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OSCILLATORY BEHAVIOUR OF SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH MIXED NEUTRAL TERMS

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ABSTRACT. The authors examine the oscillation of second-order nonlinear differential equations with mixed nonlinear neutral terms. They present new oscillation criteria that improve, extend, and simplify existing ones in the literature. The results are illustrated by some examples.

1. Introduction

This paper is concerned with the oscillatory behaviour of solutions of the second-order nonlinear differential equations with mixed neutral terms of the form

$$\left(a(t)\left(y'(t)\right)^{\alpha}\right)' + q(t)x^{\gamma}(\tau(t)) + c(t)x^{\mu}(\omega(t)) = 0 \tag{1}$$

and

$$\left(a(t)\left(y'(t)\right)^{\alpha}\right)' = q(t)x^{\gamma}(\tau(t)) + c(t)x^{\mu}(\omega(t)), \tag{2}$$

where $y(t) = x(t) + p_1(t)x^{\beta}(\sigma(t)) - p_2(t)x^{\delta}(\sigma(t))$ and $t \geq t_0 > 0$. Throughout this paper, we always assume that the following conditions are satisfied:

(i) α, β, γ, μ, and δ are the ratios of odd positive integers with $0 < \beta < 1$ and $\delta > 1$;

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(ii) $a : [t_0, \infty) \to (0, \infty)$ is a continuous function such that

$$A(t, t_0) := \int_{t_0}^{t} a^{-1/\alpha}(s)\, \mathrm{d}s \to \infty \quad \text{as } t \to \infty; \tag{3}$$

(iii) p_1, p_2, $q, c : [t_0, \infty) \to (0, \infty)$ are continuous functions;

(iv) τ, σ, $\omega : [t_0, \infty) \to \mathbb{R}$ are continuous functions such that $\tau(t) \le t$, $\sigma(t) \le t$, $\omega(t) \ge t$, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \lim_{t\to\infty} \omega(t) = \infty$.

By a *solution* of equation (1) (resp. (2)), we mean a function $x \in C\big([t_x, \infty), \mathbb{R}\big)$ for some $t_x \ge t_0$ such that $y \in C^1\big([t_x, \infty), \mathbb{R}\big)$, $a\,(y')^{\alpha} \in C^1\big([t_x, \infty), \mathbb{R}\big)$, and satisfies (1) (respectively (2)) on $[t_x, \infty)$. We only consider those solutions of (1) (respectively (2)) that exist on some half-line $[t_x, \infty)$ and satisfy the condition

$$\sup\{|x(t)| : T_1 \le t < \infty\} > 0 \quad \text{for any} \quad T_1 \ge t_x.$$

Moreover, we tacitly assume that (1) (respectively (2)) possesses such solutions. Such a solution $x(t)$ of either (1) or (2) is said to be *oscillatory* if it has arbitrarily large zeros on $[t_x, \infty)$, i.e., for any $t_1 \in [t_x, \infty)$ there exists $t_2 \ge t_1$ such that $x(t_2) = 0$; otherwise it is called *nonoscillatory*, i.e., if it is eventually positive or eventually negative. Equation (1) or (2) is said to be oscillatory if all its solutions are oscillatory.

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various differential equations, and we refer the reader to the monographs [2, 3], and the papers [1, 4–10, 14–22] and the references cited therein. With respect to neutral differential equations, the qualitative study of such equations has, besides its theoretical interest, significant practical importance. This is due to the fact that neutral differential equations arise in various phenomena including problems concerning electric networks containing lossless transmission lines (as in high-speed computers where such lines are used to interconnect switching circuits), in the study of vibrating masses attached to an elastic bar, and in the solution of vibrational problems with time delays. We refer the reader to Hale's monograph [11] for further applications in science and technology.

In reviewing the literature, it becomes apparent that results on the oscillatory behaviour of the second-order differential equations with a single sublinear neutral term are relatively scarce. For an important initial contribution for such equations, we may refer to [1, 6]. By using Riccati type transformations, in [1], the authors obtain some oscillation results for (1) in the case where $\alpha = \gamma = 1$, $p_2(t) = 0$, and $c(t) = 0$. On the other hand, the authors in [6] established some new results for the case where $\alpha = 1$, $p_2(t) = 0$, and $c(t) = 0$.

However, to the best of our knowledge, there are few results dealing with the oscillation of mixed neutral differential equations with both sublinear and superlinear neutral terms; for example, see [8], where an equation with nonnegative neutral terms was considered. The aim of the present paper is to initiate the study of the oscillatory behaviour of (1) and to provide new results that extend, generalize, and simplify existing ones in the literature, and to analyse the oscillatory and asymptotic behaviour of solutions of the corresponding equation (2) with mixed neutral terms and again under condition (3) with $\beta < 1$ and $\delta > 1$.

