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ZARISKI TOPOLOGIES ON GRADED IDEALS

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ABSTRACT. In this paper, we show there are strong relations between the algebraic properties of a graded commutative ring R and topological properties of open subsets of Zariski topology on the graded prime spectrum of R. We examine some algebraic conditions for open subsets of Zariski topology to become quasi-compact, dense, and irreducible. We also present a characterization for the radical of a graded ideal in R by using topological properties.

1. Introduction

Throughout this paper, G will be a group with identity e and R a commutative ring with a nonzero unity 1. R is said to be G-graded if $R = \bigoplus_{g\in G} R_g$ with $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$, where R_g is an additive subgroup of R for all $g \in G$. The elements of R_g are called homogeneous of degree g. Consider $\text{supp}(R, G) = \{g \in G : R_g \neq 0\}$. If $x \in R$, then x can be written as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Also, $h(R) = \bigcup_{g\in G} R_g$. Moreover, it has been proved in [6] that R_e is a subring of R and $1 \in R_e$.

Example 1.1. If S is a ring, then the polynomial ring $R = S[X]$ is a $\mathbb{Z}$-graded ring with the standard grading $R_n = SX^n$ for $n \geq 0$, and $R_n = \{0\}$ for $n < 0$.

Example 1.2. Let S be a ring and $R = S[X, X^{-1}]$ be the ring of Laurent polynomials. Then R has the standard $\mathbb{Z}$-grading $R_n = SX^n$, $n \in \mathbb{Z}$.

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Example 1.3. Consider $R = M_2(K)$ (the ring of all 2×2 matrices with entries from a field K). Then R is $\mathbb{Z}_4$-graded by

$$R_0 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}$$

and

$$R_1 = R_3 = 0.$$

Also, R is $\mathbb{Z}$-graded by

$$R_0 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \quad R_1 = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}, \quad R_{-1} = \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix}$$

and

$$R_n = 0, \quad \text{otherwise}.$$

Let P be an ideal of a graded ring R. Then P is said to be a graded ideal if $P = \bigoplus_{g \in G}(P \cap R_g)$, i.e., for $x \in P$, $x = \sum_{g \in G} x_g$, where $x_g \in P$ for all $g \in G$. The following example shows that an ideal of a graded ring need not be graded.

Example 1.4. Consider $R = \mathbb{Z}[i] = \{a + ib : a, b \in \mathbb{Z}, i^2 = -1\}$ and $G = \mathbb{Z}_2$. Then R is G-graded by $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Now, $P = \langle 1 + i \rangle$ is an ideal of R with $1 + i \in P$. If P is graded, then $1 \in P$, so $1 = a(1 + i)$ for some $a \in R$, i.e., $1 = (x + iy)(1 + i)$ for some $x, y \in \mathbb{Z}$. Thus $1 = x - y$ and $0 = x + y$, i.e., $2x = 1$ and hence $x = \frac{1}{2}$ a contradiction. So, P is not a graded ideal of R.

Lemma 1.5 ([5], Lemma 2.1)**.** *Let R be a G-graded ring. If P and K are graded ideals of R, then $P + K$ and $P \bigcap K$ are graded ideals of R.*

Let R be a G-graded ring and P a graded ideal of R. Then R/P is G-graded by $(R/P)_g = (R_g + P)/P$. Moreover, the graded radical of P is denoted by $\sqrt{P}$ and it is defined to be the set of all $x \in R$ such that for each $g \in G$, there exists a positive integer n_g satisfying $x_g^{n_g} \in P$. One can see that if x is a homogeneous element, then $x \in \sqrt{P}$ if and only if $x^n \in P$ for some positive integer n.

