sciendo FCT Fundação para a Ciência e a Tecnologia CIUHCT Ludus

How unfair is the unfair DODGEM?

Lucas Gerin

École Polytechnique, Palaiseau (France)

Abstract

We study a very simple 2-player board game called Dodgem, curiously the game is difficult to analyze when the number of tokens is not the same for the two players. We provide theoretical and experimental elements which indicate which player benefits from the asymmetry of the game.

1 Introdution

Dodgem is a simple combinatorial game invented by Colin Vout in the 1970s. It is originally played on an 3×3 board with 2 cars for both players (called Blue/Red in the sequel) but can be played on any $n \times n$ board with $n-1$ cars for each player.

The purpose of this article is to investigate with artificial brute force an innocent-looking question raised by Ian Stewart in his marvelous book [Ste09]:

> *It seems natural to use square boards, but with a rectangular board the player with fewer counters has to move them further, so the game may be playable on rectangular boards. As far as I know, they've not been considered.*

Usually the least we ask when playing a 2-player game is that the rules are fair and totally symmetrical, the only asymmetry usually tolerated is that the player who starts may have an advantage. Here we will discover that the unfair dodgem is quite interesting, and that it is not easy to determine whether it is better to play red or blue.

The rules of the game

For integers $r, b \geqslant 1$ the r-b-dodgem is played on a $(b+1) \times (r+1)$ board where for convenience board squares are labeled from $(0,0)$ (bottom-left) to (b,r) (top-right).

©2023 Lucas Gerin. This is an open access article licensed under the Creative Commons Attribution-NonCommercial-NoDerivs License (https://creativecommons.org/licenses/by-nc-nd/4.0/).

Recreational Mathematics Magazine, pp. 41–50
DOI 10.2478/rmm-2023-0002

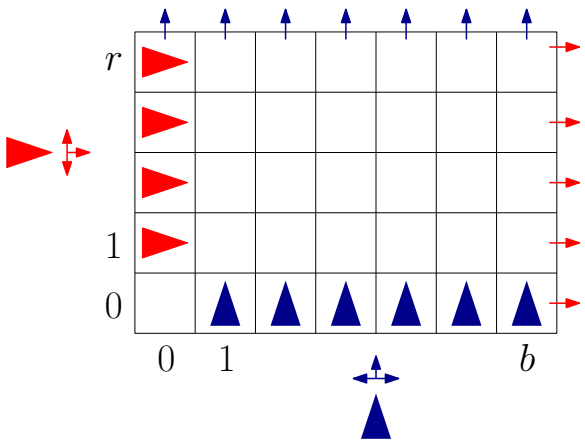


Figure 1: The board of r-b-dodgem in its initial configuration.

Initially b blue cars are parked at $(1,0),(2,0),\ldots,(b,0)$ and r red cars are parked at $(0,1),\ldots,(0,r)$ and the remaining squares are empty (see Fig.1). Players alternate, cars move one step at a time. Blue cars can go left/right/up to an empty square while red cars can go up/down/right. Blue cars can leave the board at the top, red cars at the right. The winner is the player who first moves all their cars off the board, or has all their cars blocked in by the opponent.

Unfairness

All along the article we will assume that $r \leqslant b$, we will discuss the results according to whether Red or Blue plays first. If you think a while about the question raised by I.Stewart for the strictly asymmetric case $r < b$ it is not immediate to figure out whether the game is advantageous or disadvantageous for Red. Indeed several contradictory features of the game come into play:

Advantage for Red	Advantage for Blue
(A) Red must move a total of (at least) $r(b+1)$ steps which is less than $(r+1)b$, the minimal number of moves of Blue	(B) Blue has more choices of possible moves (at least at the beginning of the game)
(C) As Red has less cars it is easier for Red to get blocked by Blue	(D) Righmost cars of Blue can move freely at the beginning of the game without being bothered by red cars.

Solving by hand

On a small board it is possible to analyze the game for perfect players by extensive research. For instance for the 1-2-`dodgem` one can prove by exploring all configurations that the first player has a winning strategy (in Fig.2 we show two trees which guarantee victory for the first player).

