Musical modes, their associated chords and their musicality

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Abstract

In this paper we present a mathematical way of defining musical modes and we define the musicality of a mode as a product of three diferent factors. We conclude by classyfing the modes which are most musical according to our definition.

1 Musical scales and modes

First we must define some concepts. In traditional western style of music there are 12 different notes in an octave. Two notes represent the same tone if they are at a distance of $12n$ semitones (a multiple of ocatves apart). The 13^{th} note restarts the cycle and it is the same tone as our first note (simply an octavehigher). Loosely speaking a scale is simply an ordered sequence of different tones within the same octave. For instance $CDEFGAB$ is the well known C major scale, $ABCDEFG$ is the A minor scale, and $EGAA^{\#}BD$ is the hexatonic blues scale in the key of E. The first note of the scale is said to be the key of the scale. For further considerations about scales we will asociate every tone with a numerical value, namely its position in the chromatic A scale ($AA^{\#}BCC^{\#}CDD^{\#}EFF^{\#}GG^{\#}$). The lexicographic order of tones will be defined as the order induced by their numerical values defined as above. The precise mathematical definition of a scale is:

Definition 1.1. *A k scale is a circular permutation of k different tones in lexicographic order.*

To exemplify the definition let's take 7 different tones in increasing order, namely $ABCDEFG$. This is simply the A minor scale. The next circular permutation of the same tones is $BCDEFGA$ and is called B Mixolydian. The following is $CDEFGAB$ and is called C major, and thus circularly permuting the notes of the A minor scale we obtain 7 different types of scales used in western music. It is well known that these scales

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have the same number of chords (harmonic triads). A song composed in the A minor key simply uses only the tones of the scale A minor. For instance a song written in the key of A minor will use the same tones as a song in the C major key. In order to understand what a musical mode is, let's examine the following two scales. Take the first one to be $CDEFGAB$ (C major) and the second one $ABC^{\#}DEF^{\#}G^{\#}$(A major). Although they start on a different note when played they sound quite similar. The reason is that the number of semitones between two consecutive notes in both of them follow the same pattern, that is classically known as w,w,h,w,w,w,h (where w = whole = 2 semitones and h = half = 1 semitone). Two scales that have a similar interval pattern are said to be of the same mode(thus the name major for both of the scales considered above). Now assume that a scale starts on the note D and has the same interval pattern (major). It's easy to see that the notes of the D major scale will be $DEF^{\#}GABC^{\#}$, and thus we obtain another scale in the major mode. Any song that is played in a certain major scale can be played in any other major scale and the two scales will have an equal number of chords. In what follows we will make more clear (mathematical) the notions of a mode and chord and we will classify modes by their musicality (a certin product of factors that we belive make the musicality of a scale).

Definition 1.2. *A k mode is a $k-$tuplet of positive numbers*

$$(a_1, a_2, \cdots, a_k, a_{k+1})$$

such that

$$\sum_{i=1}^{k-1} a_i < 12$$

and

$$a_k = 12 - \sum_{i=1}^{k-1} a_i$$

The first $k-1$ numbers in the $k-$tuplet represents the semitone interval between two consecutive notes in a given scale, while the last one is the semitone interval between the last and the first note of the scale. The semitone interval betweeen two notes is simply defined(via the numerical values of the notes defined before) as $\min(|n-m|, 12-|n-m|)$, where n, m are the two numerical values of the two notes. The reason behind this definition is that musical literature has exotic scales with more or less thatn 7 different notes and for which the interval between two consecutive notes is more than a whole step. As an example consider the blues scale based on E mentioned before, namely $EGAA^{\#}BD$. Its interval pattern (i.e mode) is $(3, 2, 1, 1, 3, 2)$. Another exotic well known example is the pentatnoic In Sen scale based in A, that is $AA^{\#}DEG$. Its mode is $(1, 3, 2, 3, 2)$. We have to note the fact that this is a purely mathematical definition and not every single scale obtained by choosing a random mode will sound musical when

played. For instance if we choose the mode of a scale to be $(1,1,1,1,1,1,1,1,1,1,1,1)$ and its key to be A then we obtain the chromatic scale based in A that consists of 11 half steps. This is NOT a very musical scale.

Theorem 1.3. *The number of k modes is $\binom{11}{k}$.*

Proof. The number of k modes is obviously equal to the number of scales with $k+1$ tones based in A. It follows that counting all posible scales with $k+1$ different tones will yield te number of k modes. To form a scale based in A we need to choose the leftover k tones(first one being A is already chosen) out of the remaining 11 tones. Thus the number of k tones is $\binom{11}{k}$. □

In order to give a precise definition of the musicality of a mode we will need to define the notion of a harmonic chord. Here we have to recall that choosing a key and a mode we obtain a scale. We will give our definition of a harmonic triad (chord) starting from a scale. It will be clear from our definition that a change of the key will only change the name of the chords and not their number.

Definition 1.4. *A harmonic interval is an interval of $3,4,5,7,8,$ or 9 semitones.*

Note: Western music has evolved to where these intervals are considered generally more pleasant to the ear than other intervals between notes. These intervals form the building blocks for major and minor chords. Jazz music and often times even classical also consider an interval of 6 semitones to be an acceptable interval for the formation of a chord.

We will adopt the next two definitions and theorem from [CF11].

