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DOMESTICATION OF MATHEMATICAL PATHOLOGIES

Abstract. Certain mathematical objects bear the name "pathological" (or "paradoxical"). They either occur as unexpected and (temporarily) unwilling in mathematical research practice, or are constructed deliberately, for instance in order to delimit the scope of application of a theorem. I discuss examples of mathematical pathologies and the circumstances of their emergence. I focus my attention on the creative role of pathologies in the development of mathematics. Finally, I propose a few reflections concerning the degree of cognitive accessibility of mathematical objects. I believe that the problems discussed in the paper may attract the attention of philosophers interested in concept formation and the development of mathematical ideas.

Keywords: paradox, mathematical pathology, concept formation, mathematical intuition.

1. Introduction

Mathematicians call an object "pathological" (or "paradoxical") if its properties stay in sharp conflict with previously accepted intuitions based on the mathematical knowledge of a given epoch. The term "pathology" has a pejorative meaning in everyday usage, as well as in the social sciences and humanities: suffice to mention social pathologies or psychopathologies. However, in mathematics the role of pathologies is evidently positive; they are creative in character, and they lead to new findings or even to the emergence of new mathematical domains (see for instance Byers 2007).

I sustain the view that calling an object "pathological" is a mood of speech only. There are no eternally existing "objective pathologies" in mathematics. Objects once called pathological become domesticated (tamed) and are then treated as "normal", full-blooded mathematical objects. Being a standard (normal, natural) mathematical object depends on its wide scope of applications and the possession of well-recognized properties shaping the mathematical intuition. As mathematicians sometimes write, standard objects are well-behaving (from a fixed point of view) and certain objects may behave better than others (for instance, differentiable functions behave better than functions that are only continuous). Being standard does not mean being in the majority; for example almost all real functions are nowhere differentiable.

Not every "strange", "bizarre", or "difficult" object is automatically classified as pathological. The necessary condition for being called "pathological" is, as already mentioned, a deep conflict with previously accepted intuitions. Pathologies often serve as counterexamples, although not every counterexample must be called pathological.

Certain pathological objects are mentioned so often in texts popularizing mathematics that they become known to the general public. Examples include Cantor's ternary set, the paradoxical decomposition of a three-dimensional ball known as the Banach-Tarski paradox, or space-filling curves. However, popular texts rarely explain in detail the circumstances of the emergence of the paradoxes in question. They recall paradoxes mainly for readers' amusement, treating them as mathematical mysteries or curiosities. The advanced reader may consult more sophisticated collections of paradoxes, counterexamples, and "strange" mathematical objects presented with appropriate commentary, for instance: Steen and Seebach 1995, Gelbaum and Olmsted 1990, 2003, Wise and Hall 1993, Kharazishvili 2006. Philosophers are certainly familiar with the celebrated book Lakatos 1976 and classical works discussing the pitfalls of mathematical intuition, for instance Poincaré 1905 and Hahn 1980, as well as the more recent work Feferman 2000, where the author discusses the role of arithmetization of analysis in the emergence of geometrical and topological pathologies and stresses the impact of selected proof methods on that process.

2. Types of pathologies

One can distinguish two main types of mathematical pathologies depending on the circumstances of their occurrence:

Unexpected pathologies. Classical examples are negative and imaginary numbers. The latter occurred as solutions of certain equations, but were initially considered unacceptable. They were initially treated not as legitimate numbers but rather only as *tools* necessary for arithmetical calculations. They were called *imaginary*, *absurd*, or *dumb* in contrast to already known, well-recognized numbers. The domestication of negative and imaginary numbers took a few centuries. Only after they were shown to be really necessary for obtaining general solutions to arithmetical problems, and they were proven to form well-defined structures (the ring of integers, the field of complex numbers), did they become fully accepted. Geometric representations of these numbers also furthered their domestication.