Graded prime ideals play an essential role in graded ring theory. A proper graded ideal P of R is said to be graded prime if whenever $x, y \in h(R)$ such that $xy \in P$, then either $x \in P$ or $y \in P$. Graded prime ideals have been admirably introduced and studied in [9]. Clearly, if P is a prime ideal of R and it is a graded ideal, then P will be a graded prime ideal. However, the converse is not true in general; a graded prime ideal is not necessarily a prime ideal, see the following example:

Example 1.6. Suppose that $F = \{x + uy : x, y \in R, u^2 = 1\}$, where R is a field. Then F is $\mathbb{Z}_2$-graded by $F_0 = R$ and $F_1 = uR$. Let $a \in h(F)$ such that $a \neq 0$. If $a \in F_0$, then $a \in R$ and since R is a field, we have a is a unit element. Suppose that $a \in F_1$. Then $a = uy$ for some $y \in R$. Since $a \neq 0$, we have $y \neq 0$, and since R is a field, we have y is a unit element, that is $zy = 1$ for some $z \in R$.

Thus, $uz \in F_1$ such that $(uz)a = uz(uy) = u^2(zy) = 1.1 = 1$, which implies that a is a unit element. Hence, F is a graded field, and then $\{0\}$ is a graded prime ideal of F. On the other hand, $\{0\}$ is not a prime ideal of F since $1+u, 1-u \in F - \{0\}$ with $(1+u)(1-u) \in \{0\}$.

The notion of graded prime ideals has led to the development of topologies on the spectrum of the graded prime ideals and so many useful connections between topologies and algebraic properties have been proved. Mainly, the graded radical of a graded commutative ring, which has an important place in the graded ring theory, was characterized by using topological concepts. After that, the Zariski topology on graded modules has also attracted considerable attention of many authors, for example, [1], [3] and [4].

In this paper, for a graded ideal P of R, we study an open subset X_P of the Zariski topology on the set of all graded prime ideals of R denoted by $\mathrm{GSpec}(R)$. We also find the relationships between open subsets of the Zariski topology and graded ideals. Then we obtain some characterizations for the graded radical of a graded ideal and graded rings by using some topological properties. For an open subset of the Zariski topology to become quasi-compact, dense, and irreducible or a graded Noetherian spectrum, some algebraic conditions have been investigated.

Now, we recall some definitions and notations as follows:

Let R be a graded ring, P be a graded ideal of R,

$$V(P) = \{B \in \mathrm{GSpec}(R) : P \subseteq B\}, \quad X_P = \mathrm{GSpec}(R) - V(P)$$

and

$$V^{\sim}(K) = V(K) - V(P),$$

where K is a graded ideal of R. Then clearly,

$$\Gamma_P = \{V^{\sim}(K) : K \quad \text{is a graded ideal of} R\}$$

satisfies the axioms for closed sets of a topological space on X_P, called the complement Zariski topology of P in R. Obviously, open subset $X_P = \mathrm{GSpec}(R)$ when $P = R$ and also $X_P = \emptyset$ when $P = \{0\}$. Let T be a subset of a topological space X. Then

1) X is said to be quasi-compact if every open cover of X has a finite subcover.
2) X is said to be irreducible if $X \neq \emptyset$ and for every decomposition $X = X_1 \bigcup X_2$ with closed subsets $X_1, X_2 \subseteq X$, either $X = X_1$ or $X = X_2$.
3) T is said to be dense in X if for every nonempty open set
$$A \subseteq X, \quad A \bigcap T \neq \emptyset.$$
4) X is said to be Noetherian if the closed subsets of X satisfy the descending chain condition.

2. Zariski topologies on graded ideals

We begin with the following proposition revealing some connections between X_P and the graded ideal P. The proof of the proposition is straightforward.

Proposition 2.1. *Let R be a graded ring, P be a graded ideal of R and $\alpha, \beta \in h(R)$. Then the following hold.*

1) $(X_P)^{\alpha} = X_P - V^{\sim}(\langle\alpha\rangle) = \mathrm{GSpec}(R) - V(\alpha P)$ *forms a basis for* X_P.
2) $(X_P)^{\alpha} \bigcap (X_P)^{\beta} = (X_P)^{\alpha\beta}$.
3) $(X_P)^{\alpha} = \emptyset$ *if and only if* $\alpha P \subseteq \sqrt{\{0\}}$.
4) *If* α *is a unit, then* $(X_P)^{\alpha} = X_P$.
5) $(X_P)^{\alpha} = (X_P)^{\beta}$ *if and only if* $\sqrt{\alpha P} = \sqrt{\beta P}$.
6) *If* $(X_P)^{\alpha} = X_P$, *then* $\sqrt{\alpha P} = \sqrt{P} \subseteq \sqrt{\langle\alpha\rangle}$.

