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INTEGRABILITY AND NON-EXISTENCE OF PERIODIC ORBITS FOR A CLASS OF KOLMOGOROV SYSTEMS

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ABSTRACT. In this article, we study the integrability and the non-existence of periodic orbits for the planar Kolmogorov differential systems of the form

$$\dot{x} = x\big(R_{n-1}(x,y) + P_n(x,y) + S_{n+1}(x,y)\big),$$

$$\dot{y} = y\big(R_{n-1}(x,y) + Q_n(x,y) + S_{n+1}(x,y)\big),$$

where n is a positive integer, R_{n-1}, P_n, Q_n and S_{n+1} are homogeneous polynomials of degree $n-1$, n, n and $n+1$, respectively. Applications of Kolmogorov systems can be found particularly in modeling population dynamics in biology and ecology.

1. Introduction and statement of the main results

The Lotka-Volterra systems, which are a class of planar polynomial differential systems of degree 2, were introduced independently by Alfred J. Lotka in 1925 [14] and Vito Volterra in 1926 [17], who initially proposed these models for studying the interactions between two species. Kolmogorov [9] in 1936 extended these systems to arbitrary dimension and degree, and expressed

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by the following differential equations of the form

$$\frac{dx_i}{dt} = x_i\, f_i(x_1, \ldots, x_n), \quad i = 1, 2, \ldots, n, \tag{1}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}^n$ are polynomials of arbitrary degree.

The Lotka-Volterra and Kolmogorov systems have been applied to model different natural phenomena, such as the time evolution of conflicting species in biology [1, 15], physics [11], chemical reactions [7], hydrodynamics [5], fluid dynamics [4], economics [16], etc.

In this paper, we put our attention to the following class of Kolmogorov systems

$$\begin{cases} \dot{x} = x\big(R_{n-1}(x,y) + P_n(x,y) + S_{n+1}(x,y)\big), \\ \dot{y} = y\big(R_{n-1}(x,y) + Q_n(x,y) + S_{n+1}(x,y)\big), \end{cases} \tag{2}$$

where n is a positive integer, R_{n-1}, P_n, Q_n and S_{n+1} are homogeneous polynomials of degree $n-1$, n, n and $n+1$, respectively. The interest in the class of Kolmogorov systems comes from two main reasons: one is its simplicity as system (2) can be written as the Riccati equation by doing a change of variables to polar coordinates, and the other is that system (2) has been used to model a large variety of real-world phenomena and as a test bed for studies on the integrability problem, see [2, 3, 8, 12, 13] for instance.

Before presenting our result, we recall the definition of the *first integral* and *invariant algebraic curve*. Let $\mathcal{W} \subseteq \mathbb{R}^2$ be an open set. A non constant $\mathcal{C}^1$ function $H : \mathcal{W} \to \mathbb{R}$ is called a first integral of system (2) in $\mathcal{W}$, if H is constant on all the orbits $\big(x(t), y(t)\big)$ of system (2) contained in $\mathcal{W}$, namely, if

$$\frac{dH}{dt} = \dot{x}\frac{\partial H}{\partial x} + \dot{y}\frac{\partial H}{\partial y} = 0,$$

in the points of $\mathcal{W}$. For more details of the definition, we refer the reader to [6, 10, 18].

Let $F = F(x, y) \not\equiv 0$ be a real polynomial. We say that an algebraic curve $F(x, y) = 0$ is the *invariant algebraic curve* of the polynomial differential system (2) if for some polynomial $K = K(x, y)$ we have

$$\dot{x}\frac{\partial F}{\partial x} + \dot{y}\frac{\partial F}{\partial y} = KF. \tag{3}$$

On the points of the algebraic curve $F = 0$, the gradient $(\partial F/\partial x, \partial F/\partial y)$ of F is orthogonal to the vector field $(\dot{x}, \dot{y})$. Hence, at every point of $F = 0$, the vector field $(\dot{x}, \dot{y})$ is tangent to the curve $F = 0$, that is to say the curve $F = 0$ is formed by trajectories of the vector field $(\dot{x}, \dot{y})$. There is a strong relation between the invariant algebraic curves and the Darboux theory of integrability, see for instance the Chapter 8 of [6].