Pathologies created on purpose. Laymen may be surprised by the fact that professional mathematicians deliberately create objects, simultaneously calling them pathological. However, this is entirely rational. Pathologies created on purpose serve as counterexamples, reveal limits to the validity of theorems, and make intuitions related to mathematical concepts more subtle. Numerous examples of pathologies of this kind occurred in the early days of general topology. Formalizations of such concepts as *connectedness* (intuitively: consisting of one "piece"), *path-connectedness* or *arc-connectedness* (intuitively: having a path between any two points), along with many others, revealed the fact that the precise definitions of the properties in question often admitted objects quite far-removed from the prototypes taken into account at the beginning of formalization. Similar difficulties appeared in the case of such concepts as *dimension* or *boundary*. Pathologies created on purpose sometimes become domesticated, too. A good example seems to be Cantor's ternary set, which found widespread application in several mathematical domains and is now considered a standard object by professionals, providing numerous valuable results. Nevertheless, Cantor's set continues to be called "pathological" in many works popularizing mathematics.

Mathematicians use many suggestive adjectives to name the objects of their interest, including standard, normal, canonical, genuine, generic, negligible, exceptional, extremal, degenerated, prototypical, and so on. There is no need to discuss these terminological details here, as I am interested only in qualifications of objects as pathological (paradoxical). Let me add only that oppositions like finite/infinite, discrete/continuous, regular/irregular, or determined/random may also be related to the impression of mathematical objects being paradoxical in character. Usually, pathologies are to be found in the second terms of the above oppositions.

What are the reasons behind and the circumstances of the label "pathological" being attached to an object? Regardless of whether a pathology occurred as unexpected or as the result of a deliberate construction, I believe the following factors should be taken into account:

Conflict with previously established intuitions. Peano and Hilbert gave examples of continuous functions whose graph fills the unit square. They are called pathological, because of the cultivated intuition that curves have a one-dimensional graph of zero area. Weierstrass provided an example of a continuous nowhere differentiable function also seen as pathological, because almost all functions encountered in applications of mathematical analysis “behave much better” as far as the existence of a derivative is concerned.

Collision of properties. Sets may be “big” with respect to cardinality (for instance, uncountable) or “dense” in the topological sense, but “small” with respect to measure, say Lebesgue measure; this is the case with Cantor’s set. However, mutually conflicting properties may also be possessed by well-recognized objects that are not necessarily pathological; for instance the set of rational numbers is a countable set of Lebesgue measure zero (hence “small” in this sense), while simultaneously being a dense subset of the set of real numbers (therefore “big” in a topological sense).

Generality of a definition. Admitting a very general definition of the concept of function resulted in the emergence of “monster-functions”, as defined by Peano, Hilbert, and Weierstrass. These “monsters” are far-removed from the prototypical functions met in everyday research practice. As already mentioned, “prototypical” or “genuine” objects do not necessarily form a majority of objects determined by a general definition.

Transfer of properties from finite to infinite case. The conviction that the whole is always greater than its part was considered based on a dogma articulated long ago by Euclid. This state of affairs is obviously true in the case of finite sets. The fact that infinite sets are equinumerous with their proper parts was considered paradoxical, for instance by Proclus, Galileo, and Bolzano. Dedekind used this very fact as a definition of infinite sets, and since then the property in question no long seems pathological.

Lack of constructivity. Many pathological objects are defined in a non-constructive way, for instance with the use of the axiom of choice. Examples include the paradoxical decomposition of the ball based on the Banach-Tarski theorem, or Vitali sets, which are not Lebesgue measurable (each Vitali set is a selector of the equivalence relation holding between two real numbers if the absolute value of their difference is a rational number).

Admitting “completely arbitrary” objects. The continuum hypothesis is undecidable in set theory when considering the totality of arbitrary sets. However, if one restricts attention to Borel sets only, then the continuum hypothesis for them is valid. Harvey Friedman has suggested that the source of incompleteness phenomena (among others, in set theory) lies in the admission of completely arbitrary objects (Friedman 1992). “Pathological” sets, escaping regular representation (as for instance sets outside well-recognized hierarchies), may thus be responsible for the incompleteness of the theory under discussion.

Undecidability. There is a problem, however, when it comes to talking about sets whose existence can be neither proved nor rejected on the basis of the axioms. Should we count them as pathological, because they are proof-theoretically inaccessible and their existence is independent of the axioms? Or should we rather suspend our judgment about them? The existence of strongly inaccessible cardinals is independent of the axioms of Zermelo-Fraenkel set theory. However, the class V_κ, where κ is the first strongly inaccessible cardinal is, in a sense, a quite natural mathematical object, and nowhere in textbooks on set theory is this class called pathological.