2. Oscillation of equation (1)

We begin with the following lemma that is essential in the proofs of our theorems.

Lemma 2.1 ([12]). *If X and Y are nonnegative, then*

$$X^\lambda + (\lambda - 1)Y^\lambda - \lambda X Y^{\lambda-1} \geq 0 \quad \text{for} \qquad \lambda > 1 \tag{4}$$

and

$$X^\lambda - (1 - \lambda)Y^\lambda - \lambda X Y^{\lambda-1} \leq 0 \quad \text{for} \quad 0 < \lambda < 1, \tag{5}$$

where equality holds if and only if $X = Y$.

For notational purposes, we let

$$A(v, u) := \int_u^v a^{-1/\alpha}(s)\, \mathrm{d}s,$$

and for any function $p \in C\big([t_0, \infty), \mathbb{R}\big)$, we set

$$g_1(t) := (\delta - 1)\delta^{\delta/(1-\delta)} p^{\delta/(\delta-1)}(t) p_2^{1/(1-\delta)}(t),$$

$$g_2(t) := (1 - \beta)\beta^{\beta/(1-\beta)} p^{\beta/(\beta-1)}(t) p_1^{1/(1-\beta)}(t),$$

$$Q(t) := \frac{q(t)}{(p_2(h_1(t)))^{\gamma/\delta}},$$

and

$$C(t) := \frac{c(t)}{(p_2(h_2(t)))^{\mu/\delta}},$$

where $h_1(t) = \sigma^{-1}(\tau(t)) \leq t$ with $\lim_{t\to\infty} h_1(t) = \infty$, $h_2(t) = \sigma^{-1}(\omega(t))$ with $\lim_{t\to\infty} h_2(t) = \infty$, $h_2'(t) > 0$, and σ^{-1} is the inverse function of σ.

Our first main result is contained in the following theorem for Eq. (1).

Theorem 2.2. *Let conditions* (i)–(iv) *and* (3) *hold. Assume that there exist functions* $p \in C\big([t_0,\infty),\mathbb{R}\big)$, $\varphi \in C\big([t_0,\infty),\mathbb{R}\big)$, *and a nondecreasing function* $\xi \in C\big([t_0,\infty),\mathbb{R}\big)$ *such that*

$$\lim_{t\to\infty}[g_1(t)+g_2(t)] = 0, \tag{6}$$

$$\varphi(t) \le t \quad and \quad \rho(t) := h_2(\varphi(t)) \ge t \tag{7}$$

and

$$h_1(t) \le \xi(t) \le t. \tag{8}$$

If there is a $\kappa_0 \in (0,1)$ *such that the first-order delay differential inequality*

$$Z'(t) + \kappa_0^{\gamma} q(t) A^{\gamma}\big(\tau(t),t_0\big) Z^{\gamma/\alpha}(\tau(t)) \le 0, \tag{9}$$

the first-order delay differential inequality

$$W'(t) + Q(t) A^{\gamma/\delta}\big(\xi(t),h_1(t)\big) W^{\gamma/\alpha\delta}(\xi(t)) \le 0, \tag{10}$$

and the first-order advanced differential inequality

$$z'(t) - \left(\frac{1}{a(t)}\int_{\varphi(t)}^{t} C(s)\,\mathrm{d}s\right)^{1/\alpha} z^{\mu/\alpha\delta}(\rho(t)) \ge 0 \tag{11}$$

have no positive solutions, then equation (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution to the equation (1), say $x(t) > 0$, $x(\sigma(t)) > 0$, $x(\tau(t)) > 0$ and $x(\omega(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. The proof if $x(t)$ is eventually negative is similar, so we omit the details of that case here as well as in the remaining proofs in this paper. Then, it follows from (1) that

$$\big(a(t)\left(y'(t)\right)^{\alpha}\big)' = -q(t)x^{\gamma}(\tau(t)) - c(t)x^{\mu}(\omega(t)) \le -q(t)x^{\gamma}(\tau(t)) < 0, \tag{12}$$

for $t \ge t_1$, and hence $a(t)\left(y'(t)\right)^{\alpha}$ is decreasing and eventually does not change its sign, say on $[t_2,\infty)$ for some $t_2 \ge t_1$. Therefore, $y'(t)$ eventually has a fixed sign on $[t_2,\infty)$, and so we shall distinguish the following four cases:

$$\text{(I)}\quad y(t) > 0 \quad \text{and} \quad y'(t) < 0, \qquad \text{(II)}\quad y(t) > 0 \quad \text{and} \quad y'(t) > 0,$$

$$\text{(III)}\quad y(t) < 0 \quad \text{and} \quad y'(t) > 0, \qquad \text{(IV)}\quad y(t) < 0 \quad \text{and} \quad y'(t) < 0.$$

First, we consider case (I). Since $y'(t) < 0$ and $a(t)\left(y'(t)\right)^{\alpha}$ is decreasing for $t \ge t_2$, we see that, for $c_1 > 0$,

$$a(t)\left(y'(t)\right)^{\alpha} \le a(t_2)\left(y'(t_2)\right)^{\alpha} := -c_1 < 0.$$

Integrating the last inequality from t_2 to t and taking (3) into account, we conclude that $\lim_{t\to\infty} y(t) = -\infty$, which contradicts the fact that $y(t)$ is eventually positive.

