

## ANALYSIS OF SIMPLY SUPPORTED LAMINATED COMPOSITE PLATE BY SEMI ANALYTICAL AND FINITE ELEMENT METHOD FOR DIFFERENT ORIENTATION ANGLES

**ARNAB Choudhury<sup>1\*</sup>, SAMAR Chandra<sup>2</sup>, SUSENJIT Sarkar<sup>2</sup>, MINTU Karmakar<sup>3</sup>**

<sup>1</sup>*Research Scholar, Department of Mechanical Engineering, Jadavpur University, Kolkata, e – mail: arnabmech001@gmail.com*

<sup>2</sup>*Faculty, Department of Mechanical Engineering, Jadavpur University, Kolkata, India, e – mail: samarju2003@yahoo.co.in, susenjit\_s@refiffmail.com*

<sup>3</sup>*Space Instrumentation Engineer in Mechanical Engineering, CESSI, IISER Kolkata, India, e – mail: mintukarmakar@gmail.com*

**Abstract:** Deflection of simply supported laminated composite plate under transverse load is found out by semi-analytical method (Classical Lamination Theory) and the results are compared with Finite element method using FEA software ANSYS. The equations developed by classical lamination theory for symmetric angle ply and cross ply laminated plate are solved by Navier and Rayleigh-Ritz method. The efficiency of Navier and Rayleigh-Ritz method in solving the problems of symmetric angle ply and cross ply laminated plate are investigated by comparing their results with ANSYS. Percentage error of the results from finite element method with Navier and Rayleigh-Ritz methods are studied and compared. The model is extended to study the effect of orientation angle, number of lamina and length to thickness ratio on the deflection of simply supported composite plates for symmetric angle ply plates of stacking sequence (orientation angle) of  $[+\theta/-\theta]_s$ . From the discussion, it is found that the percentage error depends on the variables: number of lamina, thickness of each lamina, orientation angle and mesh size. The significance of these variables in reducing the percentage error is studied by statistical tool: ANOVA and the relationship between these variables are established by regression equation. Statistical analysis is conducted by testing their significance at the 5% significance level. The comparative study and analysis of the percentage errors of the semi-analytical methods with finite element methods in solving the problem of laminated composite plates is essential to study the degree of approximation in the results.

**KEYWORDS:** Classical lamination theory, Navier method, Rayleigh-Ritz method, error, ANOVA, regression equation, angle ply, cross ply

### 1 Introduction

A composite material is defined as a new material that is composed of two or more existing materials combined at the macroscopic level and is not soluble in each other [1]. The property of the newly formed material is superior to its constituents and, in some cases, possesses a property that neither of its constituents possesses. One of the constituents is called the reinforcing phase (fibers, particles, or flakes) and the other in which it is embedded is called the matrix phase, which is generally continuous. The advantages of composite materials are high strength and stiffness, resistant to corrosion, high strength-to-weight and stiffness-to-weight ratio, low specific gravity, and high tolerance to fatigue damage, noncorrosive and temperature-dependent behavior, high chemical and impact resistance. Unlike conventional materials, the behaviour of composite materials is complex because they are composed of many layers of different or same thickness, materials and orientation stacked together. The behaviour of composite materials depends upon the thickness, orientation of fibers, stacking sequence, and materials of each ply. Therefore, many types of composite laminate can be

obtained by possible combinations of different plies or laminas in the laminate. All the plies are glued together. The study of their behaviour is complicated because delamination of the ply can occur under high heat and load. Due to the increase in thermal and or mechanical load, failure of the weakest ply can occur, which substantially reduces the strength of the laminate. So it is very essential to study the deflection and stress-strain relationship of composite laminates under load.

Composite structure problems subjected to different loadings and boundary conditions can be solved by classical lamination theory, first-order shear deformation theory, etc. But these methods are not sufficient to provide the exact solution when complex geometries, arbitrary boundary conditions, or nonlinearities are introduced. So, for solving complex problems, approximate results are considered. The finite element method is one such technique that provides an approximate solution to a complex or simple problem by using a numerical method. As a result, the finite element method can be formulated using either computer programming in MATLAB, C++, Fortran, and so on, or by using pre programmed commercial FEA packages such as ANSYS, ABAQUS, and so on.

There are several theories available for analyzing transverse deflection, stress, and strain of composite plates: classical lamination theory, first-order shear deformation theory, higher-order shear deformation theory, Zig-zag theory, and layer-wise lamination theory. A lot of papers are there based on these theories, to study the behaviour of laminated composite plate under different loadings and boundary conditions. Some of the papers among them, compared the results obtained from these theories with that obtained from finite element software i.e. ANSYS, ABAQUS etc. Some of the research papers used finite element method to solve the laminated composite plate problems by formulating their own programmes in MATLAB, C, Fortran etc. B.N. Pandya et al. [2] presented a  $C^0$  continuous displacement finite element model of a higher-order theory to determine the flexure of thick laminated composite plates subjected to transverse loads. Kam T.Y et al. [3] used a total Lagrangian finite element formulation which is based on the assumptions of Von Karman and first order shear deformation theory to perform the reliability analysis of laminated composite plate structures subjected to large deflections. Han Wanmin et al. [4] used the hierarchical finite element method to study the geometrically nonlinear analysis of laminated composite rectangular plates. C. Sridhar and K.P Rao [5] introduce a 48 d.o.f., four-node quadrilateral shell finite element for solving large deformation problems of circular composite laminated plates, and the Newton-Raphson method is used to solve this non-linear problem. M. Ganapathi et al. [6] used a new eight-node  $C^0$  membrane plate quadrilateral finite element which is based on the Reissner–Mindlin plate theory for the analysis of large deflections for static and dynamic problems of thick composite laminates, including buckling problems and the effect of coupling of membrane plate. Few papers compared the results obtained from finite element method with analytical results. G.Bao et al. [7] present an analytical solution for the buckling and bending of rectangular thin plates for different boundary conditions under in-plane compression and out-of-plane pressure loads, respectively, and the solutions are compared with finite element solutions. Y.X. Zhang and C.H. Yang [8] present a review paper on recent developments from 1990 to future in finite element analysis of free vibration and dynamics, buckling and post-buckling, geometric nonlinearity and large deformation, and failure and damage analysis problems of composite laminated plates based on the various laminated plate theories. Mauricio F. Caliri Jr. et al. [9] reviewed over 100 papers on plate and shell theories of laminated structure and explained the coupling between plate and shell theory. They also help to set a new platform for new theories and solution methods for laminated and sandwich structures. They also explained how the finite element method can simplify the complex analysis. Chalida Anakpotchanakul and Pongtorn Prombut [10] studied the influence of the

aspect ratio on vibration and bending analysis of T300/934 composite laminates with a symmetric stacking sequence of [0/90/0/90/0] s. Classical Laminated Plate Theory (CLPT) is used to model the bending and vibration behaviour of composite laminates, and the results are verified with a finite element model.

Manahan (2011) [12] presents the deflection of simply supported, symmetrically laminated composite plates subjected to a uniformly distributed load by using classical lamination theory and the finite element method. The FEA software ANSYS was used to model and perform analysis on the plate of a 4-ply composite plate by the finite element method. The equations developed by using classical lamination theory were solved by the Navier method in the MATLAB platform. Both methods were compared, and the percentage error was found. Errors were minimized by changing the orientation and increasing the number of plies. Kenneth Carroll (2013) [13] discussed the deflection of a simply supported plate under uniform pressure applied to the surface for both symmetric angle ply and cross-ply laminated plates. Two methods were used to solve the equations developed using classical lamination theory. He used the Navier method for cross-ply laminates and the Rayleigh-Ritz method for angle-ply laminates. The results were compared with the deflection results of an aluminum plate. Results were compared with the finite element modeling programme ANSYS, and a percentage error was noted, but he did not discuss the analysis of the error percentage. He analyzed different symmetric ply arrangements, ranging from four plies to sixteen plies. A failure analysis was also conducted. Qiao Jie Yang (2009) [14] studied the buckling of both simply supported and clamped edges plated subjected to uni axial compression load by CLT and FSDT and the results are compared with ANSYS and percentage error was noted. A. Choudhury et. al [16,17] studied the first ply failure analysis of laminated composite plates and beam for different mechanical, thermo mechanical and hygro-therm mechanical loads for different ply thicknesses, stacking sequences, fiber orientation angles and composite material systems by using semi analytical method. They computed and compared the strength ratio based on first ply failure load obtained from different failure theories. Also they found out the mode of failure. Javadi et al. [18] studied a three dimensional solutions for contact area in laminated composite pinned joints for symmetric and non-symmetric stacking sequences. Choudhary S, et al. [19] uses a simple higher order theory for dynamic analysis of composite plate.

Most of the research is concentrated on the analysis of laminated composite plates subjected to different loadings under different boundary conditions by different semi-analytical and finite element methods. Researchers compared the results obtained from both the methods and an error was found. But the study of the dependency of the error on different variables like orientation angle, mesh size, length to width ratio (aspect ratio) and thickness of the laminate is rare. Analysis of the error is very essential to determine the exactness of the results obtained from the semi-analytical methods. A study of the significance of the factors affecting the percentage error has not been done. In the present paper, the deflection of a laminated composite plate subjected to various loadings and boundary conditions are determined by both semi-analytical methods i.e. Classical Lamination Theory and Finite element method by using pre-programmed finite element package ANSYS, and then both the results are compared to find out the percentage error. The dependencies of the above mentioned variables on percentage error are studied by statistical method. The effects of these variables on deflection are also studied in this paper.

## **2 Semi-Analytical method**

Classical lamination theory (CLT) is used to find the deflection by the semi-analytical method. The equations developed by semi-analytical method are solved in MATLAB.

## 2.1 Classical Lamination Theory (CLT)

Classical lamination theory, an extension of classical plate theory to composite laminate is based upon Kirchhoff's Hypothesis which states that, the transverse displacement does not depend upon transverse or thickness coordinate and transverse normal and shear strains are zero i.e.

$$\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0 \quad (1)$$

### 2.1.1 Governing equation of laminated plate using CLT

Deflection of laminated composite plate under transverse load can be found out by the semi-analytical method and the result is compared with Finite element method using finite element analysis (FEA) software. Classical lamination theory is used to find the deflection by the semi-analytical method. The semi-analytical method is solved in MATLAB and the results are compared with FEA software. The governing equations of laminated plate using CLT are as [12]:

$$\begin{aligned} u(x, y, z, t) &= u_0(x, y, t) - z \frac{\partial w_0}{\partial x} \\ v(x, y, z, t) &= v_0(x, y, t) - z \frac{\partial w_0}{\partial y} \\ w(x, y, z, t) &= w_0(x, y, t) \end{aligned} \quad (2)$$

$u^0, v^0$  and  $w^0$  are the displacements along the coordinate line in a material point in  $x$ - $y$  plane and  $\frac{\partial w_0}{\partial x}, \frac{\partial w_0}{\partial y}$  are the rotations about  $x$  and  $y$  axis respectively. If the mid-point displacement can be found, then the displacement at any point in the plate can be determined by the eq. 2.