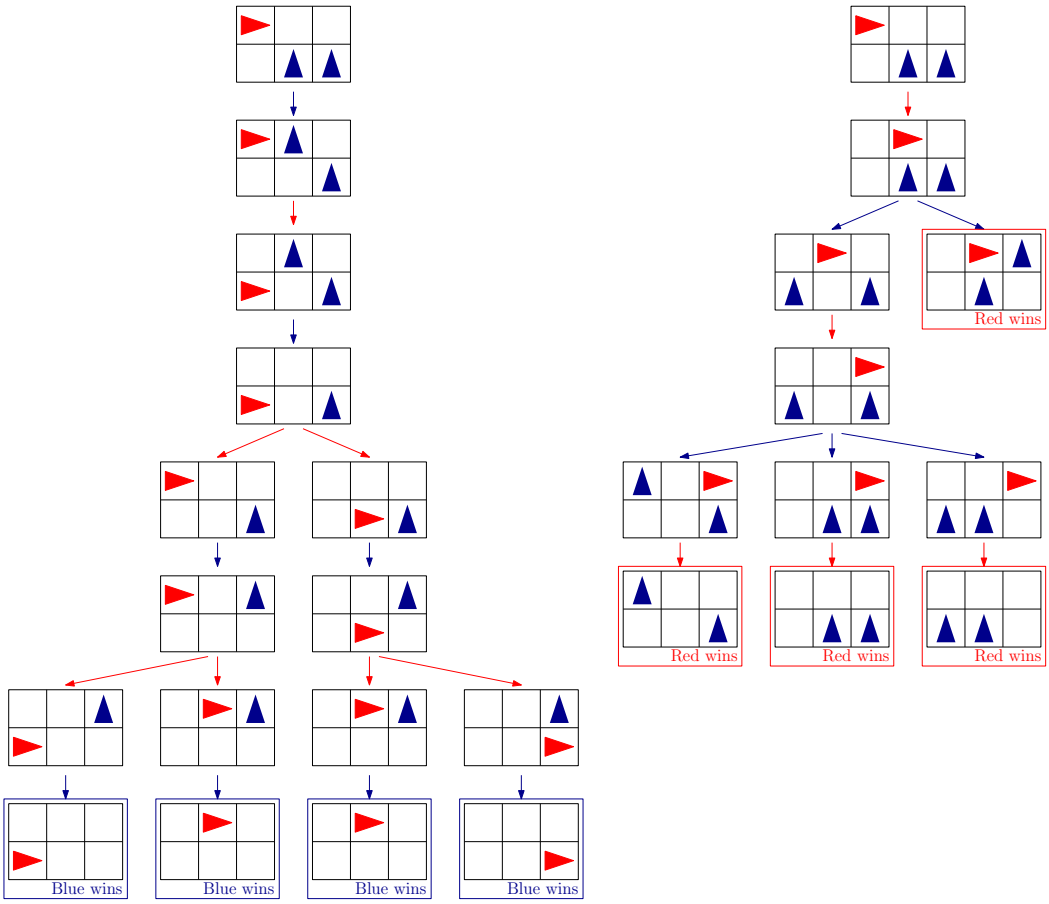


Figure 2: Analysis of the 1-2-`dodgem`: Two trees of perfect games for `Blue` when `Blue` starts (left) and for `Red` when `Red` starts (right). A red/blue arrow from a configuration to another represents a legal move for the corresponding player. No matter the moves of the opponent, these trees indicate how first player can ensure victory.

In [BCG82, p.749-750] (see also [Games, Mar06]) the status of all configurations in the 2-2-`dodgem` is given. In particular this shows that the first player has a winning

strategy. In principle one could use the same method to solve the game on larger boards, unfortunately we cannot get very far by hand. Indeed even for small r, b's the number of distinct board configurations grows very fast:

$$\begin{aligned}\text{Total \# of configurations} &= \sum_{B\leqslant b, R\leqslant r} \text{\# of configurations with } R \text{ red cars and } B \text{ blue cars} \\ &= \sum_{B\leqslant b, R\leqslant r} \binom{(r+1)(b+1)}{R} \times \binom{(r+1)(b+1)-R}{B}\end{aligned}$$

With a computer we get the following values for small r, b's :

(r, b)	# of configurations	# of reachable configurations
$(1, 1)$	21	15
$(1, 2)$	118	98
$(1, 3)$	605	520
$(2, 2)$	1423	1397
$(2, 3)$	14699	14579
$(3, 3)$	277993	277582
$(3, 4)$	4.5×10^6	?
$(4, 4)$	1.2×10^8	?
...		
$(11, 11)$	$\approx$ number of atoms on Earth	

In the above table the third column indicates the number of configurations that are reachable from the starting configuration using only legal moves. There is probably no explicit formula for this quantity, here it is found by exploring the graph of all configurations.

2 Solving `dodgem` with a computer: backward induction

We want to solve the r-b-`dodgem` given who plays first. This means that we want to find the winner (if any) assuming that both `Red` and `Blue` play the best possible move at each step.

