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The One-Cushion Escape from Snooker in a Circular Table

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Abstract

The one-cushion escape from snooker in a circular table can be viewed as a geometric problem involving the reflection of a light ray in a circular mirror. There are at most four escapes in any given configuration. We will obtain specific configurations in which there are exactly 0, 1, 2, 3 or 4 escapes. The details consist in determining the number of real solutions in the interval $(-1, 1)$ of certain polynomial equations, of degrees two, three or four.

1 Snooker in a circular table

The simplest snooker position involves only three balls: the cue-ball is at a, the ball 'on' is at β, and a direct shot from a to β is unavailable because of an intervening ball in the line segment $[a, \beta]$.

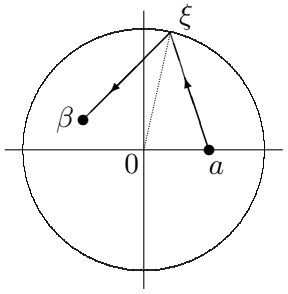


Figure 1

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A one-cushion escape from the snooker is a shot off the boundary at ξ such that

$$\angle a\xi 0 = \angle 0\xi\beta \qquad (*)$$

(Figure 1). The angles $\angle a\xi 0$ and $\angle 0\xi\beta$ are oriented angles, and by implication the points a, ξ and β are distinct. The equality of angles relation $(*)$ is symmetric in a, β. Both angles are numerically strictly less than $\pi/2$. With no loss of generality we may assume that $0 \leqslant a \leqslant 1$ and that β lies in the closed upper semi-disc $D^+ = \{z : |z| \leqslant 1, \operatorname{im} z \geqslant 0\}$.

In some positions several one-cushion escapes exist, and in some positions no one-cushion escape exists. Moreover, for an escape both angles are non-zero; for if $\angle a\xi 0 = 0 = \angle 0\xi\beta$ then the path $a \to \xi \to \beta$ contains a direct path $a \to \beta$, hence does not constitute an escape. This means that for an escape, $\xi \neq \pm 1$. This also implies that if $a = 0$ then there are no one-cushion escapes for any $\beta \in D^+$. Thus we may further assume that both a, β are non-zero. We can formulate the one-cushion escape as follows.

Definition. Denote $I = \{x : 0 < x \leqslant 1\}$, $D^+ = \{z : |z| \leqslant 1, \operatorname{im} z \geqslant 0\}$, $\partial D = \{z : |z| = 1\}$. Let $a \neq \beta$, with $a \in I$, $\beta \in D^+$, $\beta \neq 0$. A point ξ in ∂D is called a *snooker solution* for the configuration or position (a, β) if $\xi \neq \pm 1$, $\xi \neq a$, $\xi \neq \beta$ and $\angle a\xi 0 = \angle 0\xi\beta$.

We shall consistently write $\beta = b + ic$. A snooker solution for the position (a, β), or briefly a *solution*, is one that escapes the snooker. We also say that x is a solution if x is the real part of a solution ξ.

In this article we shall investigate the one-cushion escape in a circular table. On a standard 12×6 rectangular table, a simple geometrical consideration determines snooker escapes ξ. This geometrical method is illustrated in Figure 2. In that position there are four escapes, with ξ on each of the four sides of the rectangle.

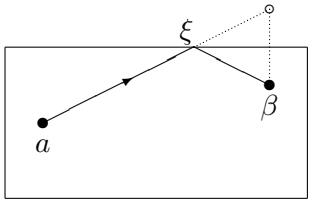


Figure 2

It is easy to find configurations in a rectangular table for which there are 0, 1, 2, 3 or 4 escapes. However, in a circular table it is not as simple to determine the possible escapes for a given configuration. Indeed it is not immediate that there are configurations admitting precisely 1, 2, 3, or 4 escapes. Our aim is to present configurations in which there are exactly 0, 1, 2, 3 or 4 escapes. We obtain these by looking at special configurations, outlined below:

(i) The simplest configuration is (a, β) with β on the real axis. This position provides examples where there are no escapes, and in addition leads to a classification of conjugate solutions.

(ii) The position (a, β) with $\beta = i$ provides examples for 1, 2, or 3 escapes.

(iii) The position (a, β) with $|\beta| = a$ provides examples for 4 escapes.

2 Configuration with no escapes

We fix $0 < a \leqslant 1$, and take $\beta = b + i0$ on the x-axis. We will determine for each b whether there are snooker escapes.