The stress- strains relations in 1-2 axes in 2 D form from Hook's law using the plane stress assumptions for unidirectional lamina [11]:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{11} \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} \quad (3)$$

where  $Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}$ ,  $Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}$ ,  $Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}$ ,  $Q_{66} = G_{12}$ ,  $\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}$ ,  $Q_{ij}$  is called reduced stiffness coefficient.

Inverting the eq. 3 we get

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{11} \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} \quad (4)$$

$S_{ij}$  are the elements of compliance matrix.

Unidirectional lamina is weak in transverse direction. So laminae are placed at angle within the laminate. The coordinate system used for angle lamina: the axis in the 1–2 coordinate systems is called the local axis or the material axes in which the axis- 1 is parallel to the fibers and the axis- 2 is perpendicular to the fibers. Local axis is also called lamina coordinate system. The global axis is the axis in the  $x$ - $y$  coordinate system. Global axis is also called laminate coordinate system because it is common to all the laminae within the laminate. The angle between the two axes is represented by an angle " $\theta$ " (Fig 1a). " $\theta$ " is called the orientation angle.

The relationship between stresses in  $x$ - $y$  and  $1$ - $2$  coordinate systems for angle lamina is given as [11]:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = [T]^{-1} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} \quad (5)$$

where  $[T]$  is called the transformation matrix.

$$[T] = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -2\cos\theta\sin\theta & \cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \quad (6)$$

Consider a laminate made of  $n$  plies shown in Fig 1(b). Each ply has a thickness of  $t_k$ . Then the thickness of the laminate  $h$  is

$$h = \sum_{k=1}^n t_k \quad (7)$$

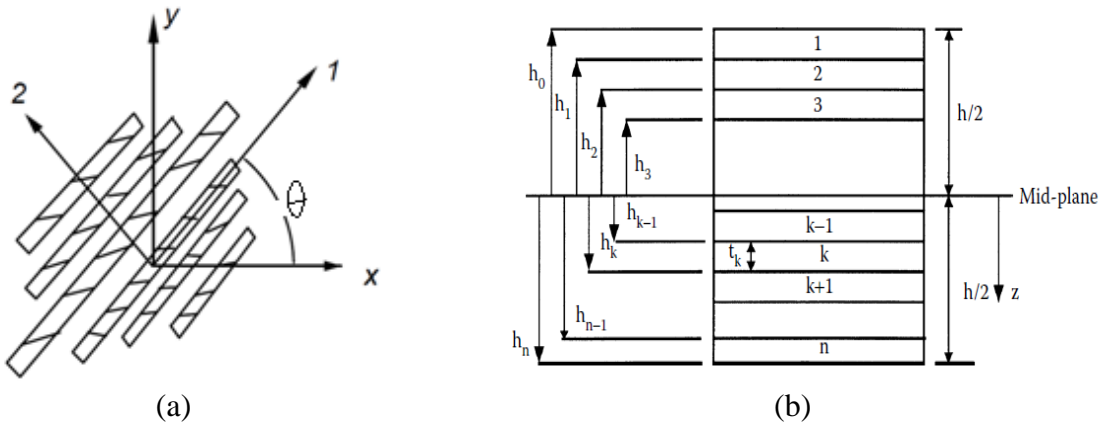


Fig. 1 (a) Lamina: Local axis (1, 2) and Global axis ( $x, y$ ) coordinate system, (b) Location of plies within laminate [11]

Then, the location of the mid plane is  $h/2$  from the top or the bottom surface of the laminate. The  $z$ -coordinate of each ply  $k$  surface (top and bottom) is given by:

$$\text{Ply 1: } h_0 = -\frac{h}{2} \text{ (top surface), } h_1 = -\frac{h}{2} + t_1 \text{ (top surface)}$$

$$\text{Ply } k: (k = 2, 3, \dots, n-2, n-1): h_{k-1} = -\frac{h}{2} + \sum_{1}^{k-1} t \text{ (top surface),}$$

$$h_{k-1} = -\frac{h}{2} + \sum_{1}^k t \text{ (bottom surface)}$$

Ply  $n$ :

$$h_{n-1} = \frac{h}{2} - t_n \text{ (top surface), } h_n = \frac{h}{2} \text{ (bottom surface)} \quad (8)$$

Force per unit length ( $N$ ) and the bending moment per unit length ( $M$ ) of a laminate are given as (fig 2):

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} k_x \\ k_y \\ k_{xy} \end{bmatrix} \quad (9)$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} k_x \\ k_y \\ k_{xy} \end{bmatrix} \quad (10)$$

$N_x, N_y$ , = normal force per unit length,  $N_{xy}$ = shear force per unit length.

$M_x, M_y$ , = Bending moment per unit length,  $M_{xy}$ = Twisting moments per unit length

$\varepsilon^0$  = Mid-plane strain of laminate in  $x$ - $y$  coordinates and  $k$  =laminate curvature.

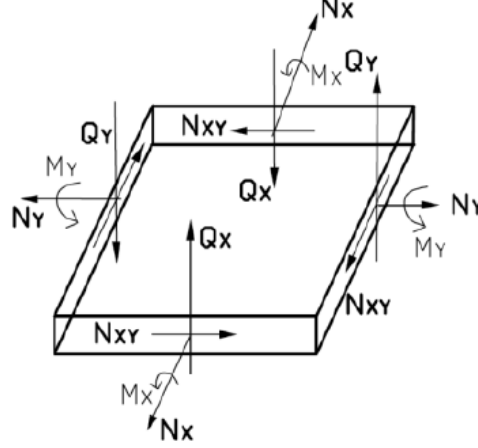


Fig 2. Plate subjected to Normal force and bending moment per unit length

Combining the eq. 9 and eq. 10:

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} \varepsilon^0 \\ k \end{bmatrix} \quad (11)$$

$$A_{ij}(\text{extensional stiffness matrix}) = \sum_{k=1}^n [\overline{Q}_{ij}]_k (h_k - h_{k-1}), \quad i = 1,2,6, j = 1,2,6$$

$$B_{ij}(\text{extension - bending coupling matrix}) = \frac{1}{2} \sum_{k=1}^n [\overline{Q}_{ij}]_k (h_k^2 - h_{k-1}^2), \quad i = 1,2,6 j = 1,2,6 \quad (12)$$

$$D_{ij}(\text{bending stiffness matrix}) = \frac{1}{3} \sum_{k=1}^n [\overline{Q}_{ij}]_k (h_k^3 - h_{k-1}^3), \quad i = 1,2,6, j = 1,2,6$$

### 2.1.2 Equation of motion

The three general equations which govern the response of laminated plates are as [12]:

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \frac{\partial^2 u_0}{\partial t^2} - I_1 \frac{\partial^2}{\partial t^2} \left( \frac{\partial w_0}{\partial x} \right) \quad (13)$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_0 \frac{\partial^2 v_0}{\partial t^2} - I_1 \frac{\partial^2}{\partial t^2} \left( \frac{\partial w_0}{\partial y} \right) \quad (14)$$

$$\begin{aligned} \frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial y \partial x} + \frac{\partial^2 M_{yy}}{\partial y^2} + N(w_0) + q \\ = I_0 \frac{\partial^2 w_0}{\partial t^2} - I_2 \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2} \right) + I_1 \frac{\partial^2}{\partial t^2} \left( \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} \right) \end{aligned} \quad (15)$$

The terms  $I_0, I_1$  &  $I_2$  are the rotary inertia terms which are neglected for static equilibrium of the plate. The eq. 11 can be written in terms of partial differentiation as:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \\ -\frac{\partial^2 w_0}{\partial x^2} \\ -\frac{\partial^2 w_0}{\partial y^2} \\ -2\frac{\partial^2 w_0}{\partial x \partial y} \end{bmatrix} \quad (16)$$

Substituting the values of normal force and bending moment from eq.16 in eq. 13 to 15, the eq. of motion are obtained as:

$$\begin{aligned} & A_{11} \left( \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial x^2} \right) + A_{12} \left( \frac{\partial^2 v_0}{\partial x \partial y} + \frac{\partial w_0}{\partial y} \frac{\partial^2 w_0}{\partial x \partial y} \right) \\ & + A_{16} \left( \frac{\partial^2 u_0}{\partial x \partial y} + \frac{\partial v_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial x^2} \frac{\partial w_0}{\partial y} + \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial x \partial y} \right) - B_{11} \frac{\partial^3 w_0}{\partial x^3} - B_{12} \frac{\partial^3 w_0}{\partial x \partial y^2} \\ & - 2B_{16} \frac{\partial^3 w_0}{\partial x^2 \partial y} + A_{16} \left( \frac{\partial^2 u_0}{\partial x \partial y} + \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial x \partial y} \right) \\ & + A_{26} \left( \frac{\partial^2 v_0}{\partial y^2} + \frac{\partial w_0}{\partial y} \frac{\partial^2 w_0}{\partial y^2} \right) + A_{16} \left( \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial x \partial y} \right) - B_{16} \frac{\partial^3 w_0}{\partial x^2 \partial y} \\ & - B_{26} \frac{\partial^3 w_0}{\partial y^3} - 2B_{66} \frac{\partial^3 w_0}{\partial x \partial y^2} - \left( \frac{\partial N_{xx}^T}{\partial x} + \frac{\partial N_{xy}^T}{\partial y} \right) = I_0 \frac{\partial^2 u_0}{\partial t^2} - I_1 \frac{\partial^3 w_0}{\partial x \partial t^2} \end{aligned} \quad (17)$$

$$\begin{aligned} & A_{16} \left( \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial x^2} \right) + A_{26} \left( \frac{\partial^2 v_0}{\partial x \partial y} + \frac{\partial w_0}{\partial y} \frac{\partial^2 w_0}{\partial x \partial y} \right) \\ & + A_{66} \left( \frac{\partial^2 u_0}{\partial x \partial y} + \frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial x^2} \frac{\partial w_0}{\partial y} + \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial x \partial y} \right) - B_{16} \frac{\partial^3 w_0}{\partial x^3} - B_{26} \frac{\partial^3 w_0}{\partial y^2 \partial x} \\ & - 2B_{66} \frac{\partial^3 w_0}{\partial x^2 \partial y} + A_{12} \left( \frac{\partial^2 u_0}{\partial x \partial y} + \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial x \partial y} \right) + A_{22} \left( \frac{\partial^2 v_0}{\partial y^2} + \frac{\partial w_0}{\partial y} \frac{\partial^2 w_0}{\partial y^2} \right) \\ & + A_{26} \left( \frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 v_0}{\partial x \partial y} + \frac{\partial^2 w_0}{\partial x \partial y} \frac{\partial w_0}{\partial y} + \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial y^2} \right) - B_{12} \frac{\partial^3 w_0}{\partial x^2 \partial y} - B_{22} \frac{\partial^3 w_0}{\partial y^3} \\ & - 2B_{26} \frac{\partial^3 w_0}{\partial y^2 \partial x} - \left( \frac{\partial N_{xy}^T}{\partial x} + \frac{\partial N_{yy}^T}{\partial y} \right) = I_0 \frac{\partial^2 v_0}{\partial t^2} - I_1 \frac{\partial^3 w_0}{\partial y \partial t^2} \end{aligned} \quad (18)$$

These are the general non linear equations which govern the response of laminated plate [3]. The equations can be simplified depending on boundary conditions, lamination scheme, type of laminate etc.

