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Flattening the curve... of Spirographs

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Abstract

The Spirograph is an old and popular toy that produces aesthetically pleasing and fascinating spiral figures. But are spirals all it can make? In playing with a software implementation of the toy, the author chanced upon a variety of shapes that it can make that are different from its well-known repertoire of spirals, in particular, shapes that have a visible flatness and not the curved spiral geometry that we are accustomed to seeing from the Spirograph. This paper reports on these explorations by the author and his delightful discovery of very elegant and simple geometric relationships between the Spirograph's structural parameters that enable those patterns.

Keywords: Spirograph, polygon, polygram, geometry, gear, trochoid, cycloid

1 The Spirograph

The Spirograph is an old toy that is used for making spiral designs with a pen on paper, by sticking the pen into a hole on a gear and making the gear roll inside or outside another gear. Figure 1a shows typical output generated by a spirograph. Figure 1b depicts terminology associated with gears that will be used in this paper.

1.1 Spirograph output geometry: Trochoids

A little bit of geometry revision [Wike]: the curve traced by the pen in a Spirograph is mathematically a trochoid. A trochoid is the path traced by a point attached to a circular disc as it rolls on another, stationary, circle or on a stationary straight line. When the rolling circle rolls on the outside of a stationary circle, the generated curve is called a epitrochoid, and when it rolls on the inside, a hypotrochoid. If the small gear inside is held stationary and the large ring gear rolled around it while in contact, like a hula-hoop, a point on the ring gear generates a peritrochoid [Nas77]. In any of these cases, if the trace point is on the periphery of the rolling disc (*i.e.*, *on* the rolling circle), the traced

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(a) (b)

Figure 1: (a) Spirograph, (b) Gear terminology relevant to discussion of the spirograph.

curve is called a cycloid or a common trochoid. The cycloid is thus a special case of a trochoid. The prefixes epi- and hypo- apply to cycloids also with the same meanings as given above.

If the trace point is chosen inside the rolling circle, the trochoid is called 'curtate' and if the trace point lies outside the rolling circle, it is called 'prolate'. These variants are illustrated in Figure 2 for a circle rolling on a straight line. It would interest the reader to know that the path traced by any point on the flange of a wheel of a railway carriage is a prolate trochoid.

1.2 Spirograph parameters

The physical Spirograph kit comes with one flat ring with gearing on both its internal and external peripheries, and a finite number of rolling discs with external gearing and with a finite number of holes in each to stick one's pen into to draw spirals. These holes have to be physically away from the edge. This leads to a fixed number of spirals obtainable from one Spirograph kit. Theoretically, of course, and in the digital simulation world, there can be infinite combinations of radii of the rolling and stationary circles. The tracing point too can be theoretically located at an infinite number of positions along the radius of the rolling circle, even extending outside the physical boundaries of the circular disk. Thus an infinite number of curve patterns are obtainable from the spirograph when not limited by physical constraints.

A spirograph — or its mathematical model — can be parametrised by 4 quantities (Figure 1b): i) the pitch circle radius R of the fixed disk's gear, ii) the pitch circle radius r of the rolling disk's gear, iii) a factor d such that $d.r$ defines the radial distance of the tracing hole from the rolling disk's centre, and iv) whether the rolling disk is rolling around the interior or the exterior of the fixed disk. However, the nature of the curves produced (as opposed to their actual size) is dependent not upon the actual gear radii but on the ratio $\alpha = R/\mathrm{r}$. In this paper, a spirograph and its trochoid will therefore be

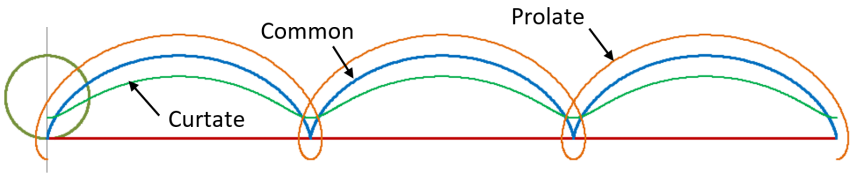


Figure 2: Curtate, common, and prolate trochoids. The common trochoid is also called a cycloid.

referred to as Sp(ei, α, d) and Tr(ei, α, d), respectively. ei would take symbolic values I or E (for Interior and Exterior), α would be a positive real number, and d a real number, positive, zero, or negative.