Logical complexity. From a formal point of view, sets definable by formulas with a finite number of quantifiers are of course well-determined. Nevertheless, if a definition contains, say, a few billion quantifiers, then the set determined by it can hardly be conceived as intuitively understood. This does not necessarily imply that such sets should be called pathological; they simply do not occur in "normal" mathematical discourse.

Lack of linguistic description. There is a distinction between *description* of an object and its *definition*. Descriptions require an effective (recursive) system of notation, while definitions do not. Certain ordinal numbers can be described and defined, while others can only be defined. Formal languages of mathematics have, as a rule, only a countable number of expressions that can name objects. The full powerset of an infinite set is uncountable, and hence almost all its elements lack names in the language of the theory dealing with them. Are these nameless objects automatically pathological, because there is no linguistic access to them, or should we (as in the case of undecidability mentioned above) suspend our judgment about their nature?

Hypothetical thinking. Asking *what-if* questions is one of the most important factors in the process of mathematical discovery. It is responsible for the invention of counterexamples, and the latter often take the form of mathematical pathology, sharply differing from objects considered to date.

3. Domestication of pathologies

By the domestication of a new mathematical concept I mean a process which ultimately results in the full acceptance of this concept in mathematical research practice. This term obviously applies not only to pathologies, but also to any kind of mathematical innovation. However, in the case of pathologies the process in question has a more spectacular form, visible in (sometimes drastic) changes in mathematical intuitions and the rapid development of new mathematical domains. And this confirms the creative role of mathematical pathologies.

An archetypal example of the domestication of objects initially considered pathological is the case of imaginary and complex numbers. The process took a few hundred years and is described in detail in many places, see for instance Kline 1972 or Stillwell 2010. I believe it would be inappropriate to discuss all mathematical details in a paper submitted to a philosophical journal, so I shall skip them here, limiting myself to a few words only. Imaginary numbers occurred unexpectedly in algebraic considerations, and Descartes later linked them to geometric impossibilities. John Wallis attempted to provide a geometric interpretation of them, but with little success. Leonhard Euler operated on complex numbers and proposed useful visualizations for them. Caspar Wessel and Jean-Robert Argand independently proposed intuitive and formally correct geometric representations. William Rowan Hamilton presented an algebraic representation of complex numbers and later Augustin-Louis Cauchy developed a theory of functions of complex variables. As for the full domestication of complex numbers, their geometric and algebraic representations seem to be primarily responsible, along with the proof of the fundamental theorem of algebra given by Karl Friedrich Gauss. In modern times complex numbers are ubiquitous in mathematics and its applications, and mathematicians tend to say that they often "behave much better" than the real numbers. However, for most of the population they remain a bit strange, not yet domesticated. One of the reasons for this situation could be the common-sense conviction that numbers should form an ordered structure, while complex numbers do not admit any ordering compatible with arithmetical operations.

Full acceptance of complex numbers by the mathematical community broke a certain psychological barrier, so to speak. Further kinds of multidimensional numbers (for instance quaternions) became accepted more quickly and from the very beginning were not called pathological. Moreover, they soon found numerous applications; for example quaternions are very useful in the description of rotations in three-dimensional space. This change of approach was without doubt also caused by the development of symbolic algebra in the nineteenth century. It is worth recalling that certain regulative methodological rules were formulated at that time, namely the *principle of permanence of equivalent forms*, proposed by George Peacock in 1833 (Peacock 1845) and later also by Hermann Hankel (Hankel 1867). Peacock's formulation in his *Treatise on algebra* runs as follows:

> Whatever algebraic forms are equivalent when the symbols are general in form, but specific in value, will be equivalent likewise when the symbols are general in value as well as in form. (Peacock 1845, 59)

This is of course only a heuristic principle and it expresses the view that while proposing new objects, being in a sense generalizations of old ones, we should keep as many "natural" properties as possible, where by "natural" we mean such properties as for instance associativity or commutativity of addition and multiplication. The principle in question could also be interpreted as forbidding postulation of the existence of objects on the basis of pure speculation, and thus as a stop sign against introducing bizarre pathologies.