## 2.2 Bending of Specially orthotropic simply supported plate

For static bending of specially orthotropic plates, coupling terms of the bending stiffness matrix (bending twisting coefficient and bending stretching coupling coefficient),  $D_{16}$  and  $D_{26}$  are taken zero. Also [B] matrix is zero for symmetric plate.

$$\begin{aligned}
& B_{11} \left( \frac{\partial^3 u_0}{\partial x^3} + \frac{\partial^2 w_0}{\partial x^2} \frac{\partial^2 w_0}{\partial x^2} + \frac{\partial w_0}{\partial x} \frac{\partial^3 w_0}{\partial x^3} \right) + B_{12} \left( \frac{\partial^3 v_0}{\partial x^2 \partial y} + \frac{\partial^2 w_0}{\partial x \partial y} \frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial w_0}{\partial y} \frac{\partial^3 w_0}{\partial x^2 \partial y} \right) \\
& + B_{16} \left( \frac{\partial^3 u_0}{\partial x^2 \partial y} + \frac{\partial^3 v_0}{\partial x^3} + \frac{\partial w_0}{\partial y} \frac{\partial^3 w_0}{\partial x^3} + 2 \frac{\partial^2 w_0}{\partial x^2} \frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial w_0}{\partial x} \frac{\partial^3 w_0}{\partial x^2 \partial y} \right) \\
& - D_{11} \frac{\partial^4 w_0}{\partial x^4} - D_{12} \frac{\partial^4 w_0}{\partial x^2 \partial y^2} - 2D_{16} \frac{\partial^4 w_0}{\partial x^3 \partial y} \\
& + 2B_{16} \left( \frac{\partial^3 u_0}{\partial x^2 \partial y} + \frac{\partial^2 w_0}{\partial x^2} \frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial w_0}{\partial y} \frac{\partial^3 w_0}{\partial x^2 \partial y} \right) \\
& + 2B_{26} \left( \frac{\partial^3 v_0}{\partial y^2 \partial x} + \frac{\partial^2 w_0}{\partial y^2} \frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial w_0}{\partial y} \frac{\partial^3 w_0}{\partial y^2 \partial x} \right) \\
& + 2B_{66} \left( \frac{\partial^3 u_0}{\partial y^2 \partial x} + \frac{\partial^3 v_0}{\partial x^2 \partial y} + \frac{\partial w_0}{\partial y} \frac{\partial^3 w_0}{\partial x^2 \partial y} + \frac{\partial^2 w_0}{\partial x \partial y} \frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial^2 w_0}{\partial x^2} \frac{\partial^2 w_0}{\partial y^2} \right. \\
& \left. + \frac{\partial w_0}{\partial x} \frac{\partial^3 w_0}{\partial y^2 \partial x} \right) - 2D_{16} \frac{\partial^4 w_0}{\partial x^3 \partial y} - 2D_{26} \frac{\partial^4 w_0}{\partial y^3 \partial x} + 4D_{66} \frac{\partial^4 w_0}{\partial x^2 \partial y^2} \\
& + B_{12} \left( \frac{\partial^3 u_0}{\partial y^2 \partial x} + \frac{\partial^2 w_0}{\partial x \partial y} \frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial w_0}{\partial x} \frac{\partial^3 w_0}{\partial y^2 \partial x} \right) + B_{22} \left( \frac{\partial^3 u_0}{\partial y^3} + \frac{\partial^2 w_0}{\partial y^2} \frac{\partial^2 w_0}{\partial y^2} \right. \\
& \left. + \frac{\partial w_0}{\partial y} \frac{\partial^3 w_0}{\partial y^3} \right) \\
& + B_{26} \left( \frac{\partial^3 u_0}{\partial y^3} + \frac{\partial^3 v_0}{\partial y^2 \partial x} + \frac{\partial w_0}{\partial y} \frac{\partial^3 w_0}{\partial y^2 \partial x} + 2 \frac{\partial^2 w_0}{\partial y^2} \frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial w_0}{\partial x} \frac{\partial^3 w_0}{\partial y^3} \right) \\
& - D_{22} \frac{\partial^4 w_0}{\partial y^4} - D_{12} \frac{\partial^4 w_0}{\partial x^2 \partial y^2} - 2D_{26} \frac{\partial^4 w_0}{\partial y^3 \partial x} \\
& - \left( \frac{\partial^2 M_{xx}^T}{\partial x^2} + 2 \frac{\partial^2 M_{xy}^T}{\partial y \partial x} + \frac{\partial^2 M_{yy}^T}{\partial y^2} \right) + N(w_0) + q \\
& = I_0 \frac{\partial^2 w_0}{\partial t^2} - I_2 \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2} \right) + I_1 \frac{\partial^2}{\partial t^2} \left( \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} \right)
\end{aligned} \tag{19}$$

The non linear terms are neglected to reduce the complexity. In the absence of thermal effects, in-plane forces, and moments, eq. 19 reduces to:

$$D_{11} \frac{\partial^4 w_0}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w_0}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w_0}{\partial y^4} = q \tag{20}$$

In this paper, simply supported boundary conditions on all the four edges of the rectangular plate are considered for study. The boundary conditions are as (fig 3):

$$\begin{aligned}
w_0(x, 0) = 0, w_0(x, b) = 0, w_0(0, y) = 0, w_0(a, y) = 0 \\
M_{xx}(0, y) = 0, M_{xx}(a, y) = 0, M_{yy}(x, 0) = 0, M_{yy}(x, b) = 0
\end{aligned} \tag{21}$$

The equation 20 can be solved by Navier method, the Levy method with state space approach and Rayleigh-Ritz method. Navier solution is used for plate when all the edges are simply supported, whereas Levy method is used when two opposite edges are simply supported and the remaining edges have any possible combination of boundary conditions. The Rayleigh-Ritz method can be used to find out the approximate solution for more general boundary conditions.

### 2.2.1 The Navier Method

The Navier method uses double trigonometric series (Fourier) to solve the equations and the choice of trigonometric function depends on the boundary condition of the problem.

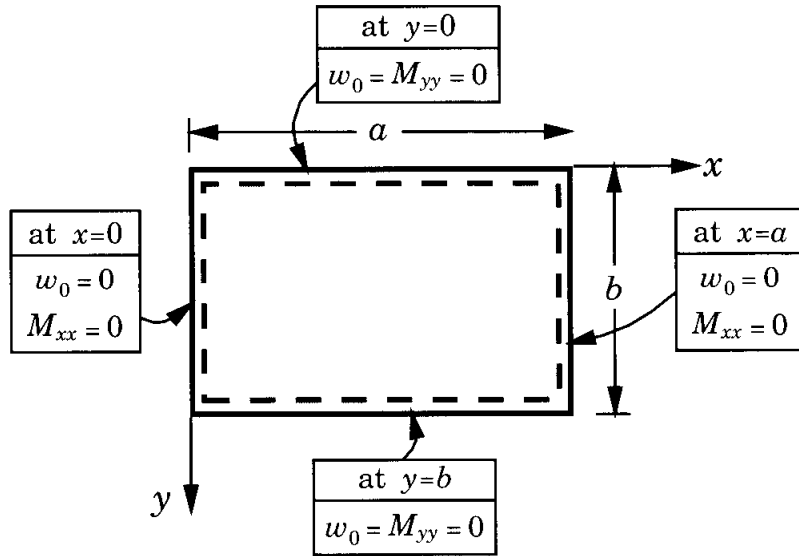


Fig. 3 Boundary conditions of simply supported rectangular plate on all sides [15]

The boundary conditions in the eq.21 are satisfied by the transverse deflection as [12]:

$$w_0(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} W_{mn} \sin \alpha x \sin \beta y \quad (22)$$

The load can be expanded as:

$$q(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Q_{mn} \sin \alpha x \sin \beta y \quad (23a)$$

where

$$Q_{mn} = \frac{4}{ab} \int_0^b \int_0^a q(x, y) \sin \alpha x \sin \beta y \, dx \, dy \quad (23b)$$

$\alpha = \frac{m\pi}{a}$ ,  $\beta = \frac{n\pi}{b}$  and  $W_{mn}$  are the coefficients to be determined such that governing eq. 20 are satisfied.  $a$  and  $b$  are the dimension of plate (fig 3).

Substituting the expansions 22-23 in the governing equation 20 gives

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \{-W_{mn}[D_{11}\alpha^4 + 2(D_{12} + 2D_{66})\alpha^2\beta^2 + D_{22}\beta^4] + Q_{mn}\} \sin \alpha x \sin \beta y = 0 \quad (24)$$

The solution of the equation becomes

$$w_0(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{Q_{mn}}{d_{mn}} \sin \alpha x \sin \beta y \quad (25)$$

where  $d_{mn} = \frac{\pi^4}{b^4} [D_{11}m^4s^4 + 2(D_{12} + 2D_{66})(mns)^2 + D_{22}(ns)^4]$  and  $s$  denotes the aspect ratio  $= \frac{a}{b}$ . For any loading condition, load can be expressed as

$$q(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Q_{mn} \sin \alpha x \sin \beta y \quad (26)$$

$Q_{mn}$  is the loading coefficient which depends upon the load. For uniformly distributed load  $Q_{mn} = \frac{16q_0}{\pi^2mn}$  for  $m, n, \text{ odd}$ ,  $q(x, y) = q_0$ , a constant.

Substituting the value of  $Q_{mn}$  and  $d_{mn}$  in eq. 25, the deflection becomes

$$w(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{16q_0 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}}{\pi^6 mn [D_{11} \left(\frac{m}{a}\right)^4 + 2(D_{12} + 2D_{66}) \left(\frac{m}{a}\right)^2 \left(\frac{n}{b}\right)^2 + D_{22} \left(\frac{n}{b}\right)^4]} \quad (27)$$

where  $m, n = 1, 3, 5, \dots$

After simplification

$$w = \frac{\alpha^4 Q_{mn}}{\pi^4 [D_{11}m^4 + 2(D_{12} + 2D_{66})(mns)^2 + D_{22}(ns)^4]} \quad (28)$$

where  $s = \frac{a}{b}$ , plate aspect ratio.

### 2.3 Bending of Symmetric angle ply simply supported plate

For static bending of Symmetric angle ply plates, bending twisting coefficient and bending stretching coupling coefficient are not zero. Also [B] matrix is zero for symmetric plate. The non linear terms are neglected to reduce the complexity. In the absence of thermal effects, in-plane forces, and moments, eq. 19 reduces to:

$$D_{11} \frac{\partial^4 w_0}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w_0}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w_0}{\partial y^4} + 4D_{16} \frac{\partial^4 w_0}{\partial x^3 \partial y} + 4D_{26} \frac{\partial^4 w_0}{\partial y^3 \partial x} = q \quad (29)$$

This eq. 29 cannot be solved by Navier solution because separations of variables are not possible and the Fourier expansion does not satisfy the eq. 29. The alternative method to solve the problem is the Rayleigh-Ritz Method which is based on the principal of total potential energy.