2 Related work on Spirographs

[Hal92] uses much the same terminology in describing spirograph figures in his interestingly titled paper *Trochoids, Roses, and Thorns — Beyond the Spirograph*. Others such as [Nas77] use d as the actual radial distance itself, not as the multiplier of r. This strips the results of much of the elegance and simplicity that this author's interpretation of d lends. [Ber14] makes it his goal to measure the length of spirograph curves. [Sim97] shows how epitrochoids are also the envelope curves of certain layouts of chords to a circle and that the number of cusps they have can be predicted. [Cav75] uses basic axioms of mathematics to prove that the number of points to the figure generated is the greatest common divisor (GCD) of the number of teeth on the two gears. [Flo92] does likewise in using the Spirograph as a tool for explaining principles of arithmetic like LCM (Least common Multiple) and GCD. [She15] writes about the physical heritage of geometry in Germany when material models used to be made to demonstrate geometric principles. [Öna84] discusses how efforts of playful and curiosity-driven exploration often lead to discovery of hidden mathematical relationships or patterns, something that this author can relate to very deeply as a lot of the findings reported in this paper are also the serendipitous outcome of exploration over months. In no published works on spirographs did this author find the outcome of this work reported.

3 The Trochoid Form

While the shapes of trochoids obtained from a spirograph are varied, the curved geometry within a single trochoid also shows regions of varying curvature, somewhere high, somewhere low, and — on cycloids — cusps, which are the pointed parts of the

curve[1] (Figure 3a). This paper looks at the relationships between the curves obtained and the radius ratios and radial tracing distances.

3.1 Cusps

In the trochoid family, cusps are encountered in cycloids, *i.e.*, on curves generated by points on the periphery of the rolling disk. It is well-known that when a circle M[2] of radius r rolls once[3] inside a circle S[2] of radius $R = \alpha\ r$ ($\alpha \in \mathbb{N}$), the resulting hypocycloid will have α cusps.

3.1.1 Cusp connectivity

How about if R is not an integral multiple of r, *i.e.*, $\alpha \notin \mathbb{Z}$? As Figure 3a shows for this case, at the end of one orbital roll of M around the inside of S, the trace point of M would not have returned to its starting point. It would take more than one orbit for that to happen. Let $R = 3.5r$. In its second orbit around S (Figure 3b), M lays new cusps mid-way between those of the first orbit. The tracing point returns to its start at the end of two orbits, producing a curve of 7 cusps as shown in Figure 3c.

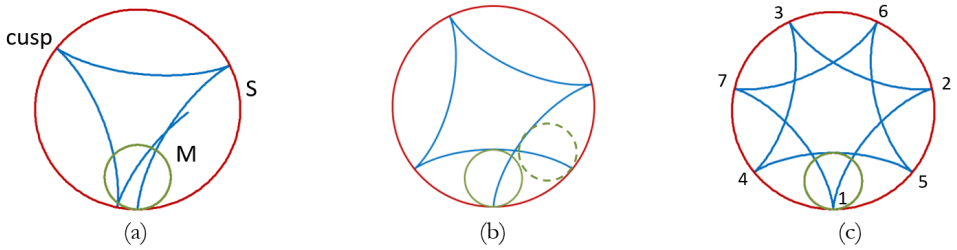


Figure 3: (a) Situation when the ratio of the two radii is not an integer: a displaced trajectory is traced in the second orbit. (b) When $R = 3.5r = 7r/2$, new cusps are introduced on the second orbit midway between the cusps of the first orbit. (c) 7 cusps are produced, in the rolling order shown.