To solve the game we will use *backward induction*: we reason backwards in time. Let us illustrate the method with Fig.2 for the 1-2-`dodgem`. We first consider board configurations for which the game ended, for example $c =$  which is a winning configuration for `Blue`. Then one can determine what `Blue` needs to play at any configuration preceding c (this is  in our case of Fig.2). This process continues backwards in the graph

until we have explored the whole graph of configurations (or at least until we have gone back to the initial configuration).

For curiosity we show in Fig.3 the complete graphs associated to 1-1-`dodgem` and 1-2-`dodgem`. Graphs are very messy for larger boards.

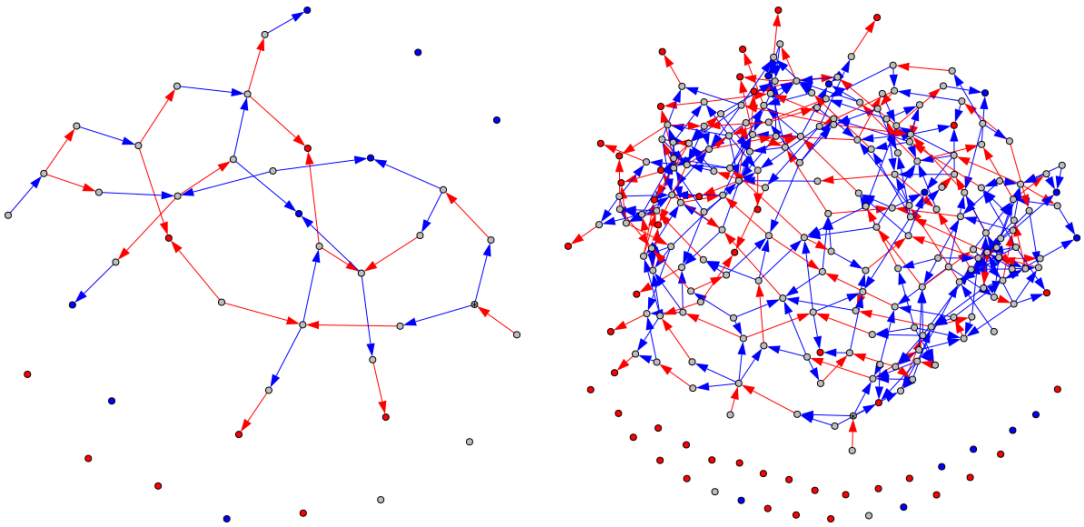


Figure 3: The graph of all configurations in the 1-1-`dodgem` (Left) and 1-2-`dodgem` (Right) when `Blue` starts. Each node represents a configuration, colors of edges indicate legal moves. Colored nodes correspond to final winning positions for `Blue`/`Red`. The lonely points all around both graphs correspond to isolated configurations. (Graphs layouts were obtained with the `python` library `iGraph`.)

Our experimentations show the following (our results will be summarized in Fig.6):

- For $(r, b) = (1, 1), (1, 2), (2, 2)$ the player who plays first has a winning strategy.
- For $(r, b) = (1, 4), (2, 3), (2, 4)$ `Blue` has a winning strategy, no matter who plays first.

In the 1990's David desJardins [des96] (then undergraduate student at Princeton) proved with a computer that perfect 3-3-`dodgem` and 4-4-`dodgem` lead to a draw[1].

[1]Notice that what desJardins denotes by 4×4 `-dodgem` is what we call 3-3-`dodgem`.

3 Some observations

By examining the experimental results obtained by backward induction we obtain some interesting observations.

3.1 Disadvantage for the first player

Computer experiments reveal that if we start at some particular configurations the second player has a winning strategy. On Fig.4 we show all possibilities (up to the symmetry `Red`/`Blue`) for the 2-2-`dodgem`. In particular this indicates that it is impossible to use the so-called *strategy-stealing* argument to solve the game of `dodgem` (even in this fair case $r = b = 2$). This applies to a symmetric game in which an extra move can never be a disadvantage to show by contradiction that the second player cannot have a winning strategy (see [Gar88, Ch.8] for a reader-friendly introduction to the strategy-stealing argument).

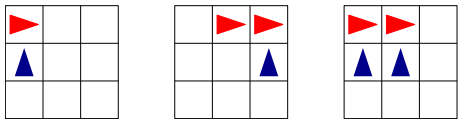


Figure 4: Configurations starting from which no one wants to play first.

3.2 The simplest draw

In Fig.5 we show configurations leading to a draw for two perfect players on the 2-3-`dodgem` (`Blue` has already moved one car off the board).