#### 2.3.1 Rayleigh-Ritz Method

The total potential energy of the laminated plate is given by [56]

$$V = \iint (D_{11} \frac{\partial^4 w^0}{\partial x^4} + 2D_{12} \frac{\partial^4 w^0}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w^0}{\partial y^4} + 4D_{66} \left(\frac{\partial^2 w^0}{\partial x \partial y}\right)^2 + 4D_{16} \frac{\partial^4 w^0}{\partial x^3 \partial y} + 4D_{26} \frac{\partial^4 w^0}{\partial y^3 \partial x} - 2pw) dx dy \quad (30)$$

Deflection of the laminated plate by the Rayleigh-Ritz Method is given by [13]

$$w = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{ij} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (31)$$

Where  $C_{ij}$  are the coefficients to be determined. Integrating the eq. 30 yield an equation which contain  $m \times n$  unknown coefficient. Total potential energy principle is used to solve the eq.30 which yields  $m \times n$  equations with a single unknown in each eq.30. After solving all the equations using any computing software, deflection is found out.

## 2.4 Semi-Analytical Model

A semi-analytical model of laminated composite plate simply supported on all the sides subjected to UDL transverse load is built up using the equations of semi-analytical method (eq. 20 for the specially orthotropic plate and eq. 29 for symmetric angle ply plate) in mathematical tool like MATLAB and MAPPLE. First five material properties are entered for each layer because of plane stress assumptions for simplification of calculation. Compliance matrix  $S_{ij}$  is calculated for each layer. Then Stiffness matrix  $[\overline{Q}_{ij}]$  is calculated for each layer by transforming and inverting a compliance matrix. Then program code calculates the distance  $h$  from the mid-plane. Then, using  $h$  and  $[\overline{Q}_{ij}]$ , bending stiffness matrix [D] is calculated by the eq.12. Then the deflection function of Navier and Rayleigh-Ritz method is calculated from the eq.28 and 31 using [D]matrix respectively.

## 2.5 Finite Element Model

A finite element model of laminated composite plate simply supported on all the sides subjected to UDL transverse load is built up in Finite element analysis (FEA) software, ANSYS. Due to symmetry, a quarter of the plate is modeled using SHELL 281 element. The model is meshed with quadrilateral mapped meshing. Simply supported boundary condition is applied on two adjacent sides of quarter of plate and symmetric condition is applied on the other two adjacent sides. Uniform pressure load is applied on the top sides of the plate and the model is solved in solution phase. Transverse deflection is noted in post processor phase for different stacking sequence. Convergence analysis is performed to determine the solution which is mesh independent. Fig 4a and 4b show the boundary conditions, quadrilateral mapped meshing and uniform pressure loading of the simply supported plate in ANSYS. Fig 4c shows the variation of deflection of [30/-30]s simply supported plate under uniform pressure loading for quarter of geometry of plate.

## 2.6 Validation of the semi-analytical & Finite element models

The model built up by semi-analytical method using Navier method solved by MATLAB tool is validated with the published literature of M.A Manahan [12] and the model built up by Finite element method of laminated composite plate under transverse load as discussed above is validated with Manahan [12] in table 1. The semi-analytical models prepared by Rayleigh-Ritz method using MAPLE tool is validated with the published literature of Kenneth Carroll [13] in table 2.

## 2.7 Numerical Problem

*A fiber reinforced laminated composite square plate simply supported on all the sides is considered. The plate is, subjected to a pressure of  $1 \text{ N/m}^2$  on the top surface. The laminas are stacked together symmetrically about the mid plane and the fibers are oriented at different orientation angle. Two type of stacking sequence is considered: symmetric cross ply and angle ply arrangement. The thickness of each lamina is 5mm. The material used is graphite-epoxy composite material (appendix). Deflection is found out by semi-analytical method and FEM and both the methods are compared. Effect of number of lamina, length to thickness ratio and orientation angle are studied.*

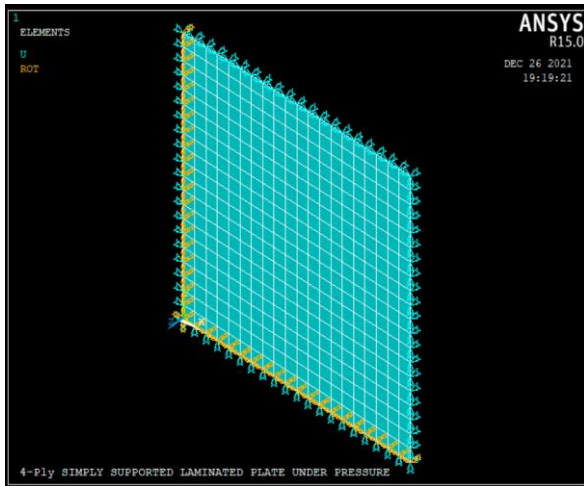


Fig. 4(a) Boundary conditions of plate simply supported on all the sides

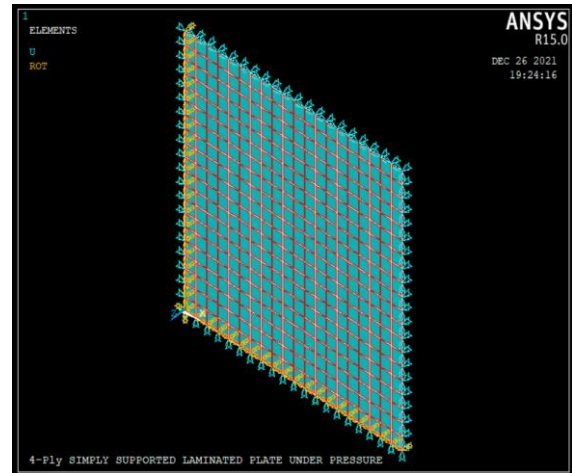


Fig. 4(b) Uniform pressure loading of plate simply supported on all the sides

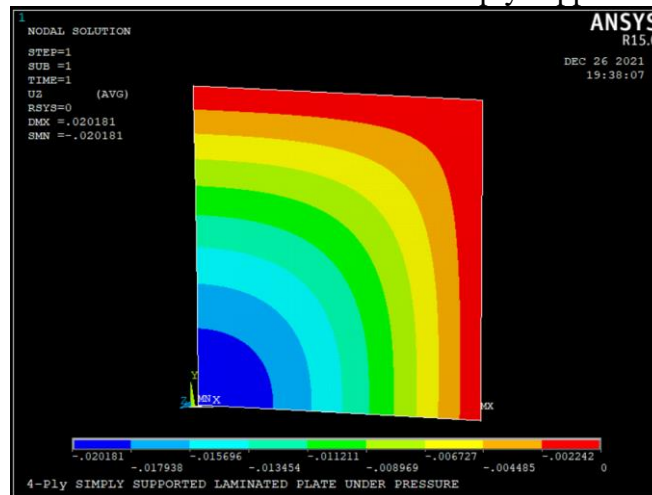


Fig. 4(c) Deflection of  $[+30/-30]_s$  composite plate

Table 1 Validation of model with published literature Manahan (2011) [12]

Stacking sequence	Mesh size	FEM deflection (m)		Navier Method deflection (m)		% error of FEM & Navier Method		Bending coefficient	
		Manahan [2011]	Present	Manahan [2011]	Present	Manahan [2011]	Present	D <sub>16</sub> (N-m)	D <sub>26</sub> (N-m)
[0 90] <sub>s</sub> (specially symmetric Orthotropic)	20 mm x 20 mm	0.0681	0.0682	0.0680	0.0680	0.1468	0.232	0.00	0.00
[60 90] <sub>s</sub> (Sym Angle ply)		0.1165	0.1170	0.0436	0.0436	62.5751	62.73	193.8	565.8

The model of semi-analytical method and FEM as discussed above is used to solve the numerical problem for four ply simply supported symmetric angle ply laminated composite plate under uniform pressure on top surface for orientation angles ranging from 0° to 90°. For symmetric angle ply laminated plate stacking sequence of  $[\theta/-\theta]_s$  is represented by orientation angle “ $\theta$ ” in the y-axis. For example stacking sequence of  $[45/-45]_s$  is

represented by  $45^\circ$  (orientation angle). Deflection is found out by using Navier and Rayleigh-Ritz method and compared with the results from FEM by using FEA software ANSYS. Percentage error is noted for each orientation angle. Also bending coefficient is determined for different orientation angle.

Table 2 Validation of model with published literature of Kenneth Carroll (2013) [13]

Stacking sequence	Rayleigh-Ritz Method deflection (in)	
	Carroll [13] (2013)	Present
$[0\ 90\ 0\ 90]_s$ (specially symmetric Orthotropic)	0.7146	0.7146
$[\pm 45\ 0\ \pm 45]_s$ (Sym Angle ply)	0.1304	0.1304

### 2.7.1 Convergence analysis

A convergence analysis is conducted in FEA software to finalize the size of the mesh. Convergence analysis is done by running the FEA code for different mesh size until a size of mesh is obtained at which same result is repeated. In this thesis, model is first validated with published literature for  $20 \times 20$  mesh size which is mentioned in the literature. Now a convergence analysis is performed with ANSYS, to find out the exact mesh size at which result is repeated. From table 3 and fig 5, it is seen that after  $40 \times 40$  mesh size same result is repeated. Therefore, at  $40 \times 40$  mesh size, a converged result is obtained.

Table 3 Convergence analysis for different mesh size

Mesh size	FEM deflection (m)
10 x 10	0.01983
20 x 20	0.02005
30 x 30	0.02013
40 x 40	0.02018
50 x 50	0.02018

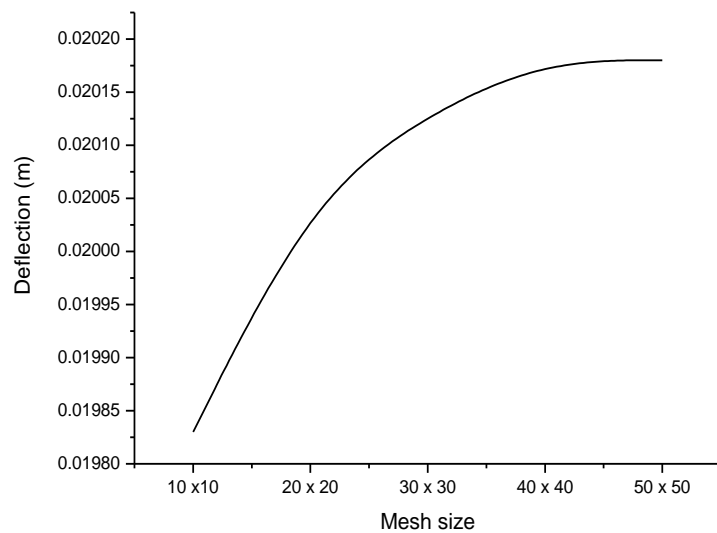


Fig. 5 Convergence analyses for different mesh size

Rayleigh- Ritz method solves  $m * n$  equations with  $m * n$  unknown coefficients using matrix elimination method. Therefore the solution of the Rayleigh-Ritz method depends on the number of equations chosen for the solution. So to make the solution of independent of the numbers of equations, convergence analysis are performed shown in table 4. So it is found out from the table that after the number of equations 49, same results are obtained, which states that the number of equations at which results converge is  $49(m = n = 7)$ .