In general, in a Spirograph parameterised as Sp(I, α, 1),

> If α can be expressed as N/D, (N, $D \in \mathbb{Z}$, and co-prime), then a hypocycloid of N cusps is produced. : Rule 1.0

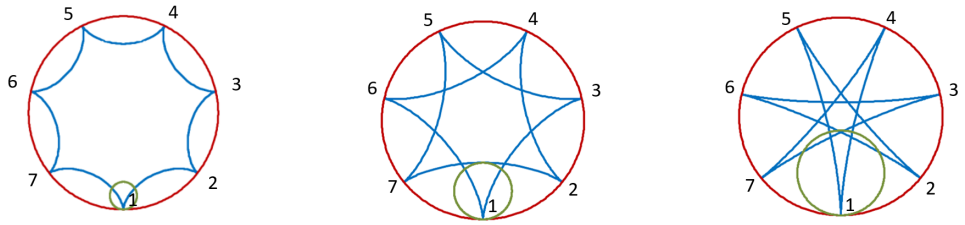
What is the significance of the denominator D? In the previous example where $3.5 = 7/2$, the denominator 2 corresponded to the number of divisions (along the fixed

[1] A cusp is a pointed tip on a curve at which a traveller along the curve would have to reverse direction to continue.

[2] M for **M**oveable, S for **S**tationary

[3] *i.e.*, the centre of M is carried one full rotation around the inside of S.

circle) made by the later orbits between the cusps produced in the first orbit. Let $\alpha = 7/3$. By Rule 1.0, a hypocycloid of 7 cusps would still be produced. But it would take 3 orbits for it to be produced and would introduce 2 more cusps uniformly between the cusps 1, 4, 7 produced during orbit 1. So $\alpha = 7/3$ will yield a connectivity of 1—4—7—3—6—2—5—1 (Figure 4c) compared to 1—3—5—7—2—4—6—1 (Figure 4b) when $\alpha = 7/2$, or 1-2-3-4-5-6-7-1 (Figure 4a) when $\alpha = 7$ (*i.e.* 7/1). In other words, when $\alpha = 7/2$, every second one of the 7 points is linked and when $\alpha = 7/3$, every third one.



(a) $\alpha = 7/1$ (b) $\alpha = 7/2$ (c) $\alpha = 7/3$

Figure 4: Hypocycloids of 7 cusps, but connected differently.

Generalising this, Rule 1.0 above can be expanded as follows:

> If α can be expressed as N/D, (N, D $\in \mathbb{Z}$, and co-prime), then the Spirograph Sp(I, α, 1) produces a hypocycloid of N cusps in which every cusp is connected to the Dth one sequentially after it around the circle. : Rule 1.1

This has far-reaching coverage as it addresses not only integer values of α, but all fractional values as well. We shall return to this later in Section 4.2.1.

It is also to be mentioned that $\alpha = 7/4$, $\alpha = 7/5$, and $\alpha = 7/6$ will produce the same curves as $\alpha = 7/3$, $\alpha = 7/2$, and $\alpha = 7/1$, respectively, except with tracing order reversed (for example, $\alpha = 7/4$ will produce a connectivity 1–5–2–6–3–7–4–1). [Hal92] calls such similar curves "congruent". We shall have occasion to return to this point later in Section 4.2.

3.1.2 Convexity-concavity

Instead of the trace point lying only on the circle M, let us consider trace points lying away from the periphery too, by adjusting the factor d. For example (Figure 5a), $d = 0$ would give the point at the centre of M and $d = 1$ the contact point of M and S. Figure 5b shows how this affects the curve that is traced.

Taking the classic Deltoid curve as example (Figure 5c) it is seen that in hypotrochoids, reducing d from 1 to 0 caused a) the cusps to retract radially inward from the fixed circle, b) the pointed cusps to become rounded (and thus ceasing to be cusps anymore,

becoming lobes or loops or rounded corners), and c) the sides to become flatter and less concave upto a point and then bulge outward and become actually convex until they became an exact circle at $d = 0$.