Let P be a proper graded ideal of a graded ring R. Then P is said to satisfy the condition $(*)$ if whenever $\sqrt{P} \subseteq \sqrt{\langle\{\alpha_i \in h(R) : i \in \Lambda\}\rangle}$, then there is a finite subset Δ of Λ such that $\sqrt{\langle\{\alpha_i \in h(R) : i \in \Lambda\}\rangle} = \sqrt{\langle\{\alpha_j \in h(R) : j \in \Delta\}\rangle}$. If R is a graded Noetherian ring, then every graded ideal of R satisfies the condition $(*)$. More specifically, if $R/\sqrt{P}$ is a graded Noetherian ring, then P satisfies the condition $(*)$ but it is obvious that the converse is not true in general.

In this paper, let us denote the finite set $\Delta = \{1, 2, \ldots, k\}$ for a positive integer k. The following proposition gives some connections between algebraic properties and topological properties.

Proposition 2.2. *Let P be a proper graded ideal of a graded ring R. Then the following hold.*

1) If X_P is quasi-compact, then there is a finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of $h(R)$ such that $\sqrt{P} = \sqrt{R\alpha_1 + R\alpha_2 + \cdots + R\alpha_n}$.
2) If P satisfies the condition $(*)$, then X_P is quasi-compact.

Proof.

1) Let $P = \langle\{\alpha_i \in h(R) : i \in \Lambda\}\rangle$. Then as $V(\langle\{\alpha_i : i \in \Lambda\}\rangle) = V(P)$, we have that $V^{\sim}(\langle\{\alpha_i : i \in \Lambda\}\rangle) = \emptyset$. Hence,

$$X_P = X_P - \emptyset = X_P - V\left(\bigcup_{i\in\Lambda} \alpha_i P\right) =$$

$$X_P - \bigcap_{i\in\Lambda} V(\alpha_i P) = \bigcup_{i\in\Lambda}\big(X_P - V(\alpha_i P)\big) = \bigcup_{i\in\Lambda}(X_P)^{\alpha_i}.$$

Thus X_P has an open cover and since X_P is quasi-compact, there is a finite set $\Delta = \{1, 2, \ldots, n\}$ such that

$$X_P = \bigcup_{i\in\Delta} (X_P)^{\alpha_i} = X_P - V^{\sim}(\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle).$$

So, $V^{\sim}(\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle) = \emptyset$, which implies that $V(\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle) \subseteq V(P)$. Therefore, $\sqrt{P} \subseteq \sqrt{R\alpha_1 + R\alpha_2 + \cdots + R\alpha_n}$, and the reverse holds, since $\alpha_1, \alpha_2, \ldots, \alpha_n \in P$.

2) Let $\{\xi_i : i \in \Lambda\}$ be an open cover of X_P. Since ξ_i can be expressed as a union of the sets of $(X_P)^{\alpha}$, we may assume that $\xi_i = (X_P)^{\alpha_i}$ for every $i \in \Lambda$. Then

$$X_P = \bigcup_{i\in\Lambda} (X_P)^{\alpha_i} = e \bigcup_{i\in\Lambda} \big(X_P - V^{\sim}(\langle \alpha_i \rangle)\big) = X_P - \bigcap_{i\in\Lambda} V^{\sim}(\langle \alpha_i \rangle) = X_P - V^{\sim}(\{\langle \alpha_i \rangle : i \in \Lambda\}).$$

