^3+1}\int_2^{\infty} \big(s+\sigma(s)\big)x(s)\Delta s + \frac{\sin t}{\mu(t)+1}. \tag{8}$$

Observe that function $a(t) = -\frac{1}{\mu^2(t)+1}$ satisfies the assumptions of Theorem 3.3 with constant $\alpha = \frac{1}{2}$. Note that function $f(t) = \frac{\sin t}{\mu(t)+1}$ is bounded and rd-continuous. Moreover, $A(t,s) = \frac{s+\sigma(s)}{t^3+1} \in C_{rd}(\mathbb{T}\times\mathbb{T},\mathbb{R})$ and one can check that

$$\sup_{t\in\mathbb{T}} \frac{1}{|a(t)|}\int_2^t \Big|\frac{s+\sigma(s)}{t^3+1}\Big| ds \leq \frac{8}{9}.$$

Hence all assumptions of Theorem 3.3 are fulfilled, and it follows that all solutions of (8) are bounded.

Remark 3. If $\mathbb{T} = \mathbb{N}$, then (8) takes the form

$$\Delta x(n) = -\frac{1}{2}x(n) + \frac{1}{n^3+1}\sum_{i=2}^{n-1}(2i-1)x(i) + \frac{\sin n}{2}.$$

If $\mathbb{T} = \mathbb{R}$, then (8) takes the form

$$x'(t) = -x(t) + \frac{2}{t^2+1}\int_2^t sx(s)ds + \sin t.$$

If $\mathbb{T} = \frac{1}{3}\mathbb{N}$, then (8) takes the form

$$x^{\Delta}(n) = \frac{9}{10} + \frac{3}{n^3+27}\sum_{k=3}^{3n-1}(k+1)x(k) + \frac{3}{4}\sin\frac{n}{3}.$$

We present our main theorem, which is a generalization of results obtained in [3].

THEOREM 3.4. *Let* (A) *hold. Assume that the following conditions are satisfied:*

(A1) *there exists* $\lambda \in \mathbb{R}^+$ *with property that* $-\lambda \in \mathcal{R}^+$ *and positive constant* L *such that*

$$|A(r,s)| \leq Le_{(-\lambda)\oplus(-\lambda)}\big(\sigma(r),t_0\big), \quad r \geq s \geq t_0,$$

(A2) $a \in \mathcal{R}^+$ *and there exists positive constant* α *such that* $-\alpha \in \mathcal{R}^+$ *and*

$$a(t) \leq -\alpha < 0, \tag{9}$$

(A3) $\alpha < \lambda$,

(A4) f *is a bounded function.*

Then equation (1) *is exponentially stable.*

P r o o f. Let $x : \mathbb{T} \to \mathbb{R}$ be a solution of homogeneous equation corresponding to equation (1). Note that if f is bounded, then x is bounded and satisfies

$$x(t) = x_0 e_a(t, t_0) + \int_{t_0}^{t} e_a\big(t, \sigma(r)\big) \int_{t_0}^{r} A(r, s)x(s)\Delta s \Delta r.$$

Therefore for an arbitrary $t > t_0$ we obtain

$$|x(t)| \le T e_a\big(t, t_0\big) + T \int_{t_0}^{t} e_a\big(t, \sigma(r)\big) \int_{t_0}^{r} |A(r, s)| \Delta s \Delta r, \tag{10}$$

where $\sup_{t\in\mathbb{T}} |x(t)| \le T$. Observe now that by property of exponential function and by Remark 1 for $r \ge s \ge t_0$ we have

$$\begin{aligned} |A(r, s)| &\le L e_{-\lambda}\big(\sigma(r), t_0\big) e_{-\lambda}\big(\sigma(r), t_0\big) \\ &\le L e_{-\lambda}\big(\sigma(r), t_0\big) e_{-\lambda}\big(r, \sigma(s)\big). \end{aligned}$$

Applying the above estimation to (10) gives

$$|x(t)| \le T e_a(t, t_0) + TL \int_{t_0}^{t} e_a\big(t, \sigma(r)\big) \int_{t_0}^{r} e_{-\lambda}\big(\sigma(r), t_0\big) e_{-\lambda}\big(r, \sigma(s)\big) \Delta s \Delta r.$$