Theorem 1. *Let $0 < a \leqslant 1$, and $\beta = b$ with $-1 \leqslant b \leqslant 1$, $b \neq 0$, $b \neq a$.*
Define intervals $J(a)$ for $0 < a \leqslant 1$ by

$$J(a) = \begin{cases} \left(\frac{a}{2a-1}, \frac{-a}{2a+1}\right), & \text{if } 0 < a \leqslant \frac{1}{3} \\ \left[-1, \frac{-a}{2a+1}\right), & \text{if } \frac{1}{3} < a \leqslant 1 \end{cases}$$

If $0 < a \leqslant 1$ and $b \in J(a)$, then there are exactly two solutions for configuration (a, b). Conversely if $0 < a \leqslant 1$ and $b \notin J(a)$, there are no solutions for (a, b).

Proof. (i) If $0 < a \neq b \leqslant 1$ and $\xi \neq \pm 1$, then the segments $[\xi, b]$ and $[\xi, a]$ lie on the same side of the radius $[\xi, 0]$ (Figure 3). Hence no ξ can be a solution.

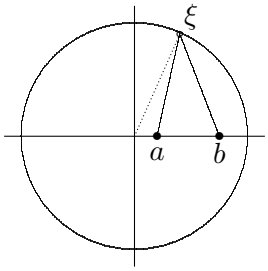


Figure 3

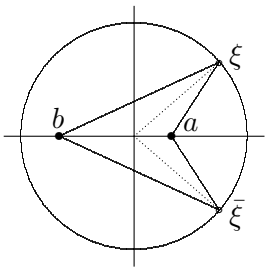


Figure 4

(ii) Suppose $-1 \leqslant b < 0 < a \leqslant 1$ (Figure 4). If ξ is a snooker solution, then $\xi \neq \pm 1$, and by the angle bisector theorem

$$\frac{|\xi - b|}{|\xi - a|} = \frac{|b|}{a}$$

Writing $\xi = x + iy$, we obtain

$$a^2\{(x-b)^2 + y^2\} = b^2\{(x-a)^2 + y^2\}$$

This simplifies to $a^2 - b^2 = 2ab(a-b)x$. Since $a \neq b$, $a \neq 0$, $b \neq 0$ we get $x = \frac{a+b}{2ab} := x^*$. Now x^* is a solution if and only if $|x^*| < 1$. Thus $x^* = x^*(a,b)$ is a solution when (a,b) lies in the set $K := \{(a,b) : 0 < a \leqslant 1, -1 \leqslant b < 0, -1 < \frac{a+b}{2ab} < 1\}$.

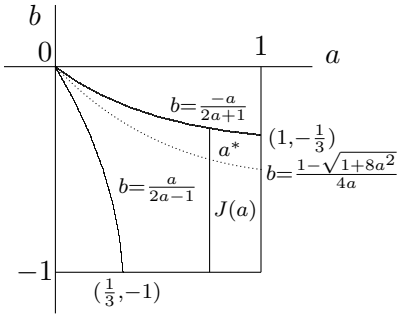


Figure 5

The region K is sketched in Figure 5, from which we may infer the intervals $J(a)$. We see that if $b \in J_a$ there are exactly two solutions ξ, $\bar{\xi}$ where $\mathrm{re}\,\xi = (a+b)/2ab$ (Figure 4). □

Example 1.

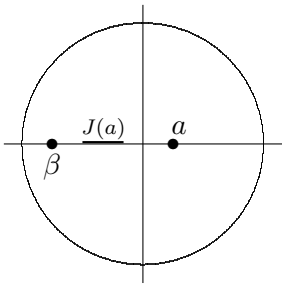


Figure 6: no escapes

In this position $0 < a \leqslant 1$ and $\beta = b$ where $b \neq a$, and $b \notin J(a)$. There are no escapes for configuration (a, β).

To understand this position, take $a = \frac{1}{4}$. Then $J(a) = J(\frac{1}{4}) = (-\frac{1}{2}, -\frac{1}{6})$. For $\beta = b \in J(a)$, escapes are given by ξ, $\bar{\xi}$ where $\mathrm{re}\,\xi = (a+b)/2ab := x^*$. When $(a,b) = (\frac{1}{4}, -\frac{1}{2})$ then $x^* = 1$, when $(a,b) = (\frac{1}{4}, -\frac{1}{6})$ then $x^* = -1$. Thus as ξ moves along the upper semi-circle from $\xi = 1$ to $\xi = -1$, the target β moves along $J(\frac{1}{4})$ from $\beta = -\frac{1}{2}$ to $\beta = -\frac{1}{6}$. All points in the x-axis off $J(a)$ are points of no escape.