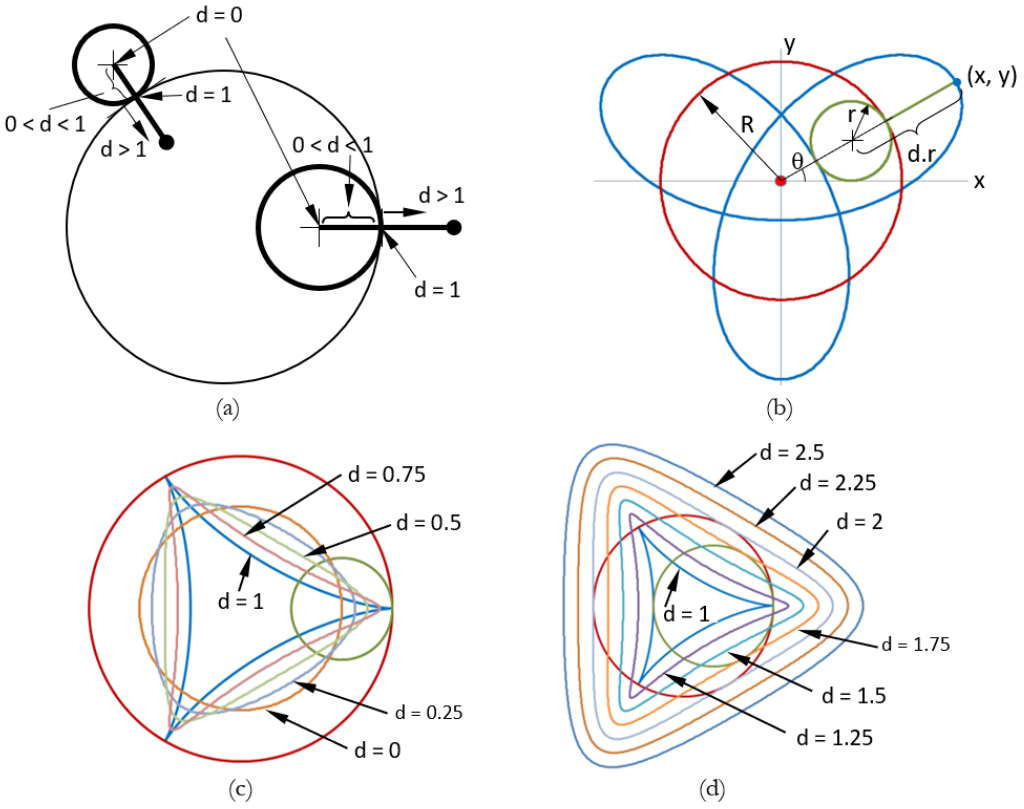


Figure 5: (a) d for hypotrochoids and epitrochoids. (b) Hypotrochoid produced by $d \neq 1$. (c) Trochoids Tr(I, 3, d) for various d. Curvature changes from concave to convex as d decreases from 1. (d) Trochoids Tr(I, 3/2, d). Curvature changes from concave to convex as d increases from 1. (Figures c and d are scaled to fit their extents.)

4 Flattening the curve

Point c above is the topic of central interest in this paper, *viz.*, that a value of d exists for each trochoid at which the sides of the trochoid are "flattest", making the trochoid resemble a polygon with rounded corners.

In Figure 5c above, for example, the curve with $d = 0.5$ displays the best flatness

compared to the curves on either of its sides with other values of d. In Figure 5d, $d = 2$ does that job.

These discoveries — that by changing d away from 1, a 'cuspy' hypocycloid could be changed to resemble a polygon with rounded corners, and that the curvatures of its sides could be minimised (and even made zero at their midpoints) to resemble extended flatness — were the starting points of many explorations by the author which culminated in this paper.