From (9) and Theorem 2.3 it follows that

$$e_a(t, t_0) \le e_{-\alpha}(t, t_0) \quad \text{for all} \quad t \ge t_0$$

and

$$e_a\big(t, \sigma(r)\big) \le e_{-\alpha}\big(t, \sigma(r)\big) \quad \text{for all} \quad t \ge \sigma(r).$$

Then, in consequence, we get

$$|x(t)| \le T e_{-\alpha}(t, t_0) + TL \int_{t_0}^{t} e_{-\alpha}\big(t, \sigma(r)\big) e_{-\lambda}\big(\sigma(r), t_0\big) \int_{t_0}^{r} e_{-\lambda}\big(r, \sigma(s)\big) \Delta s \Delta r$$

Using properties of the exponential function, we obtain

$$\begin{aligned} &e_{-\alpha}\big(t, \sigma(r)\big) e_{-\lambda}\big(\sigma(r), t_0\big) \\ &= e_{-\alpha}\big(t, \sigma(r)\big) e_{-\alpha}\big(\sigma(r), t_0\big) e_{\ominus(-\alpha)}\big(\sigma(r), t_0\big) e_{-\lambda}\big(\sigma(r), t_0\big) \\ &= e_{-\alpha}(t, t_0) e_{(-\lambda)\ominus(-\alpha)}\big(\sigma(r), t_0\big). \end{aligned}$$

By Theorem 2.3 we have

$$\int_{t_0}^{r} e_{-\lambda}\big(r, \sigma(s)\big) \Delta s = \frac{1}{\lambda}\big(1 - e_{-\lambda}(r, t_0)\big).$$

Therefore,

$$|x(t)| \leq Te_{-\alpha}(t,t_0) + \frac{TL}{\lambda}\, e_{-\alpha}(t,t_0) \int_{t_0}^{t} e_{(-\lambda)\ominus(-\alpha)}\big(\sigma(r),t_0\big)\big(1 - e_{-\lambda}(r,t_0)\big)\Delta r$$

$$\leq Te_{-\alpha}(t,t_0) + \frac{TL}{\lambda}\, e_{-\alpha}(t,t_0) \int_{t_0}^{t} e_{(-\lambda)\ominus(-\alpha)}\big(\sigma(r),t_0\big)\Delta r$$

since the expression $e_{(-\lambda)\ominus(-\alpha)}\big(\sigma(r),t_0\big)e_{-\lambda}(r,t_0)$ takes positive values. From the assumptions on constant λ and α it follows that $\big((-\lambda)\ominus(-\alpha)\big)(t) < 0$ for all $t \in \mathbb{T}$. Applying Remark 1, we obtain

$$|x(t)| \leq Te_{-\alpha}(t,t_0) + \frac{TL}{\lambda} e_{-\alpha}(t,t_0) \int_{t_0}^{t} e_{(-\lambda)\ominus(-\alpha)}(r,t_0)\Delta r$$

Obviously,

$$\int_{t_0}^{t} e_{(-\lambda)\ominus(-\alpha)}(r,t_0)\Delta r = \frac{1}{(-\lambda)\ominus(-\alpha)}\Big(e_{(-\lambda)\ominus(-\alpha)}(t,t_0) - 1\Big)$$

and it leads to the following estimation

$$|x(t)| \leq T\left(1 + \frac{L}{\lambda|(-\lambda)\ominus(-\alpha)|}\right) e_{-\alpha}(t,t_0).$$

Hence, applying that $-\alpha \in \mathcal{R}^+$ and assumption (A3), we obtain

$$|x(t)| \leq T\left(1 + \frac{L}{\lambda(\lambda-\alpha)}\right) e_{-\alpha}(t,t_0),$$

which ends the proof. □

REFERENCES

[1] ADIVAR, M.—RAFFOUL, Y. N.: *Existence of resolvent for Volterra integral equations on time scales*, Bull. Austral. Math. Soc. **82** (2010), no. 1, 139–155.