2.1 Conjugate solutions

If x is a solution, then there are potentially two corresponding solutions $\xi = x \pm iy$. If both ξ and $\bar{\xi}$ are snooker solutions then we say that these are conjugate solutions. In classifying the precise number of solutions in any configuration it is important to identify those x for which there correspond two conjugate solutions ξ and $\bar{\xi}$, and those x for which there correspond only one solution ξ or else $\bar{\xi}$. We refer to the intervals $J(a)$ as stated in Theorem 1.

Theorem 2. *Suppose* $0 < a \leqslant 1$, *and* $\beta \in D^+$. *Let* $a^* = \frac{1}{4a}\big(1 - \sqrt{1+8a^2}\big)$ *and* $L(a) = \big\{a^* + ic : 0 < c < \sqrt{1-a^{*2}}\big\}$. *Conjugate solutions exist for configuration* (a, β) *if and only if* $\beta \in J(a) \cup L(a)$.

If $\beta \in J(a)$ *then conjugate solutions are* ξ, $\bar{\xi}$, *with* $\operatorname{re}\xi = (a+b)/2ab$. *If* $\beta \in L(a)$ *then conjugate solutions are* ξ, $\bar{\xi}$ *with* $\operatorname{re}\xi = a^*$.

Proof. We divide the proof into two parts.
(i) Assume $c = 0$ so that $\beta = b$ with $-1 \leqslant b \leqslant 1$, $b \neq a$, $b \neq 0$. By Theorem 1, there exist solutions if and only if $b \in J(a)$; the solutions are conjugate ξ, $\bar{\xi}$ with $\operatorname{re}\xi = (a+b)/2ab$.

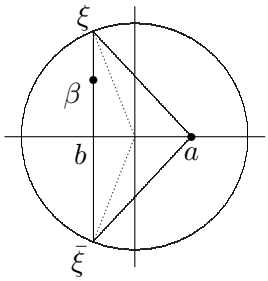


Figure 7

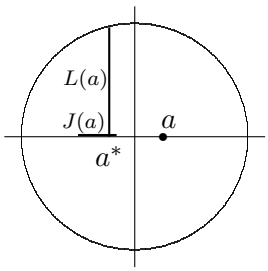


Figure 8

(ii) Assume $c > 0$ and $\beta = b + ic$ with $-1 < b < 1$. By the geometry there are no conjugate solutions when $\beta \in \partial D$, whence $0 < c < \sqrt{1-b^2}$. Moreover, by geometrical considerations, we see that conjugate solutions ξ, $\bar{\xi}$ exist for β only if $\operatorname{re}\xi = b$ (Figure 7). By Theorem 1, $b \in J(a)$ and $(a+b)/2ab = b$. This reduces to $2ab^2 - b - a = 0$, so that $b = \frac{1}{4a}(1 \pm \sqrt{1+8a^2}\,)$. By Theorem 1, $b < 0$, whence $b = \frac{1}{4a}\big(1 - \sqrt{1+8a^2}\,\big) := a^*$. Note that $a^* \in J(a)$ and $-\frac{1}{2} \leqslant a^* < 0$ (Figure 8). For each $\beta \in L(a)$ conjugate solutions are ξ, $\bar{\xi}$ where $\operatorname{re}\xi = a^*$. □

3 General equations for a snooker solution

Theorem 3. *Given configuration* (a, β), $\beta = b + ic$. *If* $\xi = x + iy$ *is a snooker solution then*

$$\text{(i)}\quad \frac{1-ax}{\sqrt{1+a^2-2ax}} = \frac{1-bx-cy}{\sqrt{1+b^2+c^2-2bx-2cy}},$$

(ii) $(a^2+b^2+c^2-2ab)x^2-2ac^2x-(a^2+b^2-2ab-a^2c^2)\neq 0$, *and*

(iii) $x^2+y^2=1$.