4.1 Rounded polygons from the Spirograph

The simple polygon (albeit with rounded corners) was a surprising — and a surprisingly bland — outcome of the spirograph which is more associated with dizzyingly intertwined spiral shapes than with simple geometric figures. However, for that reason alone — its reduction to a simple form — it was of great interest.

Investigation into the pattern of flatness as a function of d revealed the following rule:

> Any hypotrochoid Tr(I, α, d) assumes its maximum local flatness in the flanks at the value of $d = 1/(\alpha - 1)$. : Rule 2

In the example above, for $\alpha = 3$ it turned out that at $d = 1/(3 - 1) = 1/2$, the sides of the 3-sided figure became their flattest, while for $\alpha = 3/2$ this happened at $d = 1/(1.5 - 1) = 2$, confirming our observation in Figures 5c and 5d.

4.1.1 Analytical verification

This observation was verified analytically, and graphically also, as follows:

The parametric equations of the spirograph curves were derived by the author (verifiable from), using the circular position θ of the rolling gear's center (Figure 5b) as the parameter. They came out to be as follows:

$$x(\theta) = rw\cos\theta + r.d.\cos(w\theta),\ \ y(\theta) = rw\sin\theta - r.d.\sin(w\theta), \text{ where } w = \alpha - 1$$

Using the formula $C(\theta) = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}$ [Law13] for Curvature C of a curve expressed in parametric form (where '$\cdot$' is $\mathrm{d}/\mathrm{d}\theta$), we get

$$C(\theta) = \frac{1 - wd^2 - d(1-w)\cos(\theta + w\theta)}{rw(1 + d^2 - 2d\cos(\theta + w\theta))^{\frac{3}{2}}}$$

It would be expected that a point on a "flat" region would have curvature zero. This indeed came true as shown in Figure 6 in which C(θ) was plotted against θ for the case of Tr(I, 3, 0.5) and found to be zero at those points at which the trochoid was visually "flat". In addition, the derivative of curvature was plotted as well. This was also found to become zero at these points, as would be expected in longer spans of flatness.

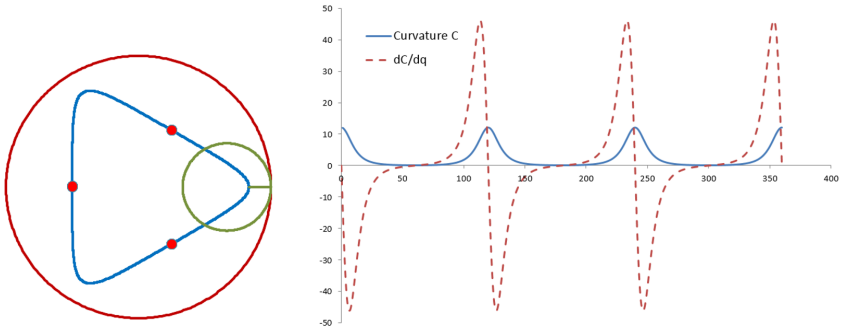
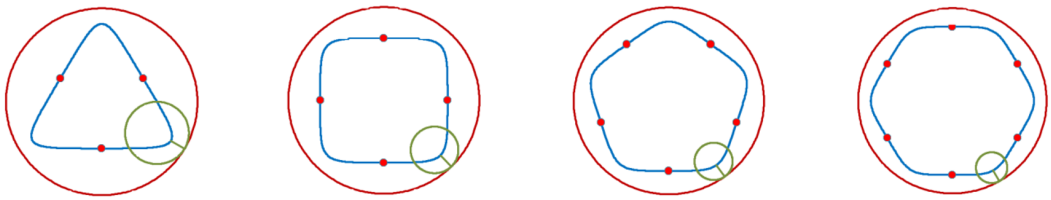


Figure 6: Curvature and derivative of curvature plotted against parameter θ for the Trochoid Tr(I, 3, 0.5). The points of extended apparent flatness, at which both are zero, are seen at $\theta = 60^o$, 180°, and 300°.