Next, we consider case (II). From the definition of $y(t)$, we have

$$x(t) = y(t) - \left[p(t)x(\sigma(t)) - p_2(t)x^{\delta}(\sigma(t))\right] - \left[p_1(t)x^{\beta}(\sigma(t)) - p(t)x(\sigma(t))\right]. \quad (13)$$

Applying (4) to $\left[p(t)x(\sigma(t)) - p_2(t)x^{\delta}(\sigma(t))\right]$ with

$$\lambda = \delta > 1,\ X = p_2^{1/\delta}(t)x(\sigma(t)) \quad \text{and} \quad Y = \left(\frac{1}{\delta}p(t)p_2^{-1/\delta}(t)\right)^{1/(\delta-1)},$$

we see that

$$\left[p(t)x(\sigma(t)) - p_2(t)x^{\delta}(\sigma(t))\right] \le (\delta-1)\delta^{\delta/(1-\delta)}p^{\delta/(\delta-1)}(t)p_2^{1/(1-\delta)}(t) := g_1(t). \quad (14)$$

Applying (5) to $\left[p_1(t)x^{\beta}(\sigma(t)) - p(t)x(\sigma(t))\right]$ with

$$\lambda = \beta < 1,\ X = p_1^{1/\beta}(t)x(\sigma(t)) \quad \text{and} \quad Y = \left(\frac{1}{\beta}p(t)p_1^{-1/\beta}(t)\right)^{1/(\beta-1)},$$

we see that

$$\left[p_1(t)x^{\beta}(\sigma(t)) - p(t)x(\sigma(t))\right] \le (1-\beta)\beta^{\beta/(1-\beta)}p^{\beta/(\beta-1)}(t)p_1^{1/(1-\beta)}(t) := g_2(t). \quad (15)$$

Using (14) and (15) in (13) gives

$$x(t) \ge \left[1 - \frac{g_1(t)+g_2(t)}{y(t)}\right]y(t) \quad \text{for } t \ge t_2. \quad (16)$$

Since $y(t) > 0$ and $y'(t) > 0$ on $[t_2, \infty)$, there exist $t_3 \ge t_2$ and a constant $c_2 > 0$ such that $y(t) \ge c_2$ for $t \ge t_3$, and so, inequality (16) can be written as

$$x(t) \ge \left[1 - \frac{g_1(t)+g_2(t)}{c_2}\right]y(t) \quad \text{for } t \ge t_3. \quad (17)$$

Now, in view of (6), for any $\kappa \in (0,1)$ there exists $t_\kappa \ge t_3$ such that

$$x(t) \ge \kappa y(t) \quad \text{for } t \ge t_\kappa. \quad (18)$$

Fix $\kappa \in (0,1)$ and choose t_κ by (18). Since $\lim_{t\to\infty}\tau(t) = \infty$, we can choose $t_5 \ge t_\kappa$ such that $\tau(t) \ge t_\kappa$ for all $t \ge t_5$. Thus, from (18) we have

$$x(\tau(t)) \ge \kappa y(\tau(t)) \quad \text{for } t \ge t_5. \quad (19)$$

Using (19) in (12) yields

$$\left(a(t)\left(y'(t)\right)^{\alpha}\right)' + \kappa^{\gamma}q(t)y^{\gamma}(\tau(t)) \le 0 \quad \text{for } t \ge t_5. \quad (20)$$

Since $y(t)$ is positive and $a(t)\left(y'(t)\right)^{\alpha}$ is decreasing for $t \ge t_5$, we see that

$$\begin{aligned} y(t) &\ge \int_{t_5}^{t} a^{-1/\alpha}(s)\left(a^{1/\alpha}(s)y'(s)\right)\mathrm{d}s \\ &\ge A(t,t_5)\left(a(t)\left(y'(t)\right)^{\alpha}\right)^{1/\alpha}. \end{aligned} \quad (21)$$

Letting $Z(t) = a(t)\left(y'(t)\right)^{\alpha} > 0$, inequality (21) takes the form

$$y(t) \geq A(t, t_5) Z^{1/\alpha}(t). \tag{22}$$

Using (22) in (20), we see that Z is a positive solution of the inequality

$$Z'(t) + \kappa^{\gamma} q(t) A^{\gamma}(\tau(t), t_5) Z^{\gamma/\alpha}(\tau(t)) \leq 0, \tag{23}$$

i.e., inequality (9) has a positive solution, which contradicts our assumption.