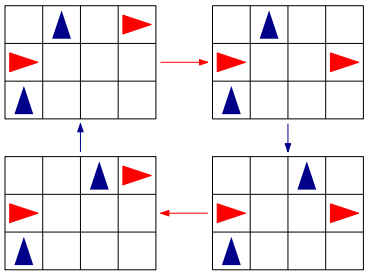


Figure 5: Starting from one of these configurations, a perfect game is never ending.

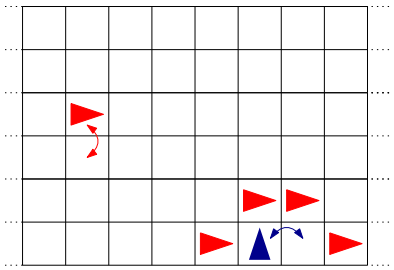
Note that this is not contradictory with the fact that `Blue` has a winning strategy

at 2-3-dodgem. Indeed if Blue plays perfectly this cycle is not reachable from initial configuration by Red.

Such cycles do not exist for smaller boards.

4 Much more is no worse (and is likely to be best)

It is quite tempting to think that in general the game is easier for Blue when b is much bigger than r but this is not so easy to prove. Indeed the following example shows a configuration for which Red can reach a draw with small r (say $r = 5$) and very large b:



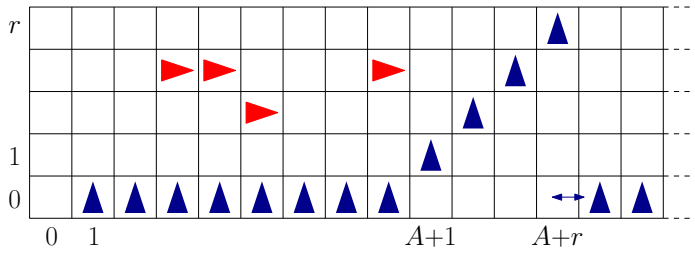
We see that Red has kidnapped a blue car.

I now present the results of my best efforts to obtain rigorous results for the general r-b-dodgem.

Theorem 1. *If $r \geqslant 2$ and $b \geqslant \frac{r^2+3r}{2}+1$ then there is a strategy for* Blue *that guarantees at least a draw (no matter who begins).*

Proof. Here is your strategy if you play Blue, no matter the moves of Red.

Set $A = r(r+1)/2$. Devote your $1+2+3+\cdots+r = A$ first moves to move your cars number $A+1, A+2, \ldots A+r$ upwards in order to form a staircase:



(While you are doing this red cars cannot bother you because the fastest one will have arrived (at most) at abscissa A by the time you have finished the staircase.) Then,

move your car number $A + r + 1$ ($= \frac{r^2+3r}{2} + 1 \leqslant b$) from left to right forever and you ensure the draw.

Something may be wrong with this strategy: `Red` can be blocked by blue cars if $r = 1$ or more generally if r is a triangular number (*i.e.* r can be written in the form $r = 1 + 2 + \cdots + m$ for some integer m) and if `Red` moves all its cars as far as possible above the staircase. In the case where r is triangular, form the staircase as below and at the next step move your car number $A + r$ at coordinates $(A + r - 1, r)$ to form a truncated staircase. `Red` will not be blocked. □

(With a little more care it is easy to lower the bound from $\frac{1}{2}(r^2+3r)+1$ *to* $\frac{1}{2}(r^2+r)+1$ *in the theorem by showing that one can actually build the staircase at* $\widetilde{A} = 1+2+\cdots+(r-1)$ *without being bothered by red cars.)*

I have tried for a while to modify the previous strategy to guarantee victory for `Blue`, possibly with the hypothesis of an even bigger b. Unfortunately I am not able to rule out the possibility that `Red` kidnaps a blue car on the left of the board.

However, to end on a positive note I propose the following exercise for the case $r = 1$:

Exercise 1. If $r = 1$ and $b \geqslant 3$ then there is a strategy for `Blue` that guarantees victory (no matter who begins).