Since the graph of Figure 6 shows that zero curvature is achieved only at the centrepoint of the rounded polygon's sides, it is obviously inaccurate to say that the trochoid Tr(I, 3, 0.5) achieves "full" flatness at $d = 1/2$. The author employs an artist's latitude in making that statement since the visual manifestation is that of a triangle with flat sides and rounded corners.

Explorations similar to that of Figure 5c were carried out with $R = 4r$ (*i.e.*, $\alpha = 4$) and d changing from 0 to 1. It was seen that the trochoid assumes the shape of a square with rounded corners at $d = 1/3$ as it transforms from a 4-cusped concave figure (the Astroid) to a circle.

Likewise, for values of α beyond 4 too, identical behaviour was noticed, namely, that hypocycloids with α cusps turned into polygonal figures when d was made the reciprocal of α - 1. Figure 7 depicts some such hypotrochoids with the points of maximum and extended flatness indicated on them and the value of d for each.



(a) $\alpha = 3$, $d = 1/2$ (b) $\alpha = 4$, $d = 1/3$ (c) $\alpha = 5$, $d = 1/4$ (d) $\alpha = 6$, $d = 1/5$

Figure 7: Simple rounded polygons — no spirals — generated by the spirograph for specific combinations of α and d. These are curtate hypotrochoids. All rounded polygons generated thus lie inside the fixed circle.

The discovery of this pattern in the relative values or α and d that gave rounded polygons as the resulting spirograph outputs is generalised in the rule:

> The spirograph Sp(I, α, d), $\alpha \in \mathbb{Z}$, will yield an α-sided rounded polygon for $d = 1/(\alpha - 1)$. This rounded polygon will be a curtate hypotrochoid and will lie entirely inside the stationary circle of the spirograph. : Rule 3

4.1.2 External rounded polygons / Rounded polygons from congruent cycloids

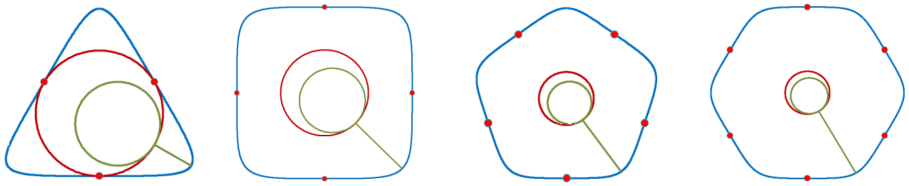
The triangle of Figure 5c is generated by a moving circle one-third the size of the fixed circle, lies completely inside the fixed circle and is a curtate hypotrochoid, *i.e.*, has a tracing point inside the moving circle. The triangle of Figure 5d, on the other hand, requires a moving circle twice as large as in Figure 5c (with $R/r = 3/2$), a tracing point outside the moving circle, (making it a prolate hypotrochoid), and lies outside the base circle. Further, the base hypocycloid from which the triangle of Figure 5d is generated is congruent to the hypocycloid from which the triangle of Figure 5c is obtained. They look the same but are traced in reverse order and have different periods.

The family of prolate polygonal hypotrochoids was explored by taking hypocycloids of other cusp counts, 4 onwards. Exploration with the hypocycloid of 4 cusps congruent to the one that gave the square with $R/r = 4/1$ revealed that with $R/r = 4/3$, a square is obtained when d is ± 3. This time a much larger square, and one lying entirely outside the stationary base circle.

Taking a cue from the pattern of these relative values of R/r and d, it was found that:

> The spirograph Sp(I, α, d) will produce an n-sided rounded polygon ($n \in \mathbb{Z}$, $n > 2$) if $\alpha = n/(n-1)$ and $d = n - 1$. The polygon will lie fully outside the fixed circle. : Rule 4

Some rounded polygons obtained using this rule are depicted in Figure 8.