Proof. The unoriented angle θ between two non-zero vectors $\mathbf{a}$, $\mathbf{b}$ is given by $\theta=\cos^{-1}\{(\mathbf{a}\cdot\mathbf{b})/|\mathbf{a}||\mathbf{b}|\}$. Equation (i) expresses the equality of non-oriented angles $\angle a\xi 0$ and $\angle 0\xi\beta$. However this equation contains two solutions ξ in which ξ, a, β lie on the same side of segment $[0,\xi]$, that is, when ξ, a, β are collinear; these are not snooker solutions. When $b=a$ then $\beta=a+ic$, hence these inadmissible solutions are obtained from $x=a$, $x^2+y^2=1$. When $b\neq a$ these inadmissible solutions are obtained from $y=\dfrac{c}{b-a}(x-a)$, $x^2+y^2=1$. Eliminating y between these latter equations we obtain the equation

$$(a^2+b^2+c^2-2ab)x^2-2ac^2x-(a^2+b^2-2ab-a^2c^2)=0 \qquad (*)$$

When $b=a$, so that $c\neq 0$, this equation becomes $x^2-2ax+a^2=0$ from which $x=a$; thus the equation $(*)$ includes the case $b=a$. Relation (ii) excludes these solutions. □

We use Theorem 3 to analyze the case $\beta=i$.

4 Configurations with 1, 2, or 3 escapes

Configurations with 1, 2 or 3 escapes may be obtained from the configuration (a,β) with $\beta=i$.

Theorem 4. *Assume* $0<a\leqslant 1$, $\beta=i$, *If* $\xi=x+iy$ *is a solution then*

$$4a^2x^3-4ax^2+(1-3a^2)x+2a=0$$

Proof. Let $\xi=x+iy$ be a snooker solution. With $b=0$, $c=1$ in Theorem 3, we obtain

$$\frac{1-ax}{\sqrt{1+a^2-2ax}}=\frac{1-y}{\sqrt{2-2y}};\quad x^2+y^2=1$$

We eliminate y between these equations.
We have $(1-2ax+a^2x^2)(2-2y)=(1-2ax+a^2)(1+y^2-2y)$.
Thus $(1-2ax)(1-y^2)+2a^2x^2-a^2(1+y^2)=-2y(a^2-a^2x^2)$.
Hence $(1-2ax)x^2+2a^2x^2-a^2-a^2(1-x^2)=-2a^2y^3$.
Therefore $-2ax^3+(1+3a^2)x^2-2a^2=-2a^2y^3$.
Thus $\left\{2ax^3-(1+3a^2)x^2+2a^2\right\}^2=4a^4(1-x^2)^3$, whence
$4a^2(a^2+1)x^6-4a(3a^2+1)x^5+(-3a^4+6a^2+1)x^4+8a^3x^3-4a^2x^2=0$.

By Theorem 3 the preceding equation includes the roots of the polynomial $(a^2+1)x^2-2ax$. This leads to a factorization

$$\big\{(a^2+1)x^2-2ax\big\}\big\{4a^2x^4-4ax^3+(1-3a^2)x^2+2ax\big\}=0$$

By the geometry of the configuration, we see that the root $x=0$ is not a snooker solution. The result follows. □

We next determine the number of roots in the interval $(-1,1)$ of the cubic in Theorem 4: we will use the following result, see Cohn [Coh82].

Harriot-Descartes rule: The equation $a_0+a_1x+\cdots+a_nx_n=0$, with real coefficients, cannot have more positive roots than the number of sign changes in the sequence $a_0,\ldots,a_n$ (ignoring zeros) and it differs from the number of these changes by an even number.

Theorem 5. *Let $0<a\leqslant 1$, $\beta=i$, and $\kappa=\sqrt{\frac{3+2\sqrt{3}}{9}}$.*

(i) *If $a=1$ then there are exactly two solutions.*

(ii) *If $0<a<\kappa$ then there is exactly one solution.*

(iii) *If $a=\kappa$ then there are exactly two solutions.*

(iv) *If $\kappa<a<1$ then there are exactly three solutions.*

Proof. By Theorem 4, solutions $\xi=x+iy$ are obtained from the cubic equation

$$f(x):=4a^2x^3-4ax^2+(1-3a^2)x+2a=0$$

(i) When $a=1$ the above equation becomes

$$2x^3-2x^2-x+1=0,\ \text{whence}\ (x-1)(2x^2-1)=0$$

Since $\xi=\pm1$ are not solutions, we deduce that the root $x=1$ is not admissible, and therefore there are two solutions $x=\pm\frac{1}{\sqrt{2}}$.