Hence, $V^{\sim}(\langle \alpha_i : i \in \Lambda \rangle) = \emptyset$, which implies that $V^{\sim}(\langle \alpha_i : i \in \Lambda \rangle) \subseteq V(P)$. In this instance, $\sqrt{P} \subseteq \sqrt{\langle \alpha_i : i \in \Lambda \rangle}$. By the condition $(*)$,

$$\sqrt{P} \subseteq \sqrt{\langle \alpha_j : j \in \Delta \rangle}$$

for some finite subset Δ of Λ, and then $V(P) = V(\langle \alpha_j : j \in \Delta \rangle)$ and $V^{\sim}(\langle \alpha_j : j \in \Delta \rangle) = \emptyset$. So,

$$X_P = X_P - V^{\sim}(\langle \alpha_j : j \in \Delta \rangle) = X_P - \bigcap_{j\in\Delta} V^{\sim}(\langle \alpha_j \rangle) = \bigcup_{j\in\Delta} \big(X_P - V^{\sim}(\langle \alpha_j \rangle)\big) = \bigcup_{j\in\Delta} (X_P)^{\alpha_j}.$$

Hence, X_P is quasi-compact. □

Definition 2.3. Let P and K be graded ideals of a graded ring R. Then the set $N_P(K)$ is defined as $N_P(K) = \bigcap \{B \in \mathrm{GSpec}(R) : K \subseteq B \text{ and } P \nsubseteq B\}$.

The next example shows that this generalization is different from the graded radical of a graded ideal.

Example 2.4. Consider $R = \mathbb{Z}$ with a trivial graduation. Then $P = \langle 6 \rangle$ and $K = \langle 10 \rangle$ are graded ideals of R such that $N_P(K) = \langle 5 \rangle$, but $\sqrt{K} = \langle 10 \rangle = K$.

By Lemma 1.5, $N_P(K)$ is a graded ideal of R. Moreover, we give some algebraic properties of the the graded ideal $N_P(K)$ as follows, the proof is straightforward.

Proposition 2.5. *Let P, K be graded ideals of a graded ring R. The following hold:*

1) *If $P = R$, then $N_P(K) = \sqrt{K}$.*
2) *If L is a graded ideal of R such that $L \subseteq K$ and $L \subseteq P$, then $N_{P/L}(K/L) = N_P(K)/L$.*
3) *$N_P(\{0\}) = N_{\sqrt{P}}(\{0\})$.*

The next proposition gives a connection between topological property of X_P and algebraic property of $N_P(\{0\})$.

Proposition 2.6. *Let P be a proper graded ideal of a graded ring R such that $\sqrt{P} \neq \sqrt{\{0\}}$. Then X_P is irreducible if and only if $N_P(\{0\})$ is a graded prime ideal of R.*

P r o o f. Suppose that $N_P(\{0\})$ is a graded prime ideal of R. Let U be a nonempty subset of X_P. Then for a graded ideal F of R, we have that

$$U = X_P - V^{\sim}(F) = \mathrm{GSpec}(R) - \big(V(P) \bigcup V(F)\big).$$

Assume that $B \in U$. Then $B \notin V(P) \bigcup V(F)$, and then $P \nsubseteq B$ and $F \nsubseteq B$. So, $N_P(\{0\}) \subseteq B$, and then $F \nsubseteq N_P(\{0\}) \subseteq B$. So, $N_P(\{0\}) \notin V(F)$ which implies that $N_P(\{0\}) \notin V(P)$ by the definition of $N_P(\{0\})$. Hence, $N_P(\{0\}) \in U$. Thus, any nonempty open subset of X_P contains $N_P(\{0\})$, which means that X_P is irreducible. Conversely, suppose that $N_P(\{0\})$ is not a graded prime ideal of R. Then there exist $x, y \in h(R) - N_P(\{0\})$ such that $xy \in N_P(\{0\})$. Since $\sqrt{P} \neq \sqrt{\{0\}}$ and $x \in h(R) - N_P(\{0\})$, we have that $V^{\sim}(\langle x \rangle) \neq \emptyset$ and $V^{\sim}(\langle x \rangle) \neq X_P$, which implies that $(X_P)^x \neq \emptyset$. Similarly, $(X_P)^y \neq \emptyset$. So,

$$(X_P)^x \bigcap (X_P)^y = (X_P)^{xy} = X_P - V^{\sim}(\langle xy \rangle) \subseteq$$
$$X_P - V^{\sim}\big(N_P(\{0\})\big) = \mathrm{GSpec}(R) - \Big(V\big(N_P(\{0\})\big) \bigcup V(P)\Big) = \emptyset,$$

which is a contradiction with the assumption. Hence, $N_P(\{0\})$ is a graded prime ideal of R. □

The next example gives an application of Proposition 2.6.