(a) $\alpha = 3/2$, $d = 2$ (b) $\alpha = 4/3$, $d = 3$ (c) $\alpha = 5/4$, $d = 4$ (d) $\alpha = 6/5$, $d = 5$

Figure 8: Rounded polygons external to the fixed circle generated with tracing points outside the rolling circle, making them prolate hypotrochoids. The figures are scaled to fit the polygons.

It was exciting to the author to realise that the relationship $d = 1/(\alpha - 1)$, discovered between d and α in the previous exercise, applied here too.

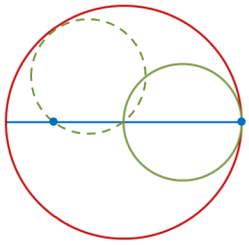
The findings from Rule 3 and Rule 4 are shown in Table 1 which gives the [R/r, d] combinations that yield rounded polygons either inside or outside the stationary circle:

Internal polygon		**(Rounded) Polygon traced**	**External polygon**	
α **(R/r)**	**d**		α **(R/r)**	**d**
3/1	1/2	Triangle	3/2	2
4/1	1/3	Square	4/3	3
5/1	1/4	Pentagon	5/4	4
6/1	1/5	Hexagon	6/5	5
		...		
$n/1$	$1/(n-1)$	n-sided polygon	$n/(n-1)$	$n-1$

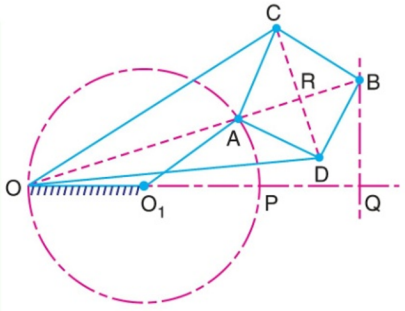
Table 1: Rounded polygon possibilities with hypotrochoids.

4.1.2.1 The straight-line tracer

What happens if α is taken as 2, *i.e.*, $R = 2r$? The induction formula above predicts that we would get a 2-sided polygon. What is a 2-sided polygon? A straight line! That indeed is what the spirograph gives if d is taken as $1/(2-1) = 1$. A circle rolling inside another circle twice its diameter (Figure 9a) is one of the few mechanisms — like the Peaucillier's Cell [Wikc] — which generates a perfect straight line (Figure 9b).



(a) $\alpha = 2, d = 1$



(b)

Picture credit: Pravinkumar Suthar, *Animation of Straight line mechanism –peaucillier, hart' s, Scottrussel, Grasshopper, watt' s, Tchebichoff*, 3 Aug 2017, https://www.youtube.com/watch?v=zgwEptukl6Q

Figure 9: (a) An exact straight line generated by the point on a circle rolling inside a circle twice its diameter: Sp(I, 2, 1). (b) The Peaucillier's mechanism for generating exact straight lines.

4.1.3 Epitrochoidal rounded polygons

What about circles rolling on the outside of other circles (generating epitrochoids by definition)? Can we expect to get similar, flattish rounded polygons from them?

The equation of an epitrochoid for the geometry shown in Figure 5a would be

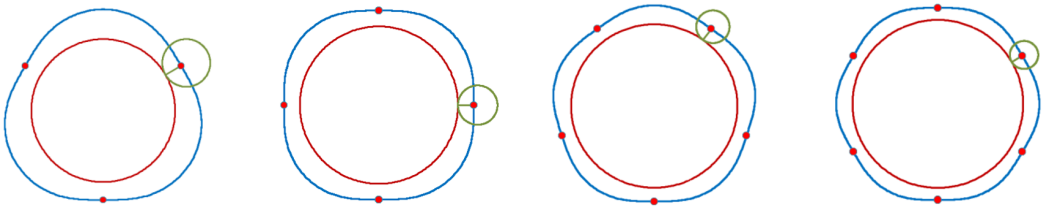
$$x(\theta) = rw\cos\theta + r.d.\cos(w\theta),\quad y(\theta) = rw\sin\theta - r.d.\sin(w\theta),\quad w = \alpha + 1$$

with curvature given by $C(\theta) = \frac{1+wd^2-d(1+w)\cos(\theta-w\theta)}{\mathrm{rw}(1+d^2-2d\cos(\theta-w\theta))^{\frac{3}{2}}}$. [Law13]