The main goal of this paper is to investigate the integrability and the periodic orbits of the Kolmogorov system (2). In order to simplify our notations in the statement of the main result, we define the trigonometric functions:

$$\begin{aligned} f(\theta) &= Q_n(\cos\theta,\sin\theta)\sin^2\theta + P_n(\cos\theta,\sin\theta)\cos^2\theta,\\ g(\theta) &= (Q_n(\cos\theta,\sin\theta) - P_n(\cos\theta,\sin\theta))\cos\theta\sin\theta,\\ A(\theta) &= \tfrac{S_{n+1}(\theta)}{g(\theta)},\ B(\theta) = \tfrac{f(\theta)}{g(\theta)},\ D(\theta) = \tfrac{R_{n-1}(\theta)}{g(\theta)},\\ U(\theta) &= B(\theta) + 2A(\theta)r_1(\theta), \end{aligned} \tag{4}$$

where

$$S_{n+1}(\theta) = S_{n+1}(\cos\theta,\sin\theta), \quad R_{n-1}(\theta) = R_{n-1}(\cos\theta,\sin\theta), \quad \text{and} \quad r_1(\theta)$$

is particular solution of Riccati equation.

The main result of this paper is stated as follows.

Theorem 1.1. *The following statements hold for the Kolmogorov system* (2)

(a) *If* $g(\theta) \not\equiv 0$*, then system* (2) *has the first integral*

$$H(x,y) = \frac{e^{\int^{\arctan(y/x)} U(s)ds}}{\sqrt{x^2+y^2} - r_1(x,y)} + \int^{\arctan(y/x)} e^{\int^s U(v)dv} A(s)ds,$$

where $g(\theta) \neq 0$.

(b) *If* $g(\theta) \equiv 0$*, then system* (2) *has the first integral* $H(x,y) = y/x$.

(c) *System* (2) *has no periodic orbits.*

2. Proof of Theorem 1.1

The proof of Theorem 1.1 can be done by using a similar ideal from the arguments of the main result of Llibre and Salhi [13, Theorem 1]. But the proof presented here contains some new technical treatments.

Before starting the proof of Theorem 1.1, we need the following lemma.

Lemma 2.1. *The curve* $Q_n(x,y) - P_n(x,y) = 0$ *is an invariant algebraic curve of system* (2).

P r o o f. We need to prove that $F(x,y) = Q_n(x,y) - P_n(x,y) = 0$ is an invariant algebraic curve of the differential system (2). Indeed, we have

$$\begin{aligned}\dot{x}\frac{\partial F}{\partial x} + \dot{y}\frac{\partial F}{\partial y} &= x(R_{n-1} + P_n + S_{n+1})\frac{\partial F}{\partial x} + y(R_{n-1} + Q_n + S_{n+1})\frac{\partial F}{\partial y} \\ &= x(R_{n-1} + S_{n+1})\frac{\partial F}{\partial x} + y(R_{n-1} + S_{n+1})\frac{\partial F}{\partial y} \\ &\quad + xP_n\frac{\partial F}{\partial x} + yQ_n\frac{\partial F}{\partial y}.\end{aligned}$$

Taking into account the following inequities:

$$x\frac{\partial P_n}{\partial x} + y\frac{\partial P_n}{\partial y} = nP_n \quad \text{and} \quad x\frac{\partial Q_n}{\partial x} + y\frac{\partial Q_n}{\partial y} = nQ_n,$$

and noting the Euler's theorem for homogeneous functions, we obtain

$$\begin{aligned}&x(R_{n-1} + S_{n+1})\frac{\partial F}{\partial x} + y(R_{n-1} + S_{n+1})\frac{\partial F}{\partial y} \\ &= (R_{n-1} + S_{n+1})\left(x\frac{\partial Q_n}{\partial x} + y\frac{\partial Q_n}{\partial y} - x\frac{\partial P_n}{\partial x} - y\frac{\partial P_n}{\partial y}\right) \\ &= (R_{n-1} + S_{n+1})(nQ_n - nP_n) \\ &= n(R_{n-1} + S_{n+1})F.\end{aligned}$$