Michael Detlefsen, writing about formalism in mathematics, points to the interconnections between the principle of permanence of forms and David Hilbert's *axiom of solvability*, the latter expressing Hilbert's epistemological optimism (there is no *ignorabimus* in mathematics; any mathematical problem can be solved or one can show that there is no solution under accepted assumptions) formulated in Hilbert 1901:

> Together, the Axiom of Solvability and the Principle of Permanence guided the progressive extension of the number-concept. The Axiom of Solvability expressed the mathematician's goal to solve problems. The Principle of Permanence acted as a constraint upon the applicability of this axiom. It required that newly introduced numbers preserve the basic laws of arithmetic. More precisely, it required that the laws governing new numbers be *consistent with* the laws governing the old ones. (Detlefsen 2005, 279)

Interesting remarks concerning the *process* of concept formation in mathematics and related to the principle of permanence of forms are presented in Buzaglo 2002. In particular, Buzaglo stresses the dynamic character of the process of extension of concepts, in contrast to synchronic descriptions of closed mathematical realms proposed in formal logic.

The infinitesimals are another example of painful domestication of a mathematical concept with a touch of paradox. Again, I am not going to recall all the mathematical details of the dispute, which are reported extensively in literature. Indivisibles have been present in mathematics since antiquity (Archimedes) and were used for instance by Johannes Kepler and Bonaventura Cavallieri in geometrical calculations. Both Isaac Newton and Gottfried Leibniz made essential use of infinitely small quantities, though interpreted in different ways. In the eighteenth century, infinitely small quantities were commomplace in calculations, but still without solid logical foundations. Beginning with the works of Augustin-Louis Cauchy, Karl

Weierstrass, Bernard Bolzano, Georg Cantor, and Richard Dedekind, the concept of limit, formulated in arithmetical terms, replaced talking about infinitely small quantities in analysis. One should not forget, however, that simultaneously non-Archimedean structures were also still being investigated, for instance by Tullio Levi-Civita, Giuseppe Veronese, Paul Du Bois-Reymond, Hans Hahn, and Otto Stolz. Thoralf Skolem constructed a non-standard model of arithmetic in 1934. A precise representation of infinitely small quantities was proposed by Abraham Robinson in the nineteen-sixties. More recently, John Horton Conway constructed the field of surreal numbers. What is interesting in the history of the efforts of domestication of the concept of infinitely small quantities is the fact that they were used for so long without a solid logical basis, and mathematicians trusted them because they led to successful calculations. Reuben Hersh has summarized this as follows:

> The infinitesimal has a fascinating history. At least as far back as Archimedes, it's been used by mathematicians who were perfectly aware that it didn't make sense. (Hersh 1997, 289)

4. Accessibility of mathematical objects

Our cognitive access to physical objects is based on perception, the values of measurements, and theoretical assumptions. Modern science admits the crucial role of theory in experiments. Advanced theories of physics treat the investigated objects ultimately as mathematical objects. We *talk* about these objects, form their *mental representations*, and conduct *reasonings* about them, all this being expressed in the language of mathematics.

I am not going to discuss here the problems of mathematical ontology. There are several standpoints in the dispute as to whether mathematics is discovered or invented in the modern philosophy of mathematics, ranging from arguments in favor of one or another solution and including proposals offering a compromise. I admit that the position of *mathematical agnosticism* seems attractive: mathematical research practice is in principle independent of the possible existence of a transcendental realm of Platonic objects. I believe that the mathematical world is a world of possibilities, governed by logical laws.

What should be understood by *cognitive access* to mathematical objects? The answer to this epistemological question depends on one's adopted perspective. Firstly, one may have in mind the process of acquisition of mathematical concepts. This is the subject of research in cognitive science, cognitive psychology, and theories of education. Secondly, one may ask how widely mathematical concepts are recognized in a society. This is studied in anthropology and theory of culture. Thirdly, the problem is of course interesting to philosophers, who are adherents of particular orientations in the philosophy of mathematics and may therefore propose a variety of solutions. Finally, one may explore how the accessibility in question could be characterized inside mathematics itself. I think that the following factors could be taken into account in the latter case:

Effectiveness of description or construction. Mathematical concepts can be situated on different levels of hierarchies based on the logical complexity of the corresponding definitions (arithmetical and analytic hierarchy, for example). This complexity is measured by the number of quantifiers needed in definitions or the number of operations used in constructions. The concept of a limit, for instance, belongs to the third level of the arithmetical hierarchy, because its definition, formulated in the ε-δ-language, requires three quantifiers. The concept of arithmetical truth (that is truth in the standard model of arithmetic) goes beyond all levels of the arithmetical hierarchy, which follows from Tarski's theorem on the undefinability of arithmetical truth in the language of first-order arithmetic.