Example 2.7. Consider $R = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then R is G-graded by $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Let $a \in \mathbb{Z}$. Then $P = \langle a \rangle$ is a graded ideal of R. Since $N_P(\{0\}) = \bigcap \{B \in \mathrm{GSpec}(R) : a \notin B\}$ is not a graded prime ideal of R, we have by Proposition 2.6 that X_P is not irreducible.

Definition 2.8. A graded ring R is said to satisfy the graded N-condition for a graded ideal P, if for any chain $N_P(K_1) \subseteq N_P(K_2) \subseteq \cdots$, where K_i is a graded ideal of R, there is a positive integer n such that $N_P(K_n) = N_P(K_{n+j})$ for all positive integers j.

Definition 2.9. The topological space X_P is said to be a graded Noetherian topological space if for every chain $V^{\sim}(K_1) \supseteq V^{\sim}(K_2) \supseteq \cdots$, where K_i is a graded ideal of R, there exists a positive integer n such that $V^{\sim}(K_n) = V^{\sim}(K_{n+j})$ for all positive integers j.

Proposition 2.10. *Let P be a proper graded ideal of a graded ring R. Then R satisfies the graded N-condition for P if and only if X_P is a graded Noetherian topological space.*

Proof. Suppose that R satisfies the graded N-condition for P. Consider the chain $V^{\sim}(K_1) \supseteq V^{\sim}(K_2) \supseteq \cdots$, where K_i is a graded ideal of R. Then $N_P(K_1) \subseteq N_P(K_2) \subseteq \cdots$ and then there is a positive integer n such that $N_P(K_n) = N_P(K_{n+j})$ for all positive integers j. So, $V^{\sim}(K_n) = V^{\sim}(K_{n+j})$ for all positive integers j. Hence, X_P is a graded Noetherian topological space. Conversely, consider the chain $N_P(K_1) \subseteq N_P(K_2) \subseteq \cdots$, where K_i is a graded ideal of R. Then $V^{\sim}(K_1) \supseteq V^{\sim}(K_2) \supseteq \cdots$ and then there is a positive integer n such that $V^{\sim}(K_n) = V^{\sim}(K_{n+j})$ for all positive integers j. So, $N_P(K_n) = N_P(K_{n+j})$ for all positive integers j. Hence, R satisfies the graded N-condition for P. □

In the rest of our paper, we study some algebraic and topological tools for graded ideals and some characterizations for graded rings. We begin with the following results that have been proved by Ortaç Önes and Mustafa Alkan in [8]. They will be needed in our study.

Proposition 2.11 ([8], Lemma 1)**.** *Let P, K and L be proper graded ideals of a graded ring R. Then*

1) *Any open set of a topological space X is of the form X_P.*
2) *$X_P \subseteq X_K$ if and only if $\sqrt{P} \subseteq \sqrt{K}$.*
3) *$X_P = X_K$ if and only if $\sqrt{P} = \sqrt{K}$.*
4) *$X_P \bigcap X_K = X_L$ if and only if $\sqrt{P} \bigcap \sqrt{K} = \sqrt{PK} = \sqrt{L}$.*

The next corollary is an immediate consequence of Proposition 2.11.