Next, we consider the cases where $y(t) < 0$ for $t \geq t_2$, i.e., cases (III) and (IV). Letting $z(t) = -y(t)$, then from the definition of $y(t)$, we see that

$$z(t) = -y(t) = -x(t) - p_1(t) x^{\beta}(\sigma(t)) + p_2(t) x^{\delta}(\sigma(t)) \leq p_2(t) x^{\delta}(\sigma(t)),$$

from which, we obtain

$$x(\sigma(t)) \geq \left(\frac{z(t)}{p_2(t)} \right)^{1/\delta},$$

or

$$x(t) \geq \left(\frac{z(\sigma^{-1}(t))}{p_2(\sigma^{-1}(t))} \right)^{1/\delta} \quad \text{for } t \geq t_2. \tag{24}$$

In case (III), we have

$$z(t) = -y(t) > 0 \quad \text{for } t \geq t_2, \quad \text{so} \quad z'(t) = -y'(t) < 0 \quad \text{for } t \geq t_2.$$

Now, it follows from (12) that

$$\left(a(t) \left(z'(t)\right)^{\alpha} \right)' \geq q(t) x^{\gamma}(\tau(t)) \quad \text{for } t \geq t_2. \tag{25}$$

Using (24) in (25) gives

$$\left(a(t) \left(z'(t)\right)^{\alpha} \right)' \geq Q(t) z^{\gamma/\delta}(h_1(t)) \quad \text{for } t \geq t_3, \tag{26}$$

where $\tau(t) \geq t_2$ for $t \geq t_3$ for some $t_3 \geq t_2$. Now, for $t_3 \leq u \leq v$, we may write

$$z(u) - z(v) = -\int_u^v a^{-1/\alpha}(s) \left(a(s) \left(z'(s)\right)^{\alpha} \right)^{1/\alpha} \mathrm{d}s \geq A(v, u) \left(-a^{1/\alpha}(v) z'(v) \right).$$

Letting $u = h_1(t)$ and $v(t) = \xi(t)$ in the last inequality, we see that

$$z(h_1(t)) \geq A\big(\xi(t), h_1(t)\big) \left(-a^{1/\alpha}(\xi(t)) z'(\xi(t)) \right). \tag{27}$$

Using (27) in (26) gives

$$\left(a(t) \left(z'(t)\right)^{\alpha} \right)' \geq Q(t) \Big[A\big(\xi(t), h_1(t)\big) \left(-a^{1/\alpha}(\xi(t)) z'(\xi(t)) \right) \Big]^{\gamma/\delta} \quad \text{for } t \geq t_3. \tag{28}$$

Letting $W(t) = a(t)\left(-z'(t)\right)^{\alpha} > 0$, we see that $W(t)$ is a positive solution of the first-order delay differential inequality

$$W'(t) + Q(t) A^{\gamma/\delta}\big(\xi(t), h_1(t)\big) W^{\gamma/\alpha\delta}(\xi(t)) \leq 0, \tag{29}$$

which contradicts the assumption that (10) has no positive solutions.

Finally, we consider case (IV). Since $y(t) < 0$ for $t \geq t_2$, as in the above, letting $z(t) = -y(t) > 0$ for $t \geq t_2$, we again arrive at (24). Using (24) in (1), we obtain

$$\begin{aligned}\big(a(t)\,(z'(t))^{\alpha}\big)' &= q(t)x^{\gamma}(\tau(t)) + c(t)x^{\mu}(\omega(t)) \\ &\geq Q(t)z^{\gamma/\delta}(h_1(t)) + C(t)z^{\mu/\delta}(h_2(t)) \\ &\geq C(t)z^{\mu/\delta}(h_2(t)). \end{aligned} \tag{30}$$

Integrating (30) from $\varphi(t)$ to t, we see that $z(t)$ is a positive solution of the first-order advanced inequality

$$z'(t) \geq \left(\frac{1}{a(t)} \int_{\varphi(t)}^{t} C(s)\,\mathrm{d}s\right)^{1/\alpha} z^{\mu/\alpha\delta}(\rho(t)), \tag{31}$$

which contradicts the assumption on inequality (11), and completes the proof of the theorem. $\square$