On the other hand, we have the inequities:

$$x\frac{\partial Q_n}{\partial x} = nQ_n - y\frac{\partial Q_n}{\partial y} \quad \text{and} \quad y\frac{\partial P_n}{\partial y} = nP_n - x\frac{\partial P_n}{\partial x}.$$

A direct computation shows that

$$\begin{aligned}xP_n\frac{\partial F}{\partial x} + yQ_n\frac{\partial F}{\partial y} &= P_n\left(x\frac{\partial Q_n}{\partial x} - x\frac{\partial P_n}{\partial x}\right) + Q_n\left(y\frac{\partial Q_n}{\partial y} - y\frac{\partial P_n}{\partial y}\right) \\ &= P_n\left(nQ_n - y\frac{\partial Q_n}{\partial y} - x\frac{\partial P_n}{\partial x}\right) \\ &\quad + Q_n\left(y\frac{\partial Q_n}{\partial y} - nP_n + x\frac{\partial P_n}{\partial x}\right) \\ &= \left(y\frac{\partial Q_n}{\partial y} + x\frac{\partial P_n}{\partial x}\right)(Q_n - P_n) \\ &= \left(y\frac{\partial Q_n}{\partial y} + x\frac{\partial P_n}{\partial x}\right)F.\end{aligned}$$

In short, we have

$$\dot{x}\frac{\partial F}{\partial x}+\dot{y}\frac{\partial F}{\partial y}=\left(n(R_{n-1}+S_{n+1})+y\frac{\partial Q_n}{\partial y}+x\frac{\partial P_n}{\partial x}\right)F.$$

Therefore, $F=0$ is an invariant algebraic curve of the polynomial differential system (2). Hence, Lemma 2.1 is proved. □

P r o o f o f T h e o r e m 1.1 (a). Doing the change of variables $x=r\cos\theta$ and $y=r\sin\theta$ system (2) becomes

$$\begin{cases}\dot{r}=R_{n-1}(\theta)r^n+f(\theta)r^{n+1}+S_{n+1}(\theta)r^{n+2},\\ \dot{\theta}=g(\theta)r^n,\end{cases}\tag{5}$$

where the functions $f(\theta)$ and $g(\theta)$ are given in (4).

If $g(\theta)\not\equiv 0$ and taking θ as new independent variable, then we write the differential system (5) in the form of a differential equation, we get the following Riccati equation

$$\frac{dr}{d\theta}=A(\theta)r^2+B(\theta)r+L(\theta),\tag{6}$$

where

$$A(\theta)=\frac{S_{n+1}(\theta)}{g(\theta)},\quad B(\theta)=\frac{f(\theta)}{g(\theta)},\quad \text{and}\quad L(\theta)=\frac{R_{n-1}(\theta)}{g(\theta)}.$$

If $L(\theta)=0$, then the Riccati equation (6) becomes a special case of the Bernoulli equation, and if further $A(\theta)=0$, it becomes a linear first order differential equation. These two cases have been studied by Llibre and Salhi in [13], see the proof of Theorem 1 in [13] for more details.