Table 4 Rayleigh- Ritz method coefficients

$m, n$	No. of equations.	Results
2	4	0.0526
3	9	0.0549
4	16	0.0560
7	49	0.0570
8	64	0.0570

### 2.7.2 Results and Discussion

Deflection of simply supported composite plate determined using Navier Method and Rayleigh-Ritz method is compared with FEA software results for symmetric angle ply composite (table 5). It is found from the table 5 that the percentage error of Navier and Rayleigh-Ritz method with FEA results is within acceptable limit for  $0^\circ$  and  $90^\circ$  but it exceeds the limit for other angles. The percentage error first increases from  $0^\circ$  to  $45^\circ$ , maximum at  $45^\circ$  and then decreases up to  $90^\circ$ . Maximum error occurs at  $45^\circ$  and minimum at  $0^\circ$  and  $90^\circ$ . This is due to the reason that coefficients  $D_{16}$  and  $D_{26}$  are zero for  $0^\circ$  and  $90^\circ$ . Also the coefficient varies in the same way as that of percentage error, i.e. as the error increases coefficient increases and decreases as they decrease which inferred that percentage error depends on the value of coefficient. Also the percentage error of Rayleigh- Ritz method is less as compared to Navier method for all the angles except  $0^\circ$  and  $90^\circ$ . This is because  $0^\circ$  and  $90^\circ$  are considered under specially orthotropic composite (cross ply composite). Therefore one of the ways to reduce the error may be by reducing the value of  $D_{16}$  and  $D_{26}$  coefficients. Navier method takes  $D_{16}$  and  $D_{26}$  coefficients as zero to solve the equations whereas Rayleigh-Ritz method considers the coefficient as non zero. That is why the percentage error of Navier method is greater than Rayleigh-Ritz method for all angles of angle ply composite except  $0^\circ$  and  $90^\circ$  whose  $D_{16}$  and  $D_{26}$  are zeros. Manahan in his paper discussed how to reduce the error by reducing the coefficients by changing the stacking sequence and number of ply but he did not discussed in detail the dependability of error percentage on different factors i.e. laminate thickness, orientation angle, number of ply and mesh size. Whereas Carroll in his paper find out the percentage error, but the dependability of percentage error on different factors were absent. For cross ply laminate bending coefficients are zero, therefore eq. 20 can be used to determine the deflection which is solved by Navier method because eq. 20 is taken by assuming the bending coefficient zero. From table 5, it is found that the percentage error of Navier method with FEA is zero for cross ply laminate and angle ply laminate of  $0^\circ$  and  $90^\circ$ , which shows that eq. 20 is effective to solve cross ply laminate and  $0^\circ$  and  $90^\circ$  angle ply laminate because they come under specially orthotropic laminate (Bending coefficient is zero). But in eq. 29 which is formed by considering bending coefficient, is used to solve cross ply laminate, it is found that the percentage error is 2.9 % which is below 5 %, therefore acceptable. But eq. 20 is more effective in solving cross ply laminate. For angle ply laminate, opposite thing happened because, bending coefficient are not zero. But using eq. 6.19 for angle ply, percentage error is reduced but not acceptable because error is greater than 5%. This may be due to the reason that, eq. 29 solved by Rayleigh-Ritz method gives approximate

solution because, its solution depends on number of equations and it solved equations by matrix elimination method. Therefore semi-analytical method formed by classical lamination theory (CLT) solved by Navier method and Rayleigh-Ritz method gives an approximate solution to problems of laminated composite plate under uniform pressure because the equations for different stacking arrangement (angle ply, cross ply laminate etc.) is derived from general plate equations by assuming some assumptions which is based upon different factors like stacking arrangement, symmetric or non symmetric conditions etc. The equations are solved by different methods like Navier method, Rayleigh-Ritz method etc. which gives an approximate solution because they solve the equations by Fourier expansion, matrix elimination etc. Also they neglect the higher order terms for simplification. So, all these assumptions make the solution approximate and dependent on different factors which are taken for solution. To study the deviation of analytical results from the actual, they are compared with FEM, and percentage error is found out. Therefore from the above discussion it may be conferred that the percentage error also depends on the above factors. The study of percentage error is discussed in next section

Table 5 Comparison of Semi-analytical method and FEM for Symmetric angle plies composite plate simply supported on all the sides under transverse uniform pressure.

Angle Orientation [+ $\theta$ / $-\theta$ ] <sub>s</sub> (DEG)	Semi-Analytical Method		Deflection (m) (FEM)	Bending Coefficient		Percentage Error	
	Deflection(m) (Navier Method)	Deflection (m) (Rayleigh-Ritz Method)		D16	D26	Navier method with FEM	Rayleigh-Ritz method with FEM
0	0.0166	0.0177	0.0166	0	0	0.00	6.33
15	0.0143	0.0160	0.0211	1.93e4	2.18e3	32.13	24.20
30	0.0112	0.0131	0.0202	2.71e4	1.00e4	44.50	35.28
45	0.0101	0.0119	0.0188	2.14e4	2.14e4	46.39	37.10
60	0.0112	0.0131	0.0202	1.00e4	2.71e4	44.42	35.19
75	0.0143	0.0160	0.0211	2.18e3	1.93e4	32.09	24.11
90	0.0166	0.0177	0.0166	0	0	0.29	6.02
Symmetric Cross ply							
[90/0] <sub>s</sub>	0.0171	0.0176	0.0171	0	0	0	2.9

### 2.7.2.1 Effect of stacking sequence (Orientation angles)

The model is now extended to study the effect of orientation angle on the deflection of simply supported composite plates for symmetric angle ply plates of stacking sequence (orientation angle) of [+ $\theta$ / $-\theta$ ]<sub>s</sub> (fig 6) in table 5. Maximum deflection takes place at of [+0/ $-0$ ]<sub>s</sub> and [+90/ $-90$ ]<sub>s</sub> orientation angle. Minimum deflection occurs at a 45° angle ([+45/ $-45$ ]<sub>s</sub>). The deflection decreases as the orientation angle increases and becomes minimum at 45° angle, then increases and becomes maximum at 90°angle (fig 6). For representation [+ $\theta$ / $-\theta$ ]<sub>s</sub>, is represented by  $\theta$  in the y axis of fig 6.

### 2.7.2.2 Effect of no. of ply (Lamina)

The model is further extended to study the effect of a number of plies or lamina on the deflection of simply supported plate. From the fig 7 and fig 8 it is seen that deflection decreases as the number of plies increases for both symmetric angle ply and cross ply composite. For cross ply composite it is seen from the fig 8 that the results of the three methods are almost same, due to the reason as discussed above. The bending coefficients D<sub>16</sub> and D<sub>26</sub> are zeros.

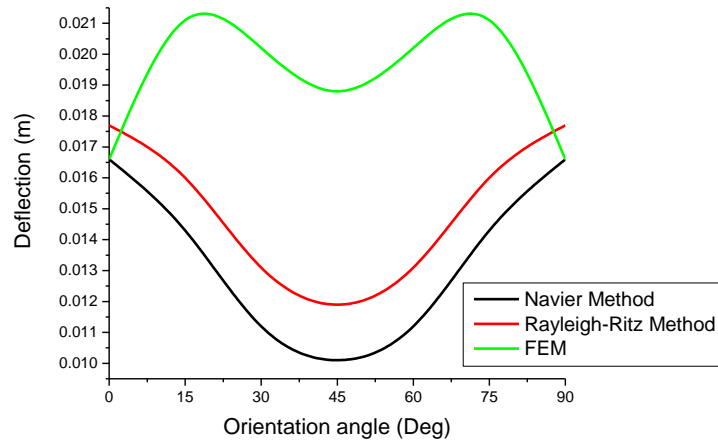


Fig. 6 Variation of deflection with orientation angle for different methods

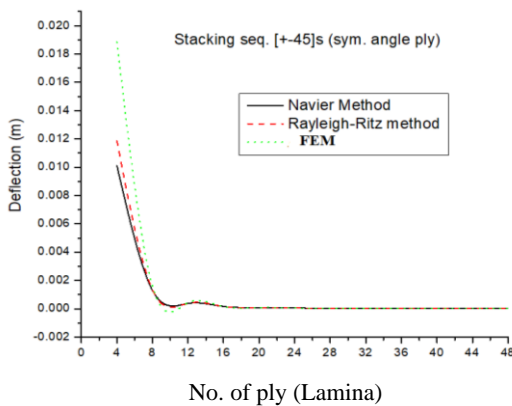


Fig. 7 variation of deflection with no. of plies for sym. Angle ply composite

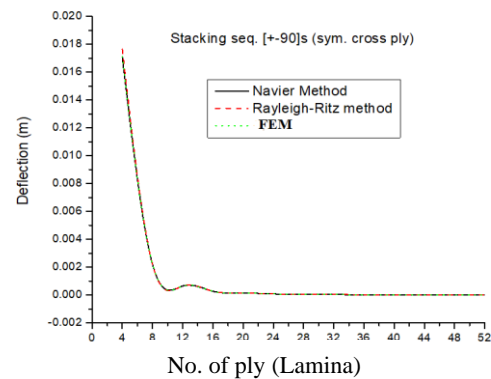


Fig. 8 variation of deflection with no. of plies for sym. cross ply composite

### 2.7.2.3 Effect of length to thickness ratio

Fig 9 shows that as the length to thickness ratio increases, deflection of the plate increases. Length to thickness ratio increases can be varied in two ways: keeping the thickness constant and vary the length or vice versa. When thickness is kept constant and length is increased, ratio increases at the same time deflection also increases. On the other hand, when the length is kept constant and thickness decreases, ratio increases and deflection increases. It shows that deflection increases with the increase in length of the plate and decrease in thickness of the plate.

## 2.8 Analysis of Percentage error of laminated composite Plate

As discussed above, the results from FEM are compared with semi-analytical method (Navier Method) solved in MATLAB platform for different orientation angles (or stacking sequence), thickness of the lamina, no. of ply or lamina for simply supported laminated symmetric angle ply composite plate under the uniform distributed load. As discussed above stacking sequence of symmetric angle ply laminate of  $[\theta/-\theta]_s$  is represented by “ $\theta$ ” (orientation angle) in this thesis. Percentage error is the difference in the results of FEM (FEA software) with semi-analytical method with respect to semi-analytical results. In the above discussion, it is seen that deflection depends on the number of lamina or ply, thickness of the lamina and orientation angle. The variable Length to width ratio is not considered because square laminated plate ( $L/W=1$ ) is taken for analysis and mesh size is eliminated in the

present analysis by convergence. The effect of these variables on %age error is discussed in this section to determine the significance of the variables in reducing the percentage error.