Corollary 2.3. *Let conditions* (i)–(iv) *and* (3) *hold. Assume that there exist functions* $p \in C\big([t_0,\infty),\mathbb{R}\big)$, $\varphi \in C\big([t_0,\infty),\mathbb{R}\big)$, *and nondecreasing function* $\xi \in C\big([t_0,\infty),\mathbb{R}\big)$ *such that* (6)–(8) *are satisfied. If*

$$\int_{t_0}^{\infty} q(s)A^{\gamma}\big(\tau(s),t_0\big)\,\mathrm{d}s = \infty \qquad \textit{if} \quad \gamma < \alpha, \tag{32}$$

$$\int_{t_0}^{\infty} Q(s)A^{\gamma/\delta}\big(\xi(s),h_1(s)\big)\,\mathrm{d}s = \infty \qquad \textit{if} \quad \gamma < \alpha\delta, \tag{33}$$

and

$$\int_{t_0}^{\infty} \left(\frac{1}{a(u)} \int_{\varphi(u)}^{u} C(s)\,\mathrm{d}s\right)^{1/\alpha} \mathrm{d}u = \infty \qquad \textit{if} \quad \mu > \alpha\delta, \tag{34}$$

then equation (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution to the equation (1), say $x(t) > 0$ $x(\sigma(t)) > 0$, $x(\tau(t)) > 0$ and $x(\omega(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Proceeding as in the proof of Theorem 2.2, we again arrive at (23) for $t \geq t_5$, (29) for $t \geq t_3$, and (31) for $t \geq t_2$, respectively. Using the fact that $Z(t) = a(t)\,(y'(t))^{\alpha}$ is positive and decreasing, and noting that $\tau(t) \leq t$, we have

$$Z(\tau(t)) \geq Z(t),$$

and so, inequality (23) can be written as

$$Z'(t) + \kappa^\gamma q(t) A^\gamma\big(\tau(t), t_5\big) Z^{\gamma/\alpha}(t) \le 0,$$

or

$$\frac{Z'(t)}{Z^{\gamma/\alpha}(t)} + \kappa^\gamma q(t) A^\gamma\big(\tau(t), t_5\big) \le 0 \quad \text{for } t \ge t_5. \tag{35}$$

An integration of (35) from t_5 to ∞ gives

$$\int_{t_5}^{\infty} q(s) A^\gamma\big(\tau(s), t_5\big)\, \mathrm{d}s \le \frac{1}{\kappa^\gamma} \frac{Z^{1-\frac{\gamma}{\alpha}}(t_5)}{1-\frac{\gamma}{\alpha}} < \infty,$$

which contradicts (32). Using similar arguments, the remainder of proof follows from the facts that $\xi(t) \le t$, $\rho(t) \ge t$, and inequalities (29) and (31); we omit the details. □

3. Oscillation of equation (2)

In this section we examine the behaviour of solutions of equation (2).

Theorem 3.1. *Let conditions* (i)–(iv) *and* (3) *hold. Assume that there exist functions $p \in C\big([t_0,\infty),(0,\infty)\big)$ and $\eta \in C\big([t_0,\infty),\mathbb{R}\big)$ such that* (6) *holds,*

$$\eta(t) \le t, \quad \omega'(t) \ge 0 \quad \textit{and} \quad \pi(t) := \omega(\eta(t)) \ge t, \tag{36}$$

and either

$$\int_{t_0}^{\infty} q(s)\, \mathrm{d}s = \infty \quad \textit{or} \quad \int_{t_0}^{\infty} \left(\frac{1}{a(u)} \int_u^{\infty} q(s)\, \mathrm{d}s \right)^{1/\alpha} \mathrm{d}u = \infty, \tag{37}$$

are satisfied. In addition, if there is a $\kappa_0 \in (0,1)$ such that the first-order advanced differential inequality

$$z'(t) - \kappa_0^{\mu/\alpha} \left(\frac{1}{a(t)} \int_{\eta(t)}^{t} c(s)\, \mathrm{d}s \right)^{1/\alpha} z^{\mu/\alpha}(\pi(t)) \ge 0, \tag{38}$$

and the first-order delay differential inequality

$$Z'(t) + Q(t) A^{\gamma/\delta}\big(h_1(t), t_0\big) Z^{\gamma/\alpha\delta}(h_1(t)) \le 0, \tag{39}$$

have no positive solutions. If $x(t)$ is a solution to the equation (2), *then either $x(t)$ is oscillatory or* $\liminf_{t\to\infty} |x(t)| = 0$.