We now assume that system (6) has one particular solution $r_1(\theta)$, we want to find the general form of its solutions. Let

$$r(\theta)=r_1(\theta)+\frac{1}{u(\theta)},\tag{7}$$

where $u(\theta)$ is unknown function. Substituting (7) into (6) and after a simple calculation we obtain

$$\frac{du}{d\theta}+U(\theta)u+A(\theta)=0,\tag{8}$$

where $U(\theta)=B(\theta)+2A(\theta)r_1(\theta)$. Solving this equation, we have

$$u(\theta)=e^{-\int^{\theta}U(s)ds}\left(-\int^{\theta}e^{\int^{s}U(v)dv}A(s)ds+k\right),$$

where k is an arbitrary constant. Therefore, we obtain

$$r(\theta)=r_1(\theta)+\frac{e^{\int^{\theta}U(s)ds}}{(-\int^{\theta}e^{\int^{s}U(v)dv}A(s)ds+k)}.$$

Going back to the cartesian coordinates, we obtain the following first integral of system (2)

$$H(x,y) = \frac{e^{\int^{\arctan(y/x)} U(s)ds}}{\sqrt{x^2+y^2}-r_1} + \int^{\arctan(y/x)} e^{\int^s U(v)dv} A(s)ds. \qquad \square$$

Proof of Theorem 1.1 (b). Suppose that $g(\theta) \equiv 0$. Then, according to system (5) we obtain that $\dot{\theta} = 0$. It is obvious that the straight lines through the origin of coordinates of system (2) are invariant by the flow of this system. Therefore, $H = y/x$ is a first integral of the system. The proof of statement (b) of Theorem 1.1 is completed. $\square$

Proof of Theorem 1.1 (c). It is easy to check that the equilibrium points of system (2) are located at $(0,0)$, or $(x,0)$ or $(0,y)$, or in somewhere of the open four quadrants. Clearly, the axes x and y are formed by trajectories of system (2), surrounding the equilibria located on these axes cannot be periodic orbits. Let γ be a periodic orbit surrounding an equilibrium which located in one of the open quadrants, and let $k_\gamma = H(\gamma)$.

Assume that $g(\theta) \not\equiv 0$, then the curves $H = k$ with $k \in \mathbb{R}$, which are formed by the trajectories of the differential system (2), can be written as

$$r(\theta) = r_1(\theta) + \frac{e^{\int^\theta U(s)ds}}{\left(-\int^\theta e^{\int^s U(v)dv} A(s)ds + k\right)},$$

where $g(\theta) \neq 0$. Therefore the periodic orbit γ is contained in the curve

$$r(\theta) = r_1(\theta) + \frac{e^{\int^\theta U(s)ds}}{\left(-\int^\theta e^{\int^s U(v)dv} A(s)ds + k_\gamma\right)}.$$

But this curve cannot contain the periodic orbit γ contained in one of the open quadrants because this curve at most have a unique point on every ray $\theta = \theta^*$ for all $\theta^* \in [0, 2\pi)$. When there exists θ^* such that $g(\theta^*) = 0$, from Lemma 2.1 we know that the curve $y = \tan(\theta^*)x$ is an invariant algebraic curve for system (2), so that the curve is not a periodic orbit. This complete the proof of statement (c). $\square$

3. Illustrative Examples

The following two examples are provided to illustrate our results.

Example. If we choose

$$R_1(x,y) = x - y, \qquad P_2(x,y) = -x^2, \qquad Q_2(x,y) = -xy,$$

and

$$S_3(x,y) = x^2 y,$$

then system (2) is reduced to

$$\begin{cases} \dot{x} = x\left(x - y - xy + x^2 y\right), \\ \dot{y} = y\left(x - y + x^2 + x^2 y\right). \end{cases} \tag{9}$$

Writing system (9) to polar coordinates by using $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\begin{cases} \dot{r} = R_1(\theta) r^2 + f(\theta)\, r^3 + S_3(\theta)\, r^4, \\ \dot{\theta} = g(\theta)\, r^2, \end{cases} \tag{10}$$

where

$$R_1(\theta) = \cos\theta - \sin\theta, \quad f(\theta) = -\overset{4}{\cos}\theta - \overset{3}{\sin}\theta\,\cos\theta, \quad S_3(\theta) = \overset{2}{\cos}\theta\,\sin\theta,$$

and

$$g(\theta) = \cos\theta\,\sin\theta\left(\overset{2}{\cos}\theta - \sin\theta\cos\theta\right).$$