[2] AKIN-BOHNER, E.—BOHNER, M.—AKIN, F.: *Pachpatte inequalities on time scales*, J. Inequal. Pure Appl. Math. **6** (2005), no. 1, Article 6.

[3] BEREZANSKY, L.—MIGDA, M.—SCHMEIDEL, E.: *Some stability conditions for scalar Volterra difference equations*, Opuscula Math. **36** (2016), no. 4, 459–470.

[4] BOHNER, M.—PETERSON, A.: *Dynamic Equations On Time Scales*. Birkhäuser, Basel, 2001.

[5] BOHNER, M.—PETERSON, A.: *Advances in Dynamic Equations on Time Scales*. Birkhäuser, Boston, Ma, 2003.

[6] DIBLÍK, J.—RŮŽIČKOVÁ, M.—SCHMEIDEL, E.: *Asymptotically periodic solutions of Volterra difference equations*, Tatra Mt. Math. Publ. **43** (2009), 51–61.

[7] DU, N. H.—TIEN, L. H.: *On the exponential stability of dynamic equations on time scales*, J. Math. Anal. Appl. **331** (2007), 1159–1174.

[8] GIREJKO, E., MACHADO, L.—MALINOWSKA, A. B.—MARTINS, N.: *Krause's model of opinion dynamics on isolated time scales*, Math. Methods Appl. Sci. **39** (2016), no. 18, 5302–5314.

[9] GIREJKO, E.—MACHADO, L.—MALINOWSKA, A. B.—MARTINS, N.: *On consensus in the Cucker-Smaletype model on isolated time scale*, Discrete Contin. Dyn. Syst. **11** (2018), no. 1, 77–89.

[10] KARPUZ, B.: *Volterra theory on time scales*, Results Math. **65** (2014), no. 3–4, 263–292.

[11] KULIK, T.—TISDELL, C. C.: *Volterra integral equations on time scales: basic qualitative and quantitative results with applications to initial value problems on unbounded domains*, Int. J. Difference Equ. **3** (2008), no. 1, 103–133.

[12] MESSINA, E.—VECCHIO, A.: *Stability and Convergence of Solutions to Volterra Integral Equations on Time Scales*, Discrete Dyn. Nat. Soc. **2015**, Article ID 612156, 6 pp.

[13] MIGDA, J.—MIGDA, M.: *Asymptotic behavior of solutions of discrete volterra equations*, Opuscula Math. **36** (2016), no. 1, 265–278.

[14] MIGDA, M.—DUTKIEWICZ, A.: *Asymptotic behavior of solutions of second-order difference equations of Volterra type*, Turkish J. Math. **43** (2019), no. 5, 2203–2217.

[15] MIGDA, M.—RŮŽIČKOVÁ, M.—SCHMEIDEL, E.: *Boundedness and stability of discrete Volterra equations*, Adv. Difference Equ. **47** (2015), 1–11.

[16] OSTASZEWSKA, U.—SCHMEIDEL, E.—ZDANOWICZ, M.: *Emergence of consensus of multi-agents systems on time scales*, Miskolc Math. Notes **20**, no. 2 (2019), 1201–1214.

[17] SANTOS, I. L. D.: *On Volterra integral equations on time scales*, Mediterr. J. Math. **12** (2015), no. 2, 471–480.

[18] SCHMEIDEL, E.—OSTASZEWSKA, U.—ZDANOWICZ, M.: *Exponentially stable solution of mathematical model of agents dynamics on time scales*, Adv. Difference Equ. **231** (2019), 1–19.

Received October 25, 2022

Urszula Ostaszewska
Faculty of Mathematics
University of Bialystok
Ciołkowskiego St. 1M
15-245 Białystok
POLAND
E-mail: uostasze@math.uwb.edu.pl

Ewa Schmeidel
Małgorzata Zdanowicz
Institute of Computer Science
University of Bialystok
Ciołkowskiego St. 1M
15-245 Białystok
POLAND
E-mail: e.schmeidel@uwb.edu.pl
mzdan@math.uwb.edu.pl