Categoricity. Objects that admit categorical description, that is a unique (up to isomorphism) characterization, may be considered more accessible. The ordering of rational numbers is the only (up to isomorphism) dense linear ordering without endpoints. Natural numbers are characterized categorically in a second-order language (but not in a first-order one!). The standard model of first-order arithmetic is its only recursive model (Tennenbaum's theorem). Real numbers are uniquely characterized as a completely ordered field. Complex numbers form the only algebraically closed field of characteristic zero, whose transcendence degree over the rational numbers equals continuum. These structures are widespread in many areas of mathematics, and are understood there in principally the same way, due to their unique characterizations.

Multiplicity of representations. An object is more accessible if it admits many formal representations. Rational numbers may serve as an example. They are represented as a minimal field of characteristic zero, a dense subset of real numbers, a tree-like structure (Calkin-Wilf tree or Stern-Brocot tree), and they also admit several geometric representations. A multiplicity of representations is not in conflict with categorical description, the former corresponding to different points of view concerning a given object. Mathematical objects are not simply *seen*, but are rather *seen as* something (Sierpińska 1994). Let us observe in the margins that distinct representations of the same object may differ in descriptive complexity. Dirichlet's function, that is the characteristic function of the set of rational numbers, can also be represented as a limit of limits of continuous functions and therefore as a function belonging to the second Baire class.

Fruitfulness of applications. This concerns applications in mathematics itself as well as applications in the sciences. All quantitative descriptions of physical phenomena require number systems, functions and concepts associated with them (derivative, integral, and so on). "The unreasonable effectiveness of mathematics in science", though remaining mysterious from a philosophical point of view, nevertheless forces us to believe in the credibility of mathematical tools.

5. Concluding remarks

This paper is addressed to philosophers and cognitive scientists interested in concept formation and the development of mathematical ideas. The above remarks are certainly well-known to professional mathematicians. However, mathematicians rarely share their philosophical reflections with the general public, which is a pity, because people do not understand what mathematicians are doing and why the results of their creative work are so important for the development of culture.

One of the most famous appearances of mathematical pathologies was the discovery of incommensurable magnitudes by the Pythagoreans. The Pythagorean school believed the universe to be governed by the rules of numbers, and all numbers then known (natural and positive rational numbers) represented commensurable magnitudes. The discovery that the diagonal of a unit square is not commensurable with its side was like the appearance of a ghost from outside the known universe, and was very likely one the reasons for which pure algebra did not develop in ancient Greece, while geometrical considerations were held in high esteem. The full theory of real numbers was created as late as in the nineteenth century.

Contrary to the opinion expressed by one of the referees of this paper, I do not think that "zero is the most pathological mathematical object". The introduction of zero was forced by the use of a positional numeral system. The very idea of nothingness was already present in the philosophical reflection of ancient cultures. Zero is an example of mathematical *innovation*, emerging quite naturally from mathematical research practice. The paper Gaifman 2004 contains more remarks on the distinction between innovations and non-standard mathematical objects.

An interesting problem, in my opinion at least, is whether mathematical objects treated as paradoxical have real counterparts in nature. Fractals, for instance, are purely infinitary objects, and it would seem that physical objects may only be partial approximations of fractals. Supertasks discussed in philosophical literature are processes in which an infinite number of operations is executed in a finite amount of time (for instance the Thomson lamp or Laraudogoitia's supertask). Arguments against the physical reality of supertasks are known, see for instance Romero 2014. Heller considers an entire hierarchy of possible worlds, differing with respect to their degree of "mathematical content", including irrational worlds, softly and strongly malignant worlds, and mathematically friendly worlds (Heller 2010). The degrees in question correspond to the complexity of the mathematics needed for their description. It would seem that a higher level of a world's malignancy is related to the pathological character of its mathematical inhabitants.

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