We next assume that $0<a<1$. Now $f(-1)=-(a+1)^2<0$, $f(0)=2a>0$, $f(1)=(a-1)^2>0$. Hence $f(x)$ has at least one zero in the interval $(-1,0)$. The coefficients of the polynomial $f(-x)$ are $-4a^2$, $-4a$, $3a^2-1$, $2a$, and these have signs $(-,-,\epsilon,+)$ where $\epsilon=0,\pm$. The number of sign changes is one. By Harriot-Descartes' rule, $f(x)$ has exactly one negative root, which we have seen lies in $(-1,0)$. Denote this negative root by x_1. The existence of zeros of $f(x)$ in the interval $(0,1)$ is determined by the location of the minimum point of $f(x)$.

The function f has a local maximum at $(s_1,f(s_1))$ given by

$$s_1=\frac{2-\sqrt{1+9a^2}}{6a},\quad f(s_1)=\frac{1+27a^2+\sqrt{(1+9a^2)^3}}{27a}$$

and a local minimum at $(s_2, f(s_2))$ given by

$$s_2 = \frac{2+\sqrt{1+9a^2}}{6a}, \quad f(s_2) = \frac{1+27a^2-\sqrt{(1+9a^2)^3}}{27a}$$

Set $g(x) = \dfrac{1+27x^2-\sqrt{(1+9x^2)^3}}{27x}$, defined for $x > 0$. We find that $g(x)$ has exactly one zero given by $x = \kappa = \sqrt{\frac{3+2\sqrt{3}}{9}} \approx 0.8475$; moreover $0 < \kappa < 1$. Hence $g(x)$ keeps constant sign in $0 < x < \kappa$ and constant sign in $\kappa < x < \infty$. We note that $g(\frac{1}{3}) > 0$ and $g(1) < 0$. There are three cases.

(ii) Suppose that a satisfies $0 < a < \kappa$. Then $g(a) > 0$, so $f(x)$ has its minimum value above the x-axis. This implies that $f(x)$ has no positive zeros. In this configuration there is exactly one snooker solution, denoted by x_1 in part (i).

(iii) Suppose that $a = \kappa$. Then $f(s_2) = g(\kappa) = 0$, and furthermore, $f(x)$ has a double root at s_2, where

$$s_2 = \frac{2+\sqrt{1+9\,\kappa^2}}{6\kappa} = \frac{3+\sqrt{3}}{2\sqrt{3+2\sqrt{3}}} = \sqrt{\frac{(3+\sqrt{3})^2}{4(3+2\sqrt{3})}} = \sqrt{\frac{\sqrt{3}}{2}}$$

We see that $0 < s_2 < 1$. In this configuration there are exactly two snooker solutions, namely, x_1 and s_2.

In this case we can determine the exact value of the negative solution denoted x_1. First $x_2 = x_3 = \sqrt{\frac{\sqrt{3}}{2}}$ is a double root of $f(x)$. Using the relation between the product of the roots of a cubic and the coefficients of the cubic, we have $x_1x_2x_3 = \frac{-1}{2\kappa}$, whence $x_1 = -\sqrt{2\sqrt{3}-3}$.

(iv) Suppose that a satisfies $\kappa < a < 1$. Then $g(a) < 0$. Now $f(x)$ attains a local minimum value $f(s_2)$ at $s_2 = (2+\sqrt{1+9a^2})/6a$, and $f(s_2) = g(a)$. In order to determine whether $s_2 < 1$, we first calculate that the inequality $0 < s_2 < 1$ holds when $a > (4+\sqrt{7})/9 := \delta \approx 0.7384$. Since $a > \kappa$ and $\kappa > \delta$ we conclude that indeed $0 < s_2 < 1$. That is, $f(x)$ attains a minimum at $(s_2, g(a))$, where $g(a) < 0$. Recall that $f(0) > 0$ and $f(1) > 0$. We deduce that $f(x)$ has a zero x_2 in the interval $(0, s_2)$ and a zero x_3 in $(s_2, 1)$. In this configuration there are three solutions x_1, x_2, x_3. □

Example 2. We illustrate the cases $0 < a < \kappa$, $a = \kappa$, $\kappa < a < 1$, in Figure 9, these having precisely 1, 2, and 3 escapes respectively.