Corollary 2.12 ([8], Corollary 2)**.** *Let P and K be proper graded ideals of a graded ring R. Then $X_P \bigcap X_K = \emptyset$ if and only if $\sqrt{P} \bigcap \sqrt{K} = \sqrt{PK} = \sqrt{\{0\}}$.*

Proposition 2.13 ([8], Proposition 3)**.** *Let P be a proper graded ideal of a graded ring R. Then X_P is dense in X if and only if $\sqrt{PK} \neq \sqrt{\{0\}}$ for every proper graded ideal $K \nsubseteq \sqrt{\{0\}}$.*

If P is a graded ideal of a G-graded ring R, then $\sqrt{P}$ is not necessarily a graded ideal of R; see [7, Exercises 17 and 13 on pp. 127-128]. However, in [2, Lemma 2.13], it has been proved that if P is a graded ideal of a $\mathbb{Z}$-graded ring R, then $\sqrt{P}$ is a graded ideal of R. Assuming that R is a

$\mathbb{Z}$-graded ring, the next proposition gives a characterization for the graded ring $R/\sqrt{\{0\}}$ by using topological properties. The following lemma will be needed in the proof of the proposition.

Lemma 2.14. *Let R be a G-graded ring, I be an ideal of R and J be a graded ideal of R such that $J \subseteq I$. Then I is a graded ideal of R if and only if I/J is a graded ideal of R/J.*

Proof. Suppose that I is a graded ideal of R. Clearly, I/J is an ideal of R/J. Let $x + J \in I/J$. Then $x \in I$ and since I is graded, $x = \sum_{g \in G} x_g$, where $x_g \in I$ for all $g \in G$ and then $(x + J)_g = x_g + J \in I/J$ for all $g \in G$. Hence, I/J is a graded ideal of R/J. Conversely, let $x \in I$. Then $x = \sum_{g \in G} x_g$, where $x_g \in R_g$ for all $g \in G$ and then $(x_g + J) \in (R_g + J)/J = (R/J)_g$ for all $g \in G$ such that

$$\sum_{g \in G} (x + J)_g = \sum_{g \in G} (x_g + J) = \left(\sum_{g \in G} x_g \right) + J = x + J \in I/J.$$

Since I/J is graded, $x_g + J \in I/J$ for all $g \in G$ which implies that $x_g \in I$ for all $g \in G$. Hence, I is a graded ideal of R. □

A graded essential ideal P is a graded ideal that has nonzero intersection with every other graded nonzero ideal, or, equivalently, if $aP = \{0\}$ implies $a = 0$ for all $a \in h(R)$ ([6]).

Proposition 2.15. *Let R be a $\mathbb{Z}$-graded ring. The following are equivalent:*

1) $\sqrt{\{0\}}$ *is a graded prime ideal of* R.
2) $\mathrm{GSpec}(R)$ *is irreducible.*
3) *Every graded ideal of* $R/\sqrt{\{0\}}$ *is graded essential.*
4) *Every open subset of* $\mathrm{GSpec}(R)$ *is dense.*

Proof. It is easy to prove that (1) ⇔ (2).

(3) ⇒ (4): Let X_P and X_K be open subsets for graded ideals P and K of R. Then by Lemma 1.5, $P + \sqrt{\{0\}}$ and $K + \sqrt{\{0\}}$ are graded ideals of R, and then by Lemma 2.14, $(P + \sqrt{\{0\}})/\sqrt{\{0\}}$ and $(K + \sqrt{\{0\}})/\sqrt{\{0\}}$ are graded ideals of $R/\sqrt{\{0\}}$. Then

$$\sqrt{\{0\}} \neq \sqrt{\left(P + \sqrt{\{0\}}\right) \bigcap \left(K + \sqrt{\{0\}}\right)} = \sqrt{\left(P + \sqrt{\{0\}}\right)\left(K + \sqrt{\{0\}}\right)} = \sqrt{PK + \sqrt{\{0\}}},$$

and hence $\sqrt{PK} \neq \sqrt{\{0\}}$ which implies that X_P is dense.

By using similar argument, one can prove that (4) ⇒ (2) ⇒ (3). □

Remark 2.16. Proposition 2.15 has been proved in [8, Theorem 4], for any G-graded ring, where G is any group, but this is not true because their proof depended on saying that $\sqrt{P}$ is a graded ideal of R, and this is not true in general by [7, Exercises 17 and 13 on pp. 127–128]. On the other hand, it has been proved in [2, Lemma 2.13], that if P is a graded ideal of a $\mathbb{Z}$-graded ring R, then $\sqrt{P}$ is a graded ideal of R. So, Proposition 2.15 is a correction for [8, Theorem 4]. In fact, we do not know whether the general result in Proposition 2.15 is true for any group G, but there is a fault in the proof and the former proof works for $\mathbb{Z}$-graded rings.