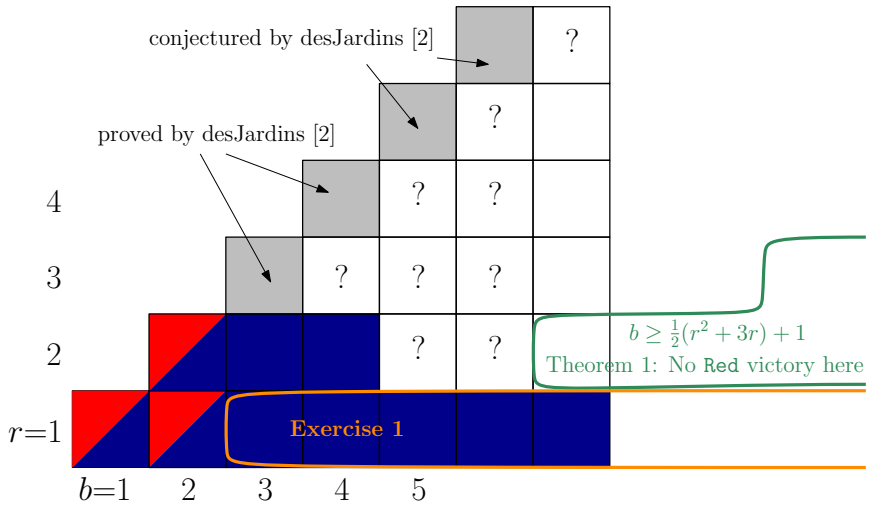


Figure 6: Summary of results for perfect players. Legend: top-left (resp. bottom-right) triangle of each case indicates the color of the winner if `Red` (resp. `Blue`) starts, grey indicates a draw.

5 Random players: Monte-Carlo simulations

At least for small boards the case of the r-b-`dodgem` is solved for perfect players. I now discuss the case of less skilled players, with random simulations. I chose a very simplistic modelling, `Blue` and `Red` play as follows:

- Each player moves uniformly at random one of its cars which can go forward (*i.e.* upwards for `Blue`, rightward for `Red`) ;
- If there is no car that may move forward then the player plays uniformly at random one of the remaining legal moves (of course if there is no legal move the game is over).

For each case I ran 10000 simulations in order to estimate the probability of winning with a 95% confidence interval smaller than $\pm 1\%$ (see for example Sec.VII.5 in [Fel68]). The results[1] are shown in Fig.7. In particular we observe that the table is much redder than the table of perfect players. It seems that for random players the dominating property in the table of p.42 is Ⓐ.

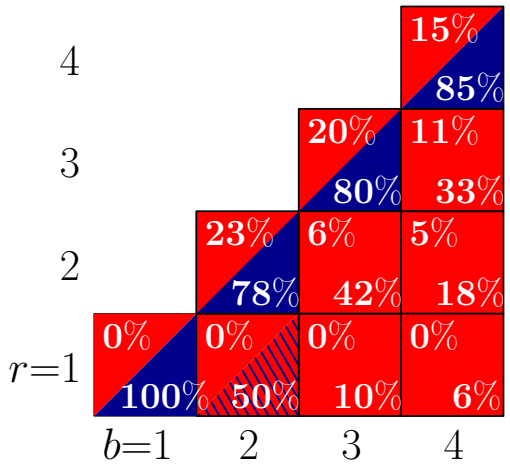


Figure 7: Frequencies of victories for `Blue` when both players play randomly. It is shown in the top-left (resp. bottom-right) triangle of each case the frequency of victories of `Blue` when `Red` (resp. `Blue`) starts. Color indicates the player above 50%.

[1] For example, `Blue` won 2322 of the 10000 games in the 2-2-dodgem when `Red` starts. This gives a 95% confidence interval smaller than 0.2322 ± 0.01 that I rounded in 23% in the table for more readibility.

Conclusion

In the end, how unfair is the unfair `dodgem`? Which color would you prefer to play with if $r < b$? When b is much larger than r our study confirms the intuition that you would better play `Blue`. When b, r are pretty close the answer is less clear and depends on how smart do you think you are. If you don't want to learn graphs like Fig.3 by heart you may indeed prefer to play `Red`.

References

[BCG82] E. R. Berlekamp, J.H. Conway, R.K. Guy, *Winning Ways for Your Mathematical Plays*, volume 3, AK Peters/CRC Press, 1982.

[des96] D. desJardins, Unpublished work, 1996. See `https://www.ics.uci.edu/~eppstein/cgt/dodgem.html`.

[Fel68] W. Feller, *An introduction to probability theory and its applications*, vol. 1, 3rd edition, Wiley, 1968.

[Games] GamesCrafters group (UC Berkeley). `http://gamescrafters.berkeley.edu/games.php?game=dodgem`.

[Gar88] M. Gardner, *Hexaflexagons and other mathematical diversions: The first Scientific American book of puzzles and games*, University of Chicago Press, 1988.

[Mar06] T. W. Marlow, "The Game of Dodgem", *Games and Puzzles Journal*, issue 16, 2006. Available at `https://www.mayhematics.com/k/games4.htm`.

[Ste09] I. Stewart, *Professor Stewart's hoard of mathematical treasures*, Basic Books, 2009.