Proof. Let $x(t)$ be a nonoscillatory solution to the equation (2), say $x(t) > 0$, $x(\sigma(t)) > 0$, $x(\tau(t)) > 0$, and $x(\omega(t)) > 0$ for $t \geq t_1$, for some $t_1 \geq t_0$. Then, it follows from (2) that

$$\left(a(t)\left(y'(t)\right)^{\alpha}\right)' = q(t)x^{\gamma}(\tau(t)) + c(t)x^{\mu}(\omega(t)) \geq q(t)x^{\gamma}(\tau(t)) > 0, \tag{40}$$

for $t \geq t_1$, and hence $a(t)\left(y'(t)\right)^{\alpha}$ is increasing and eventually does not change its sign on $[t_2, \infty)$ for some $t_2 \geq t_1$. Therefore, $y'(t)$ eventually has a fixed sign on $[t_2, \infty)$, and so we shall distinguish the following four cases:

$$\text{(I)} \quad y(t) > 0 \quad \text{and} \quad y'(t) < 0, \qquad \text{(II)} \quad y(t) > 0 \quad \text{and} \quad y'(t) > 0,$$

$$\text{(III)} \quad y(t) < 0 \quad \text{and} \quad y'(t) > 0, \qquad \text{(IV)} \quad y(t) < 0 \quad \text{and} \quad y'(t) < 0.$$

First, we consider case (I): In this case, we claim that $\liminf_{t\to\infty} x(t) = 0$. To prove this, we assume that there exists a constant $b > 0$ such that $x(t) > b$. Using this in (40), we see that

$$\left(a(t)\left(y'(t)\right)^{\alpha}\right)' \geq b^{\gamma}q(t). \tag{41}$$

Integrating (41) from t_2 to ∞, we see that

$$\int_{t_2}^{\infty} q(s)\,\mathrm{d}s \leq \frac{-a(t_2)(y'(t_2))^{\alpha}}{b^{\gamma}} < \infty,$$

which contradicts the first part of (37). If we integrate (41) from t to u and letting $u \to \infty$, we obtain

$$-y'(t) \geq \left[b^{\gamma}\frac{1}{a(t)}\int_{t}^{\infty} q(s)\,\mathrm{d}s\right]^{1/\alpha}. \tag{42}$$

Integrating (42) from t_2 to ∞ and using the second part of (37), we again arrive at the desired contradiction.

Next, we consider case (II). Proceeding exactly as in the proof of Theorem 2.2, we again arrive at (18). Using (18) in (2), we see that

$$\left(a(t)\left(y'(t)\right)^{\alpha}\right)' \geq \kappa^{\mu}c(t)y^{\mu}(\omega(t)) \quad \text{for } t \geq t_5. \tag{43}$$

Integrating (43) from $\eta(t)$ to t, we see that $y(t)$ is a positive solution of the first-order advanced differential inequality

$$y'(t) \geq \kappa^{\mu/\alpha}\left(\frac{1}{a(t)}\int_{\eta(t)}^{t} c(s)\,\mathrm{d}s\right)^{1/\alpha} y^{\mu/\alpha}(\pi(t)), \tag{44}$$

which contradicts the assumption on inequality (38).

In case (III), since $y(t) < 0$ for $t \geq t_2$, as in the proof of case (III) in Theorem 2.2, we let $z(t) = -y(t) > 0$ and see that (24) holds. Using (24) in (40), we obtain

$$\big(a(t)\,(z'(t))^{\alpha}\big)' + Q(t) z^{\gamma/\delta}(h_1(t)) \leq 0 \quad \text{for } t \geq t_2. \tag{45}$$

Using the fact that $z'(t) = -y'(t) < 0$ and $a(t)\,(z'(t))^{\alpha}$ is decreasing, and taking into account (3), as in the proof of case (I) in Theorem 2.2, we contradicts the fact that $z(t)$ is eventually positive.

Finally, for case (IV), letting $z(t) = -y(t) > 0$, we again arrive at (45). Using the fact that $z(t) > 0$ and $a(t)\,(z'(t))^{\alpha}$ is decreasing, we see that

$$z(t) \geq \int_{t_2}^{t} a^{-1/\alpha}(s)\,\big(a(s)\,(z'(s))^{\alpha}\big)^{1/\alpha}\,\mathrm{d}s \geq A(t,t_2)\left(a^{1/\alpha}(t) z'(t)\right). \tag{46}$$

Letting $Z(t) = a(t)\,(z'(t))^{\alpha} > 0$, in (46) yields

$$z(t) \geq \; A(t,t_2) Z^{1/\alpha}(t). \tag{47}$$

Substituting (47) into (45), we see that $Z(t)$ is a positive solution of the first--order delay differential inequality

$$Z'(t) + Q(t) A^{\gamma/\delta}\big(h_1(t), t_2\big) Z^{\gamma/\alpha\delta}(h_1(t)) \leq 0, \tag{48}$$

which contradicts our assumption on inequality (39) and completes the proof. □

It is well-known from [13] (see also [2, Lemma 2.2.9]) that if

$$\liminf_{t\to\infty} \int_{\zeta(t)}^{t} R(s)\,\mathrm{d}s > \frac{1}{e}, \tag{49}$$

then the first-order delay differential inequality

$$x'(t) + R(t) x(\zeta(t)) \leq 0 \tag{50}$$

where $R,\ \zeta \in C\big([t_0,\infty),\mathbb{R}\big)$ with $R(t) \geq 0$, $\zeta(t) \leq t$, and $\lim_{t\to\infty} \zeta(t) = \infty$, has no eventually positive solutions. If $\zeta(t) \geq t$, and $\zeta'(t) \geq 0$, we have the following result (see [2, Lemma 2.2.10]): If

$$\liminf_{t\to\infty} \int_{t}^{\zeta(t)} R(s)\,\mathrm{d}s > \frac{1}{e}, \tag{51}$$

then the first-order advanced differential inequality

$$x'(t) - R(t) x(\zeta(t)) \geq 0 \tag{52}$$

has no eventually positive solutions.