The next proposition has been introduced in [8], Theorem 5, but we introduce a different proof.

Proposition 2.17. *Let P_i be a proper graded ideal of a graded ring R for all $i \in \Lambda$. Then $\bigcup_{i\in\Lambda} X_{P_i} = X_K$ for any graded ideal K of R if and only if $\sqrt{K} = \sqrt{\sum_{i\in\Lambda} P_i}$.*

P r o o f. Certainly,

$$\begin{aligned}
\bigcup_{i\in\Lambda} X_{P_i} = X_K, \quad &\text{if and only if} \quad \bigcup_{i\in\Lambda}\big(X - V(P_i)\big) = X_K,\\
&\text{if and only if} \quad X - \left(\bigcap_{i\in\Lambda} V(P_i)\right) = X - V(K),\\
&\text{if and only if} \quad \bigcap_{i\in\Lambda} V(P_i) = V(K),\\
&\text{if and only if} \quad V\left(\bigcup_{i\in\Lambda} P_i\right) = V(K),\\
&\text{if and only if} \quad V\left(\sum_{i\in\Lambda} P_i\right) = V(K),\\
&\text{if and only if} \quad \sqrt{\sum_{i\in\Lambda} P_i} = \sqrt{K}.
\end{aligned}$$

□

The next proposition has been given in ([8], Theorem 10), but we obtain it by combining Proposition 2.6 and Proposition 2.17.

Proposition 2.18. *Let P_i be a graded ideal of a graded ring R for all $1 \le i \le n$. Then $X = \bigcup_{i=1}^{n} X_{P_i}$, where X_{P_i} is irreducible if and only if $R = \sum_{i=1}^{n} P_i$ and $N_{P_i}(\{0\})$ is a graded prime ideal of R.*

We close our paper with the following question.

QUESTION 2.19. We expect that Proposition 2.6 provides a lead to characterize the irreducible components of the Zariski topology on GSpec(R). We likewise wonder if the irreducible components of the Zariski topology on GSpec(R) are characterized by the graded ideal $N_P(\{0\})$?

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REFERENCES

[1] ABU-DAWWAS, R.: *Zariski topology on the spectrum of graded pseudo prime submodules*, Bol. Soc. Parana. Mat. **39** (2021), no. 3, 17–26.
[2] ABU-DAWWAS, R.—BATAINEH, M.: *Graded r-ideals*, Iranian J. Math. Sci. Inform. **14** (2019), no. 2, 1–8.
[3] AL-ZOUBI, K.—JARADAT M.: *The Zariski topology on the graded classical prime spectrum of a graded module over a graded commutative ring*, Mat. Vesnik, **70** (2018), no. 4, 303–313.
[4] AL-ZOUBI, K.—JARADAT M.: *The Zariski topology on the graded primary spectrum over graded commutative ring*, Tatra Mt. Math. Publ. **74** (2019), 7–16.
[5] FARZALIPOUR, F.—GHIASVAND, P.: *On the union of graded prime submodules*, Thai J. Math. **9** (2011), no. 1, 49–55.
[6] NĂSTĂSESCU, C.—VAN OYSTAEYEN, F.: *Methods of graded rings*. In: *Lecture Notes in Mathematics Vol. 1836*, Springer-Verlag, Berlin, 2004.
[7] NORTHCOTT, D. G.: *Lessons on Rings, Modules and Multiplicities*. Cambridge University Press, London, 1968.
[8] ÖNES, O.—ALKAN, M.: *The relationships between graded ideals and subspaces*. In: *1st International Conference on Mathematical and Related Sciences (ICMRS 2018)*, https://doi.org/10.1063/1.5047900.
[9] REFAI, M.—HAILAT, M.—OBIEDAT, S.: *Graded radicals and graded prime spectra*, Far East Journal of Math. Sci. (FJSM) 2000, Specila Volume, Part I , 59–73.

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