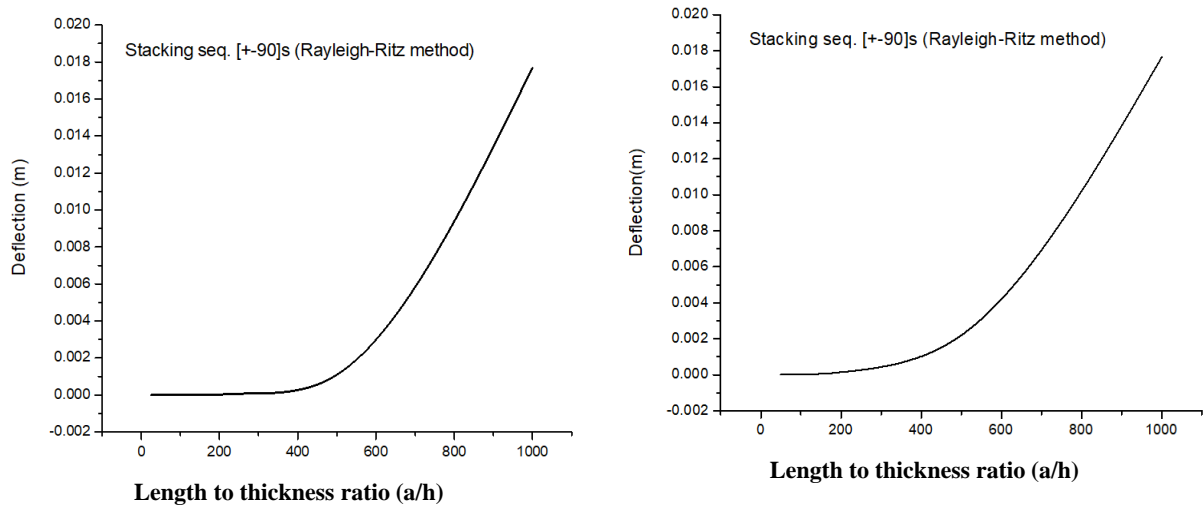


Fig. 9 Variation of deflection with length to thickness ratio using two ways: Keeping thickness constant, length varies (left) and length constant, thickness varies (right)

### 2.8.1 Thickness of lamina (or Ply)

The variation of percentage error in thickness per lamina is investigated keeping the other factors constant i.e. orientation angle and number of lamina. From the table 6 and fig 10, it is seen that the effect of thickness per lamina is very less significant on %age error because the variation of the graph is almost constant. The significance of thickness per lamina is studied for different orientation angle and number of lamina by ANOVA and its relation with %age error is established by regression analysis in the next section for detailed investigation.

### 2.8.2 Number of Lamina (or Ply)

Another important variable is the number of lamina (or ply). The significance of this variable is studied keeping the other variable viz. Orientation angle and thickness per ply constant. From the fig 11 and table 7 it is seen that no. of lamina has a significant effect on %age error. As the no. of lamina increases %age error decreases. It shows a sharp decrease from 4 to 8 and after 8, the variation in curve is less. Since the thickness per lamina is constant, the increase in the number of lamina increases the laminate thickness. Therefore, in other words, it may be say that as the laminate thickness increases, % error decreases which are already seen in the case of beam in the previous section.

### 2.8.3 Orientation angle (Stacking sequence)

Another important variable is orientation angle. In this section symmetric arrangement of angle ply laminated composite plate is taken. The percentage error varies as the orientation angle changes from 0 deg to 90 deg. From the fig 12 it is seen that for angle 0 deg and 90 deg, the %age error is zero because the bending stiffness D16 and D26 is zero as discussed above in details. Also it is found that the curve first increases as the angle increases up to 45 deg and then decreases to zero at 90 deg. Therefore 45 deg angle posses the highest %age error. So while selecting the levels for ANOVA statistical analysis, angle 0 deg and 90 deg are excluded because %age error is negligible.

Table 6. Effect of thickness per lamina for 4 ply square plate (L/W=1), angle= [45/-45]s

Sl. No.	Thickness of Lamina	% Error
1	0.00125	41.59
2	0.0025	41.61
3	0.00375	42.02
4	0.005	41.58
5	0.00625	43.53
6	0.0075	41.53
7	0.00875	40.95
8	0.01	38.46

Table 7 Effect of No. of lamina for square plate (L/W=1), angle= [45/-45]s, thickness/lamina is 5mm

Sl. No.	Thickness /Lamina	% Error
1	0.00125	41.59
2	0.0025	41.61
3	0.00375	42.02
4	0.005	41.58
5	0.00625	43.53
6	0.0075	41.53
7	0.00875	40.95
8	0.01	38.46

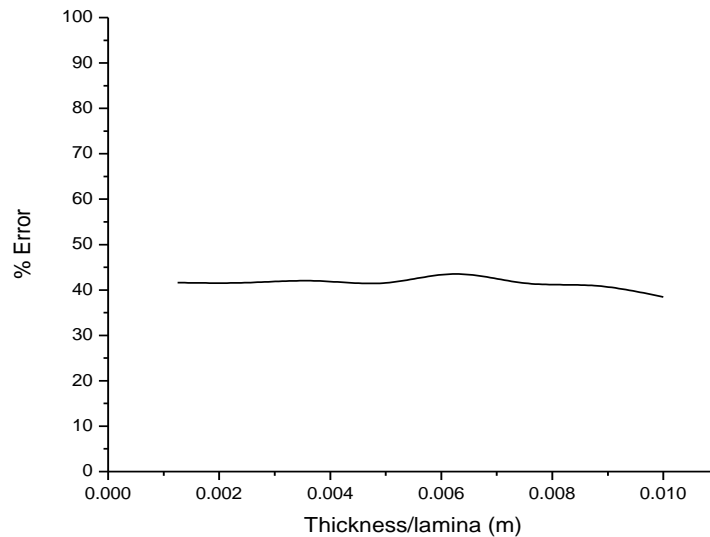


Fig. 10 Variation of thickness per lamina with % error for 4 ply, 45 deg angle square plate subjected to transverse load of 1 N/m

### 2.8.4 Statistical Analysis of Percentage Error in deflection

To understand the effect of these variables, No. of Lamina (A), thickness per lamina (B) and orientation angle (C) on percentage error (E), data from the table 8 are analyzed using analysis of variance (ANOVA) statistical tool. Three levels of these variables are chosen for creating an ANOVA table (table 9). In this analysis, No. of Lamina (A), thickness per lamina (B) and orientation angle (C) is independent variables whereas percentage error (E) is dependent variable or response factor. The significance level was based on the P-value, i.e. Significant if  $P < 0.05$  and insignificant for  $P > 0.05$  (5% significance level or 95% confidence level). MATLAB tool is used to solve the ANOVA table.

Both one way and two ways ANOVA test is conducted on response variable: percentage error and the output are presented in table 10 and 11 respectively. In both the tables it is found that the variable thickness per lamina is insignificant with probability greater than 5%. The variable number of the lamina and orientation angle is significant with probability of zero.

From one way ANOVA table, it may be predicted that the significance of factor A is more than B because the probability value of A is less than B, though the probability of both are less than 0.05.

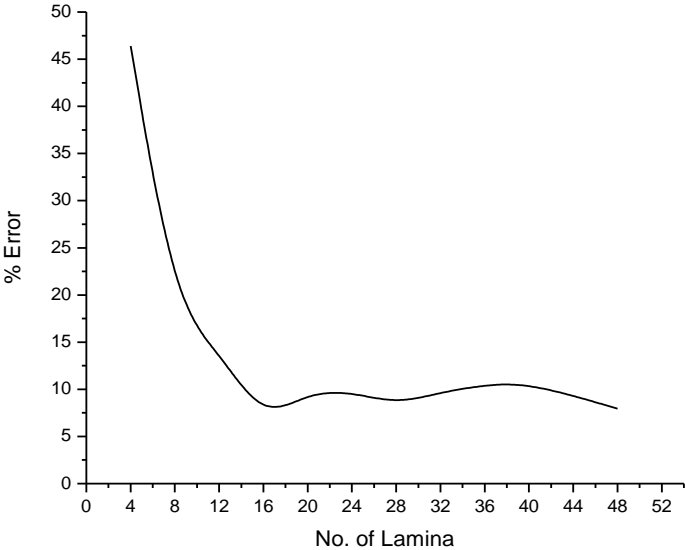


Fig. 11 Variation of No. of lamina with % error for 45 deg angle square plate subjected to transverse load of 1 N/m

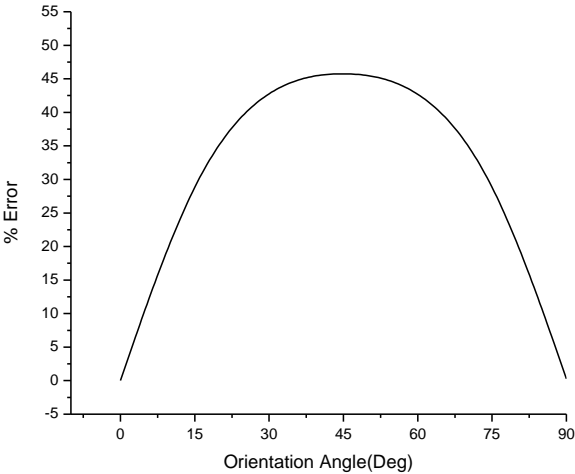


Fig. 12 Variation of Orientation angle with % error for square plate subjected to transverse load of 1 N/m of thickness per lamina of 5mm.

The interaction between No. of lamina (A) and orientation angle (C) i.e. AC is significant with a probability value zero since A and C is significant. The interaction of thickness per lamina (B) with A and C (AB & BC) is not significant with probability value 0.3875 and 0.4663 respectively. The interaction between thickness per lamina (B) and orientation angle (C) i.e. BC is much higher than AB, which shows that the significance of the number of lamina (A) on response is more than orientation angle (C).

**2.8.5 (a) Regression equation**

In this thesis Design Expert 13.0 software has been applied to the data to obtain the mathematical equations for % error. In the present study responses, % age errors (E) are functionally of No. of Lamina (A), thickness per lamina (B) and orientation angle (C). The

regression equations developed by RSM, used for predicting responses of % error (E) in terms of these variables are given as:

$$E = 10.99 - 9.27125 * A + 0.52375 * B + 2.9275 * C + 1.595 * A*B - 4.1675 * A*C + 0.1625 * B*C + 10.6925 * A^2 + 4.3425 * B^2 - 9.27 * C^2 \quad (32)$$

For given levels of each factor, the equation in terms of coded factors can be used to make predictions about the response. The high levels of the factors are coded as +1 and the low levels of the factors are coded as -1 by default. By comparing the factor coefficients, the coded equation can be used to determine the relative impact of the factors.

### 2.8.5 (b) Validation of model

The adequacy of the developed model for % error has been tested using the statistical analysis of variance (ANOVA) technique, which revealed that the regression is significant with linear and quadratic terms for % error at 95% confidence level because its p-value is less than 0.05. The signal to noise ratio is measured with adequate precision. It is preferable to have a ratio of more than four. This model has a signal-to-noise ratio of 8.09, which suggests a good signal. Therefore, this model can be used to navigate the design space. The developed model has also been validated by the normal probability plot of the residuals for % error as shown in Fig. 13 and it is found that the residuals fall on the straight line indicating that the errors are distributed normally and the developed mathematical relationship is accurate.