Thus, from Theorem 3.1, we have the following oscillation result for equation (2).

Corollary 3.2. *Let conditions* (i)–(iv) *and* (3) *hold. Assume that there exist functions* $p \in C([t_0,\infty),\mathbb{R})$ *and* $\eta \in C([t_0,\infty),\mathbb{R})$ *such that* (6) *and* (36) *hold. If condition* (37),

$$\liminf_{t\to\infty} \int_t^{\pi(t)} \left(\frac{1}{a(u)} \int_{\eta(u)}^{u} c(s)\,\mathrm{d}s \right)^{1/\alpha} \mathrm{d}u > \frac{1}{e} \quad \text{if } \mu = \alpha, \tag{53}$$

and

$$\liminf_{t\to\infty} \int_{h_1(t)}^{t} Q(s) A^{\gamma/\delta}(h_1(s), t_0)\,\mathrm{d}s > \frac{1}{e} \quad \text{if } \gamma = \alpha\delta, \tag{54}$$

are satisfied, then a solution $x(t)$ *of equation* (2) *is either oscillatory or satisfies* $\liminf_{t\to\infty} |x(t)| = 0$.

P r o o f. The proof is straightforward and we omit the details. □

Corollary 3.3. *Let conditions* (i)–(iv) *and* (3) *hold. Assume that there exist functions* $p \in C([t_0,\infty),\mathbb{R})$ *and* $\eta \in C([t_0,\infty),\mathbb{R})$ *such that* (6) *and* (36) *hold. If condition* (37),

$$\int_{t_0}^{\infty} \left(\frac{1}{a(u)} \int_{\eta(u)}^{u} c(s)\,\mathrm{d}s \right)^{1/\alpha} \mathrm{d}u = \infty \quad \text{if } \mu > \alpha, \tag{55}$$

and

$$\int_{t_0}^{\infty} Q(s) A^{\gamma/\delta}(h_1(s), t_0)\,\mathrm{d}s = \infty \quad \text{if } \gamma < \alpha\delta, \tag{56}$$

are satisfied, then a solution $x(t)$ *of equation* (2) *is either oscillatory or satisfies* $\liminf_{t\to\infty} |x(t)| = 0$.

P r o o f. The proof is similar to the proof of Corollary 2.3, and hence we omit the details. □

We conclude this paper with some examples to illustrate the above results and some suggestions for future research.

4. Examples

Example 1.
Consider the equation

$$(ty'(t))' + (1+t^2)x^{1/3}(t/4) + (4t)^{\mu/3}x^{\mu}(2t) = 0, \quad t \geq 1, \tag{57}$$

with

$$y(t) = x(t) + \frac{1}{t}x^{1/3}(t/2) - tx^3(t/2).$$

Here

$$\alpha = 1, \quad \gamma = 1/3, \quad \beta = 1/3, \quad \delta = 3, \quad \tau(t) = t/4,$$
$$\sigma(t) = t/2, \quad \omega(t) = 2t, \quad a(t) = t, \quad q(t) = 1+t^2, \quad c(t) = (4t)^{\mu/3}$$

with $\mu > 3$ is the ratio of positive odd integers,

$$p_1(t) = 1/t \quad \text{and} \quad p_2(t) = t.$$

Then, it is easy to see that conditions (i)–(iv) and (3) hold. Letting $p(t) = 1$, we see that condition (6) holds. Letting $\xi(t) = 2t/3$ and $\varphi(t) = t/2$, we see that

$$h_1(t) = \sigma^{-1}(\tau(t)) = t/2 \leq 2t/3,$$

and

$$\rho(t) := h_2(\varphi(t)) = 4t > t,$$

i.e, conditions (7) and (8) hold. Since

$$A(t,t_0) = A(t,1) = \int_1^t \frac{\mathrm{d}s}{s} = \ln t,$$

we see that

$$\int_{t_0}^{\infty} q(s)A^{\gamma}\big(\tau(s),t_0\big)\,\mathrm{d}s = \int_1^{\infty}(1+s^2)\left(\ln\frac{s}{4}\right)^{1/3}\mathrm{d}s = \infty,$$