Definition 1.5. *A harmonic subset of a scale S is a subset of notes in S such that the interval between any two of its notes is a harmonic interval.*

Theorem 1.6. *The maximum number of notes in a harmonic subset is three.*

Proof. Let $1 \leqslant n_1 < n_2 < \cdots < n_s \leqslant 12$, be the ordered numerical values of the notes in our harmonic subset and assume that $s > 3$. By the very definition of a harmonic subset $|n_i - n_j| \in \{3,4,5,7,8,9\}$ and looking at the first 4 numerical values of the tones within our set $n_1 < n_2 < n_3 < n_4$ we conclude that $n_4 - n_1 = 9$. It follows that $n_2 - n_1 = 3, n_3 - n_1 = 3$, and $n_4 - n_3 = 3$, which in turn implies that $n_3 - n_1 = 6 \notin \{3,4,5,7,8,9\}$, which contradicts our hypothesis. □

The above theorem allows us to give the following definition.

Definition 1.7. *A $3-$toned harmonic subset of a scale is called a chord.*

Remark: The reader could easily verify that the only types of harmonic chords $n_1 < n_2 < n_3$ are:

- $n_2 - n_1 = 3$ & $n_3 - n_2 = 4$ (major)
- $n_2 - n_1 = 4$ & $n_3 - n_2 = 3$ (major)
- $n_2 - n_1 = 3$ & $n_3 - n_2 = 5$ (minor)
- $n_2 - n_1 = 5$ & $n_3 - n_2 = 3$ (minor)
- $n_2 - n_1 = 4$ & $n_3 - n_2 = 4$ (augumented)

The following statement is the motivation of the classification of modes in the next section.

Theorem 1.8. *If two scales belong to the same mode then they have the same number of chords and semi-tones. Moreover if a mode is obtained as a circular permutation of another mode, the two modes will have the same number of chords and semi-tones.*

For a proof of this see [CF11]. The previous Theorem will allow us, in the next section, to give a mathematical definition the definition of the "musicality" of mode.

2 Classification of musical modes by their musicality

In this section we present an attempt of defining the "musicality" of a mode. We will consider two modes to be equivalent if one is a circular permutation of the other one.

Definition 2.1. *The chord density of a k mode, for $k \geqslant 3$, is the ratio between the number of chords of a scale within that mode and $\binom{k}{3}$ and it is denoted $\mathcal{D}$.*

The motivation behind this definition is that most of the people will consider a scale to be more musical if it has more chords. In general a scale with more notes will likely have more chords, so in order to compare the chord richness of two scales with different number of notes we need to normalize the number of harmonic triads by the number of all posible combinations of three distinct notes within the scale.

Definition 2.2. *The tonal variety of a mode is $\mathcal{V} = \dfrac{k\sqrt{k}}{10}$, where k the number of notes in the mode.*

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Obviously the more notes you have available to write a piece of music, the better the music will sound. So the "musicality" of a scale should be direct proportional to the number of different notes. If we choose $v = \frac{k}{10}$ the most musical scales will be the ones with only 3 different tones, which is a fact that contradicts our intuition. If we choose $v = \frac{k^2}{10}$, then the most musical will be the ones with 11 notes, again something that contradicts our intuition about musicality.

Definition 2.3. *The anti-chromaticity of the mode is the number $\mathcal{A}$ of intervals larger than a semitone within the mode.*

The motivation of "anti-chromaticity" is that in general the most pleasently sounding scales will have less semitone intervals.

Definition 2.4. *If a mode has tonal variety $v = \frac{k\sqrt{k}}{10}$, chord density $\mathcal{D}$, and anti-chromaticity $\mathcal{A}$, then its musicality is $\mathcal{M} = \mathcal{D} \cdot \mathcal{V} \cdot \mathcal{A}$.*

We wish to find the k-modes which are most musical according to our definition. The program will search through all of the possible scales, count the number of chords in this scale, and keep track of those scales with the maximum number of chords. The program will also check to only keep a representative scale of the set of all equivalent scales for a particular mode, equivalence being defined to be the same mode or a circular permutation of a mode (as described above). In the end we will only have a few nonequivalent scales which are representatives of the modes with the maximum number of chords, and this will also give us the musicality of the mode as defined above. To find all the combinations (scales), we used a modification of the algorithm found in [Ros12] p. 438.

Number of of tones	Scale representation	Number of chords	$\mathcal{A}$	Musicality
3	1 4 8	1	3	1.559
3	1 4 9	1	3	1.559
3	1 5 9	1	3	1.559
4	1 3 6 10	2	4	1.6
5	1 2 4 6 9	3	4	1.34
5	1 2 5 6 9	4	3	1.34
5	1 2 5 6 10	4	3	1.34
5	1 2 5 7 10	3	4	1.34
5	1 2 5 8 10	3	4	1.34
5	1 2 6 9 11	3	4	1.34
6	1 2 5 6 9 10	8	3	1.76
7	1 2 4 6 7 9 11	6	5	1.587
8	1 2 3 5 6 8 10 11	9	4	1.455
9	1 2 3 5 6 7 9 10 11	15	3	1.446
10	1 2 3 4 5 6 7 9 10 11	17	2	.896
11	1 2 3 4 5 6 7 8 9 10 11	21	1	.464
12	1 2 3 4 5 6 7 8 9 10 11 12	28	0	0.0

Let's us examine more closely the results of our algorithm displayed in the previous table. If we look at the (only!) 7 notes scale in the table and we choose the key to be B then we get our familiar B Mixolydian, which is a circular permutation of the C major scale. The scale with 6 notes is a circular permutation of a variant of the blues scale(one that is more musical than the blues scale but harder to be played) and the 1,2,6,9,11 scale is only a variant of the In Sen scale. Thus we can see that our definition of musicality of a mode is consistent to the musicality as perceived by most of the people.

References

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[Ros12] Kenneth H. Rosen : *Discrete Mathematics and Its Applications*, Seventh Edition, McGraw-Hill, 2012.