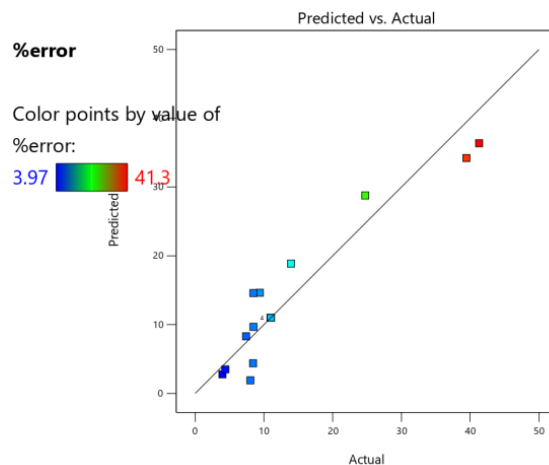


Fig. 13 Residual plot of % age error

Table 8 ANOVA table

SL NO.	A	B	C	M	A	% error
1	4	0.005	10	0.0155	0.0168	8.39
2	4	0.005	40	0.0102	0.0144	41.30
3	4	0.005	70	0.0131	0.0164	25.23
4	4	0.0075	10	0.0046	0.0050	8.46
5	4	0.0075	40	0.003	0.0043	42.77
6	4	0.0075	70	0.0039	0.0049	24.74
7	4	0.01	10	0.0019	0.0021	10.84
8	4	0.01	40	0.0013	0.0018	39.46
9	4	0.01	70	0.0016	0.002	25.00
10	12	0.005	10	5.73E-04	5.96E-04	3.97
11	12	0.005	40	3.77E-04	4.13E-04	9.53

12	12	0.005	70	4.86E-04	5.22E-04	7.41
13	12	0.0075	10	1.70E-04	1.77E-04	4.21
14	12	0.0075	40	1.12E-04	1.24E-04	10.99
15	12	0.0075	70	1.57E-04	1.55E-04	1.08
16	12	0.01	10	7.17E-05	7.48E-05	4.39
17	12	0.01	40	4.71E-05	5.28E-05	12.03
18	12	0.01	70	6.07E-05	6.59E-05	8.48
19	20	0.005	10	1.24E-04	1.29E-04	4.18
20	20	0.005	40	8.14E-05	8.91E-05	9.40
21	20	0.005	70	1.05E-04	1.12E-04	6.70
22	20	0.0075	10	3.67E-05	3.36E-05	8.42
23	20	0.0075	40	0.00002	2.69E-05	11.47
24	20	0.0075	70	3.11E-05	3.36E-05	8.03
25	20	0.01	10	1.55E-05	1.62E-05	4.66
26	20	0.01	40	1.02E-05	1.16E-05	13.94
27	20	0.01	70	1.31E-05	1.43E-05	8.99

Table 9 Variables for statistical analysis with levels.

Variables	Unit	Notation	Level		
			1	2	3
No. of lamina	unit less	A	4	12	20
Thickness/ply	mm	B	0.005	0.0075	0.01
Angle	deg	C	10	40	70

Table 10 Results for one way ANOVA

Source	Sum of Square	Degree of freedom	Mean Square	F	Probability (P)	Remarks
No. of Lamina (A)	1842.0985	2	921.0492	25.4411	3.19e-06 < 0.05	significant
Thickness per Lamina (B)	7.8137	2	3.9068	0.1079	0.898 > 0.05	In significant
Angle (C)	993.6062	2	496.8031	13.7226	1.77e-4 < 0.05	significant
Error	724.0627	20	36.2031			
Total	3567.5812	26				

### 2.8.5 (c) Effect of Process Parameters

Fig 14 and 15 show the variation of factors with the response in surface plot and 2D plot respectively. From the plot 15:B it is seen that the variation of % error with thickness per the lamina is almost constant, because the graph is almost horizontal which shows that the thickness per lamina is insignificant. Also the 2D plot of the no. of lamina (fig 15 A) and orientation angle (fig 15 B) with %age error shows the same nature of graph as discussed in section 2.8.2 and 2.8.3. In the graph 15, the black line is the mean of upper limit and lower limit blue line. The 3 D surface plot (fig 14) shows the combined effect of the parameters on the % error. From the fig 14(a) and 14(c), it shows that the effect of thickness per lamina on % error is not significant because the curve is smoothly decreasing in fig.14 (a) and increasing

to a peak value and then decreasing in fig 14(c) as discussed above for no. of lamina and orientation angle. But for fig 14(b) the curve is bending which shows the combined effect both significant variable numbers of lamina and orientation angle. Fig. 15 is the 2 dimensional representation of the variation of parameters with the % age error of fig 14.

Table 11 Results for two way ANOVA

Source	Sum of Square	Degree of freedom	Mean Square	F-value	P-value	Remark
A	1842.0985	2	921.0492	253.1965	0 < 0.05	Significant
B	7.8137	2	3.9068	1.07399	0.386 > 0.05	In Significant
C	993.6062	2	496.8031	136.5712	0 < 0.05	Significant
A*B	17.2164	4	4.3041	1.1832	0.3875 > 0.05	In Significant
A*C	663.3711	4	165.8427	45.5902	0 < 0.05	Significant
B*C	14.3736	4	3.5934	0.9878	0.466 > 0.05	In Significant
Error	29.1014	8	3.6376			
Total	3567.5812	26				

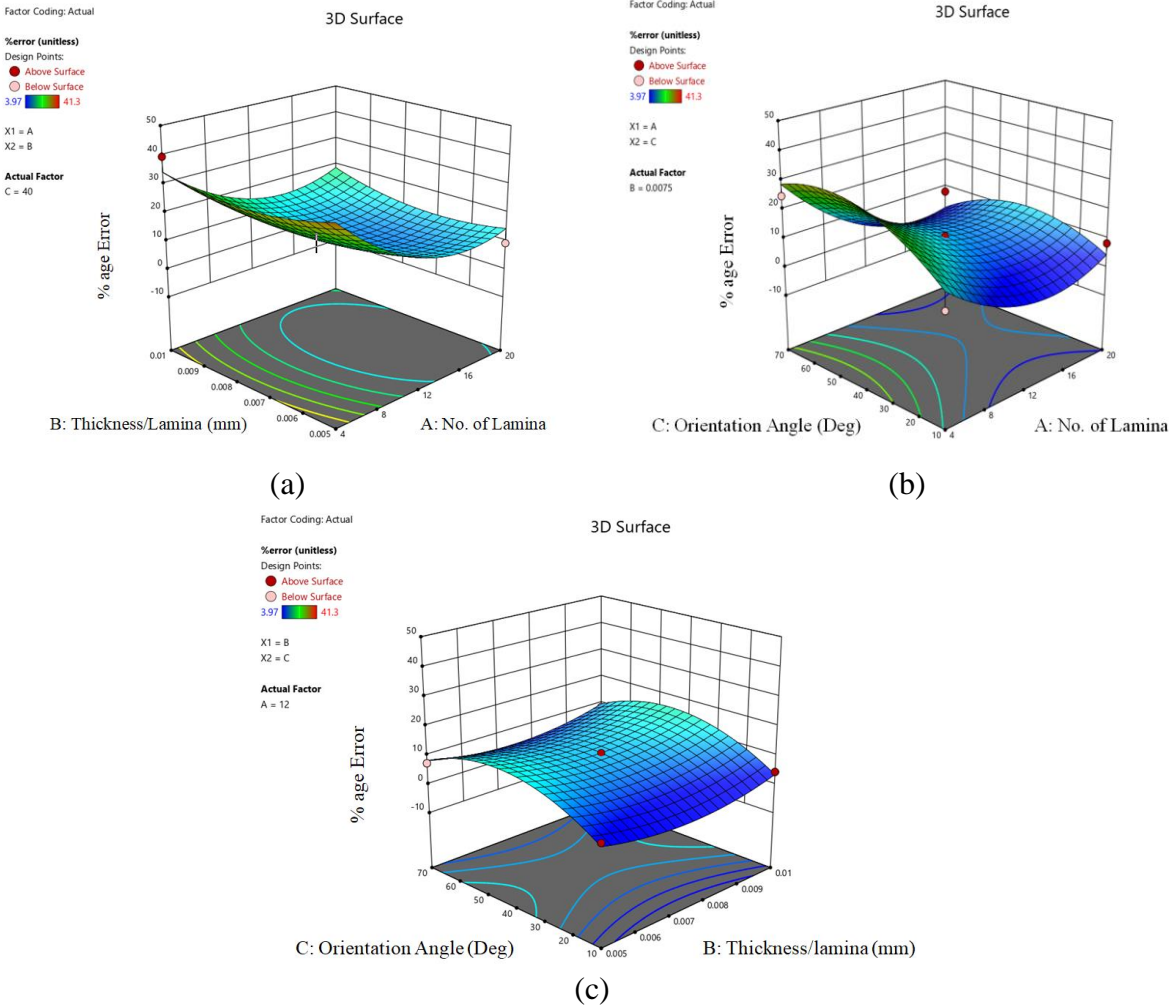


Fig. 14 Surface plot of factors with response (% age Error)

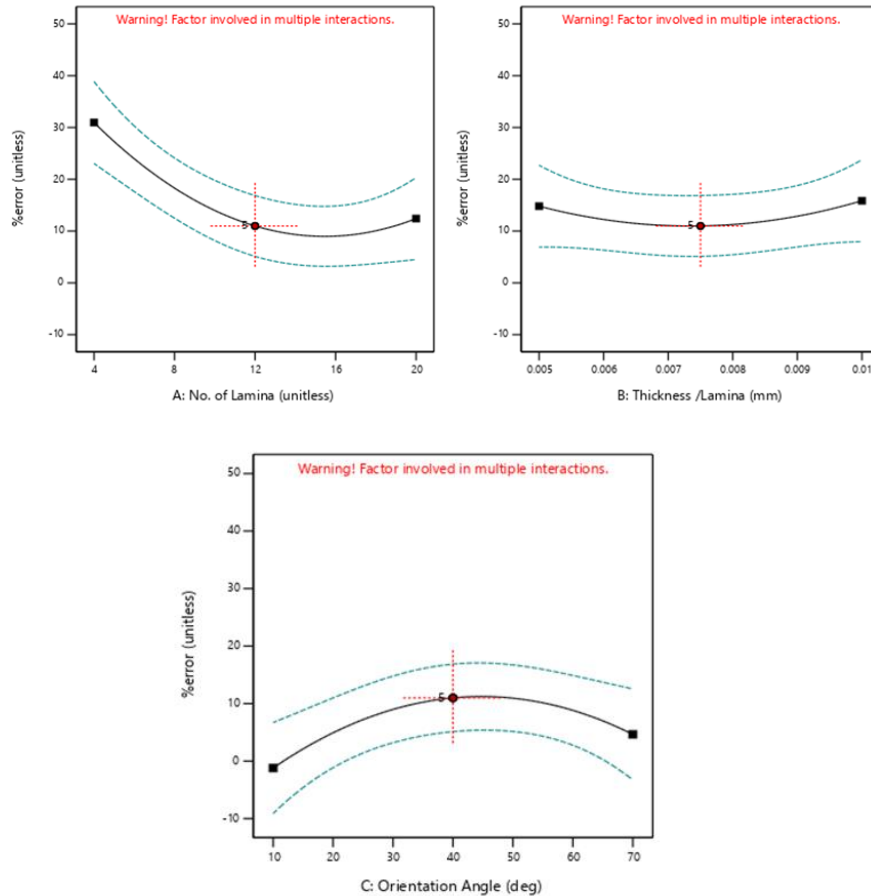


Fig. 15 2 D plot of factors with response (% age Error)

### 2.8.6 Results & Discussion

It may be concluded that the percentage error for laminated composite plate under transverse load depends on the number of lamina and orientation angle (or stacking sequence). If the thickness of each lamina is kept constant, then increase in the number of lamina means increase in total thickness of the laminate. So, number of lamina can be predicted as laminate thickness. Therefore, in other way it can be said that, % error of the plate depends on laminate thickness and orientation angle. Mesh size is an insignificant variable because mesh size can be converged by convergence analysis. Statistical analysis is conducted to reduce the number of variables by testing their significance at 5% significance level. Therefore the percentage error may be reduced by proper optimization of these variables.