$$\int_{t_0}^{\infty} Q(s)A^{\gamma/\delta}\big(\xi(s),h_1(s)\big)\,\mathrm{d}s = \int_1^{\infty}\frac{2^{1/9}(1+s^2)}{s^{1/9}}\left(\ln\frac{4}{3}\right)^{1/9}\mathrm{d}s = \infty,$$

and

$$\int_{t_0}^{\infty}\left(\frac{1}{a(u)}\int_{\varphi(u)}^{u} C(s)\,\mathrm{d}s\right)^{1/\alpha}\mathrm{d}u = \int_1^{\infty}\left(\frac{1}{u}\int_{u/2}^{u}\mathrm{d}s\right)\mathrm{d}u = \int_1^{\infty}\frac{1}{2}\,\mathrm{d}u = \infty,$$

i.e., conditions (32)–(34) hold. Thus, by Corollary 2.3, equation (57) is oscillatory.

Example 2.

Consider the equation

$$\left(t^{1/3}\left(y'(t)\right)^{1/3}\right)' = (1+t^3)x^{1/5}(t/6) + x^{\mu}(3t), \quad t \ge 1, \tag{58}$$

with

$$y(t) = x(t) + \frac{1}{t^6}x^{1/7}(t/3) - tx^5(t/3).$$

Here

$$\alpha = 1/3, \qquad \gamma = 1/5, \quad \beta = 1/7, \qquad \delta = 5,$$

μ is the ratio of positive odd integers,

$$\tau(t) = t/6, \qquad \sigma(t) = t/3, \quad \omega(t) = 3t, \qquad a(t) = t^{1/3},$$
$$q(t) = 1+t^3, \qquad c(t) = 1, \quad p_1(t) = 1/t^6, \qquad p_2(t) = t.$$

Then, it is easy to see that conditions (i)–(iv), (3) and the first part of (37) hold. Letting $p(t) = 1$, we see that condition (6) holds. Letting $\eta(t) = 2t/3$, we see that

$$\pi(t) := \omega(\eta(t)) = 2t > t,$$

i.e, condition (36) holds. Since

$$A(t,t_0) = A(t,1) = \int_1^t \frac{\mathrm{d}s}{s} = \ln t,$$

we see that

$$\int_{t_0}^{\infty} Q(s)A^{\gamma/\delta}(h_1(s),t_0)\,\mathrm{d}s = \int_1^{\infty} \frac{2^{1/25}(1+s^3)}{s^{1/25}}\left(\ln\frac{s}{2}\right)^{1/25}\mathrm{d}s = \infty,$$

and

$$\int_{t_0}^{\infty}\left(\frac{1}{a(u)}\int_{\eta(u)}^{u} c(s)\,\mathrm{d}s\right)^{1/\alpha}\mathrm{d}u = \int_1^{\infty}\left(\frac{1}{u^{1/3}}\int_{2u/3}^{u}\mathrm{d}s\right)^{3}\mathrm{d}u$$
$$= \frac{1}{27}\int_1^{\infty} u^2\,\mathrm{d}u = \infty,$$

i.e., conditions (55) and (56) hold. Thus, by Corollary 3.3, a solution $x(t)$ of equation (58) is either oscillatory or satisfies

$$\liminf_{t\to\infty}|x(t)| = 0.$$

Remark 1. There is a number of directions for future research that can be based on the results we obtained here. For example, could there be two different delays in the neutral term, that is, could we have

$$y(t) = x(t) + p_1(t)x^{\beta}(\sigma_1(t)) - p_2(t)x^{\delta}(\sigma_2(t))$$

with

$$\sigma_1(t) \le t \qquad \text{and} \qquad \sigma_2(t) \le t?$$

Here we asked that

$$0 < \beta < 1 \qquad \text{and} \qquad \delta > 1.$$

What if

$$\beta > 1 \qquad \text{and} \qquad 0 < \delta < 1,$$

or any other combination? Another possibility would be to consider the case where

$$\lim_{t\to\infty} A(t,t_0) = \lim_{t\to\infty} \int_{t_0}^{t} a^{-1/\alpha}(s)\,\mathrm{d}s < \infty.$$

It would be of interest also to extend the results here, or for the modifications mentioned above, to higher-order equations such as

$$\left(a(t)\left(y^{(n-1)}(t)\right)^{\alpha}\right)' + q(t)x^{\gamma}(\tau(t)) + c(t)x^{\mu}(\omega(t)) = 0$$

and

$$\left(a(t)\left(y^{(n-1)}(t)\right)^{\alpha}\right)' = q(t)x^{\gamma}(\tau(t)) + c(t)x^{\mu}(\omega(t)),$$

where $n \ge 3$, and the functions a, c, q, and y are as in this paper.

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