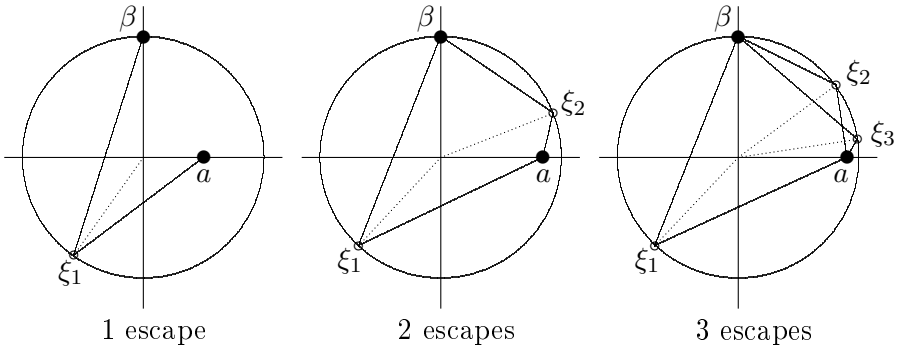


Figure 9

5 Configurations with 4 escapes

The next configuration assumes that $|\beta| = a$. As we shall see, these configurations admit either two solutions ξ_1, ξ_2 that are the endpoints of a diameter, or four solutions that are the vertices ξ_1, ξ_2, ξ_3, ξ_4 of a kite.

Theorem 6. *Suppose* $0 < a \leqslant 1$ *and* $|\beta| = a$. *Then there are either two or four solutions.*

Proof. The perpendicular bisector of the segment $[a, \beta]$ passes through the origin, and meets the boundary at solutions denoted ξ_1, ξ_2 (Figure 10).

Now consider a configuration $OACB$, where $A = a$, $B = \beta$, and C is a solution ξ (Figure 11).

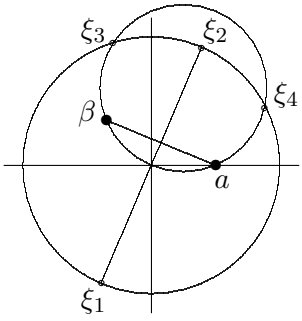


Figure 10

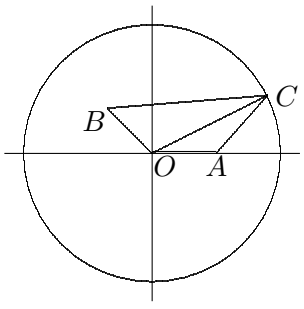


Figure 11

By the sine formula we have

$$\frac{OB}{\sin OCB} = \frac{OC}{\sin OBC} \quad \text{and} \quad \frac{OC}{\sin OAC} = \frac{OA}{\sin OCA}$$

Since $OB = OA$ and $\angle OCB = \angle OCA$, the above equations imply that $\sin OBC = \sin OAC$, so that $\angle OBC = \angle OAC$ or $\angle OBC + \angle OAC = \pi$.

Suppose that $\angle OBC = \angle OAC$. Then $\triangle OBC \equiv \triangle OAC$ and C lies on perpendicular bisector of the segment AB.

Suppose that $\angle OBC + \angle OAC = \pi$. Then the quadrilateral $OACB$ is cyclic. The circle through A, O, B meets the boundary ∂D in no points, one point ξ_2, or two points shown as ξ_3, ξ_4 in Figure 10.

We conclude that there are either two solutions ξ_1, ξ_2, or four solutions ξ_1, ξ_2, ξ_3, ξ_4. □

Example 3.

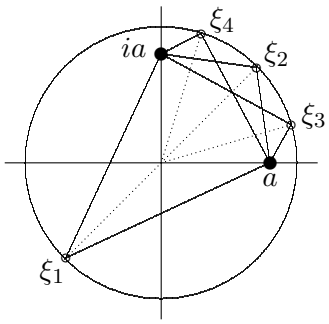


Figure 12: 4 escapes

Figure 12 shows the configuration (a, β) where $\frac{1}{\sqrt{2}} < a < 1$, $\beta = ia$, in which there are four solutions.

Example 4.

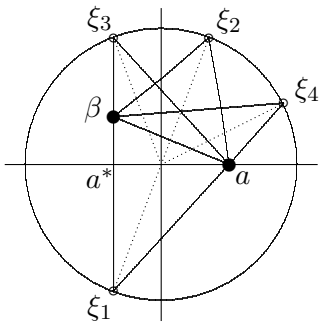


Figure 13: 4 escapes

Figure 13 shows a configuration with four solutions, obtained by an application of Theorems 2 and 6. The configuration (a, β), $\beta = b + ic$, can be obtained as follows. Select a in $(0, 1)$. Use Theorem 2 to find $b = a^*$ such that $\beta = b + ic$ admits conjugate solutions, and that $a^* > -a$. Select c with $b^2 + c^2 = a^2$.

References

[Coh82] P. M. Cohn, *Algebra Volume 1* (2nd edn.), Wiley, 172–173, 1982.