By taking θ as the new independent variable, system (10) can be written in the form of the following Riccati differential equation

$$\frac{dr}{d\theta} = A(\theta) r^2 + B(\theta) r + L(\theta), \tag{11}$$

where

$$A(\theta) = \frac{1}{\cos\theta - \sin\theta},$$
$$L(\theta) = \frac{1}{\cos^2\theta\,\sin\theta},$$

and

$$B(\theta) = -\frac{\csc\theta\,\sec\theta\left(\cos^4\theta + \cos\theta\,\sin^3\theta\right)}{\cos^2\theta - \cos\theta\,\sin\theta}.$$

One can easily verify that $r_1(\theta) = 1/\cos\theta$ is a particular solution of equation (11). Letting

$$r(\theta) = \frac{1}{\cos(\theta)} + \frac{1}{u(\theta)}, \tag{12}$$

where $u(\theta)$ is unknown function, then equation (11) becomes

$$\frac{du}{d\theta} + U(\theta) u + A(\theta) = 0,$$

where

$$U(\theta) = \frac{2}{\cos^2\theta - \cos\theta\sin\theta} - \frac{\csc\theta\,\sec\theta\left(\cos^4\theta + \cos\theta\,\sin^3\theta\right)}{\cos^2\theta - \cos\theta\,\sin\theta}.$$

The general solution of this equation is

$$u(\theta) = \frac{c_1 \sqrt{\tan^2\theta + 1}}{(\tan\theta - 1)\tan\theta} + \frac{\sqrt{\tan^2\theta + 1}\,\sqrt{\frac{1}{\cos^2\theta}}\,(\sin\theta - \theta\,\cos\theta + \cos\theta\,\ln(\cos\theta))}{(\tan\theta - 1)\tan\theta}, \tag{13}$$

where c_1 is an arbitrary constant. Substituting (13) into (12) and going back to the cartesian coordinates, we obtain the following first integral of system (9)

$$H(x,y) = -\frac{w(x,y)\left(x^4 - y^4 + x^2y^2\,v(x,y)\,\eta(x,y) - (xy - y^2)\left(x^2 - y^2(v(x,y)\,\eta(x,y) - 1)\right)\right)}{2x^2y(x-y)\,\eta(x,y)\big(v(x,y) - \eta(x,y)\big)},$$

where

$$v(x,y) = \sqrt{x^2 + y^2}, \qquad \eta(x,y) = \sqrt{1 + \frac{y^2}{x^2}},$$

and

$$w(x,y) = \log\left(\frac{x - y}{x\sqrt{1 + \frac{y^2}{x^2}}}\right) - \log\left(\frac{y}{x\sqrt{1 + \frac{y^2}{x^2}}}\right).$$

It follows from statement (c) of Theorem 1.1 that system (9) has no periodic orbits.

Example. If we take

$$R_1(x,y) = x + y, \qquad P_2(x,y) = -xy, \qquad Q_2(x,y) = -xy,$$

and

$$S_3(x,y) = x^3 + y^3,$$

then system (2) is reduced to

$$\begin{cases} \dot{x} = x\left(x + y - xy + x^3 + y^3\right), \\ \dot{y} = y\left(x + y - xy + x^3 + y^3\right). \end{cases} \tag{14}$$

In polar coordinates (r,θ) it takes the form

$$\begin{cases} \dot{r} = R_1(\theta)\,r^2 + f(\theta)\,r^3 + S_3(\theta)\,r^4, \\ \dot{\theta} = g(\theta)r^2, \end{cases}$$

where

$$R_1(\theta) = \cos\theta + \sin\theta, \qquad f(\theta) = -\cos\theta\,\sin\theta, \qquad S_3(\theta) = \overset{3}{\cos}\,\theta + \overset{3}{\sin}\,\theta$$

and

$$g(\theta) \equiv 0.$$

Therefore, according to Theorem 1.1, system (14) has the first integral

$$H = x/y$$

and also has no periodic orbits.

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