### Conclusion

It may be concluded from the discussion of this chapter:

1. Deflection of simply supported composite plate determined using Navier Method and Rayleigh-Ritz method is compared with FEM using FEA software results for symmetric angle ply and cross plies composite.
2. Navier method is more efficient to solve cross ply composite problem, whereas Rayleigh-Ritz method is used to solve angle ply composite problem.
3. Navier method is used to solve the plate equations which assume bending coefficient as zero. For cross ply composite and  $0^\circ$  and  $90^\circ$  angle ply composite bending coefficient is

zero. But for angle ply composite bending coefficient is not zero. Therefore Rayleigh-Ritz method is more efficient to solve this type of problem.

4. It is seen that the percentage error of Navier and Rayleigh method with FEA results is within acceptable limits for  $0^\circ$  and  $90^\circ$  but it exceeds the limit for other angles.

5. Navier method takes  $D_{16}$  and  $D_{26}$  coefficients as zero to solve the equations, whereas Rayleigh-Ritz method considers the coefficient as non zero. That is why the percentage error of Navier method is greater than Rayleigh-Ritz method except  $0^\circ$  and  $90^\circ$  whose  $D_{16}$  and  $D_{26}$  are zeros.

6. It is seen that deflection decreases as the number of plies increases for both symmetric angle ply and cross ply composite.

7. It is found from the analysis that as the length to thickness ratio increases, deflection of the plate increases.

8. Percentage error for laminated composite plate under transverse load depends on laminate thickness and orientation angle.

9. Mesh size is an insignificant variable because mesh size can be converged by convergence analysis. Statistical analysis is conducted to reduce the number of variables by testing their significance at the 5% significance level. Therefore the percentage error may be reduced by proper optimization of these variables.

10. The percentage error with FEA software may be reduced, if the general governing equation is used to solve the problem without neglecting the bending coefficients. Second, if the problem is solved by higher order theories. ANSYS results may be improved by improving the model by modeling each lamina separately and joined them using a function.

## References

- [1] Isaac, M. D., Ori, I. "Engineering Mechanics of Composite materials", Oxford University Press, Inc., New York, **2007**. ISBN 978-0195150971.
- [2] Pandya, B. N., Kant, T. "Finite element analysis of laminated composite plates using a higher-order displacement model", Composite Science and Technology 32(2), pp. 137 – 155, **1988**.
- [3] Kam, T.Y., Lin, S.C., Hsiao, K.M. "Reliability analysis of nonlinear laminated composite plate structures", Composite Structure 25(1–4), pp. 503 – 510, **1993**.
- [4] Han, W., Petyt, M., Hsiao, K.-M. "Investigation into geometrically nonlinear analysis of rectangular laminated plates using the hierarchical finite element method", Finite Element Analysis Des, 18(1–3), pp. 273 – 288, **1994**.
- [5] Sridhar, C., Rao, K.P. "Large deformation finite element analysis of laminated circular composite plates", Computers and structures 54(1), pp. 59 – 64, **1995**.
- [6] Ganapathi, M., Polit, O., Touratier, M. "C0 eight-node membrane shear- bending element for geometrically non-linear (static and dynamic) analysis of laminates", Int. Journal for Numerical Methods in Engineering 39(20), pp. 3453 – 74, **1996**.
- [7] Bao, G., Jiang, W., Roberts, J. C. "Analytic and finite element solutions for bending and buckling of orthotropic rectangular plates", Int. of Journal of solids and structures 34 (14), pp. 1797 – 1822, **1997**.
- [8] Zhang, Y.X., Yang, C.H. "Recent developments in finite element analysis for laminated composite plates", Composite structure 88(1), pp. 147 – 157, **2009**.
- [9] Mauricio, F. C. Jr., Antonio, J.M. Ferreira, Volnei, T. "A review on plate and shell theories for laminated and sandwich structures highlighting the Finite Element Method", Int. Journal of Composite structures 156, pp. 63 – 67, **2016**.

- [10] Anakpotchanakul, Ch., Prombut, P. “Influence of aspect ratios on vibration and bending of composite laminates“, *Materials today proceedings* 17(4), pp. 1588 – 1594, **2019**.
- [11] Kaw, A. K. “Mechanics of Composite material“, second edition, Taylor & Francis, **2006**. ISBN 978-0-8493-1343-1
- [12] Manahan, M. A. “A Finite Element Study of the Deflection of Simply Supported Composite Plates Subject to Uniform Load“, M.E Thesis, Rensselaer Polytechnic Institute Hartford, Connecticut, **2011**.
- [13] Carroll, K. “Maximum Deflection Analysis of Simply Supported Aluminum and Composite Plates Under Uniform Pressure Using Rayleigh-Ritz Method and Finite Element Method“, M.E Thesis, Rensselaer Polytechnic Institute Hartford, Connecticut, **2013**.
- [14] Yang, Q. J. (): “Simplified Approaches to Buckling of Composite Plates“, Thesis for Master of Science, University of Oslo, **2009**.
- [15] Reddy, J. N. “Mechanics of Laminated composite plates and shells- Theory and analysis“, CRC Press, London, New York, **1997**. ISBN 0-8493-1592-1
- [16] Choudhury, A., Mondal, S.C., Sarkar, S. “Failure Analysis of Laminated composite plate under hygro thermomechanical load and optimization”, *IJAME* 24(3), pp. 509 – 526, **1999**.
- [17] Choudhury, A., Mondal, S.C., Sarkar, S. “Failure Analysis of Laminated composite beam for different boundary condition under thermomechanical load”, *IJAME* 25(1), pp.12 – 26, **2020**.
- [18] Javadi, H., Rajabi, I., Yavari, V., Kadivar, M. H. “Three-dimensional solutions for contact area in laminated composite pinned joints with symmetric and non-symmetric stacking sequences“, *Strojnícky časopis – Journal of Mechanical Engineering* 58(6), pp. 327 – 339, **2007**.
- [19] Choudhary, S. S., Tungikar, V.B. “A simple higher-order theory for dynamic analysis of composite plate“, *Strojnícky časopis – Journal of Mechanical Engineering* 61 (3), pp. 169 – 182, **2010**.

## BRIEF NOMENCLATURE

Symbols	Details
$E$	Young’s modulus
$E_1$	longitudinal Young’s modulus (in direction 1)
$E_2$	transverse Young’s modulus (in direction 2)
$\nu_{12}$	Major Poisson’s ratio
$\nu_{21}$	Minor Poisson’s ratio
$G_{12}$	in-plane shear modulus (in plane 1–2)
$G$	Shear modulus
$\tau$	Shear stress at plane (MPa)
$\nu$	Poisson’s ratio
$\sigma_x, \sigma_y, \tau_{xy}$	Stress in global coordinate system
$\sigma_1, \sigma_2, \tau_{12}$	Stress in local coordinate system
$\varepsilon_x, \varepsilon_y, \gamma_{xy}$	Strain in global coordinate system
$\varepsilon_1, \varepsilon_2, \gamma_{12}$	Strain in local coordinate system
$\varepsilon$	Strain tensor
$\theta$	Orientation angle (degree)

$\gamma$	Shear strain at plane (MPa)
$\phi_x$	Rotations of the transverse normal about the $x$ axis
$\phi_y$	Rotations of the transverse normal about the $y$ axis
$x, y, z$	Global coordinate system
1,2,3	Local coordinate system
$wbar$	Non dimensional deflection
$w_0$	Displacements along the coordinate line of a material point on the $x$ - $y$ plane
$w_{max}$	Maximum deflection
$M_{max}$	Maximum bending moment
$\sigma$	Stress tensor (MPa)
$M$	Bending moment
$T$	Transformation matrix
$K$	Shear correction coefficient
$h$	Total thickness of laminate
$z$	Distance from neutral axis
$E_{xx}^b I_{yy}$	Bending stiffness
$Q_{ij}$	Reduced stiffness coefficient
$S_{ij}$	Compliance matrix
[R]	Reuter matrix
$N_x, N_y,$	Normal force per unit length
$N_{xy}$	Shear force per unit length
$M_x, M_y,$	Bending moment per unit length
$M_{xy}$	Twisting moments per unit length
$\epsilon^0$	Mid-plane strain of laminate in $x$ - $y$ coordinate
$k$	Laminate curvature
$A_{ij}$	Extensional stiffness matrix
$B_{ij}$	Extension – bending coupling matrix
$D_{ij}$	Bending stiffness matrix
$u, v, w$	Displacement in $x, y, z$ directions
$I_0, I_1 \& I_2$	Rotary inertia terms
SR	Strength ratio
$q$	Applied Distributed Force
FEM	Finite element model
CLT	Classical Lamination theory
FSDT	First Order shear deformation theory
FEA	Finite element analysis

#### Properties of composite material [11]

Property	Symbol	Unit	Graphite/Epoxy	Glass/Epoxy
Longitudinal elastic modulus	$E_1$	GPa	181	38.6
Transverse elastic modulus	$E_2$	GPa	10.3	8.27
Major Poisson's ratio	$\nu_{12}$		0.28	0.26
Shear modulus	$G_{12}$	GPa	7.17	4.14
Ultimate longitudinal tensile strength	$(\sigma_1^T)_{ult}$	MPa	1500	1062
Ultimate longitudinal compressive strength	$(\sigma_1^C)_{ult}$	MPa	1500	610
Ultimate transverse tensile strength	$(\sigma_2^T)_{ult}$	MPa	40	31
Ultimate transverse compressive strength	$(\sigma_2^C)_{ult}$	MPa	246	118
Ultimate in-plane shear strength	$(\tau_{12})_{ult}$	MPa	68	72