Using these expressions, the explorations done with hypotrochoids were conducted with epitrochoids also and various combinations of α and d were tried to see what yields rounded flattish polygons. It was found that there did exist values of d which gave local flatness behaviour, with the difference that the span of flatness was much smaller than for the hypotrochoid polygons, and the rounded "corner" radii were very large. The mathematical relationship observed was that for integer values of α the epitrochoid achieved maximum local flatness at $d = 1/(\alpha+1)$, as opposed to $1/(\alpha-1)$ for hypotrochoids. This is summarised in the rule:

> A spirograph Sp(E, α, d) will produce an α-sided rounded polygon with $d = 1/(\alpha+1)$ if $\alpha \in \mathbb{Z}$. This polygon will lie entirely outside the fixed circle.
> : Rule 5

Examples of such polygon-resembling epitrochoids are shown in Figure 10. Points of exact zero curvature (and zero derivative of curvature) are indicated.



(a) $\alpha = 3$, $d = 1/4$ (b) $\alpha = 4$, $d = 1/5$ (c) $\alpha = 5$, $d = 1/6$ (d) $\alpha = 6$, $d = 1/7$

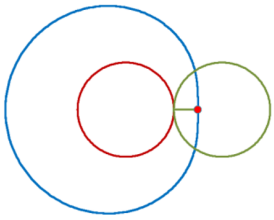
Figure 10: "Polygons" made by epitrochoidal Spirograph configurations. Note the large "corner" radii.

These are curtate epitrochoids, since $d < 1$. There were no values of $d > 1$ for which a rounded polygon was found to be generated by the spirograph for any value of α if the rolling circle rolls on the exterior of the stationary circle. In other words, no prolate epitrochoidal polygons can be found. Table 2 lists the values of R/r and d at which rounded polygons are obtained as epitrochoids.

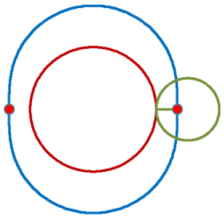
(Rounded) Polygon traced	α **(R/r)**	**d**
Triangle	3	1/4
Square	4	1/5
Pentagon	5	1/6
Hexagon	6	1/7
...		
n-sided polygon	n	$1/(n+1)$

Table 2: Rounded polygon possibilities with epitrochoids.

Figure 11 shows the figures obtained for values $\alpha = 1$ and 2. Although not polygons, they were extremely interesting for their simplicity and elegance, and they do possess local regions of extended flatness. One ($\alpha = 1$) resembles a bean and the other ($\alpha = 2$) a capsule.



(a) $\alpha = 1$, $d = 1/2$



(b) $\alpha = 2$, $d = 1/3$

Figure 11: Two simple but unique shapes with flat extents obtained as epitrochoids, resembling a) a bean flattened on one side, and b) a capsule.

4.2 Stars. Or Polygrams.

Another surprisingly uncomplicated and simple, yet elegant outcome that the experiments with the Spirograph yielded was a family of stars!

For a given fixed circle and for a given n, a maximum of $n-1$ hypocycloids of n cusps (some identical to each other) are possible to generate, by using $\alpha = n/i$ and $d = 1$, where i can vary between 1 and $n-1$, and such that n and i are co-prime. Thus, for $n = 5$, a maximum of 4 hypocycloids of 5 cusps can be generated, since 1, 2, 3, 4 are all co-prime with 5. But for $n = 6$, only 2 hypocycloids of 6 cusps can be generated since only 1 and 5 are co-prime with 6.

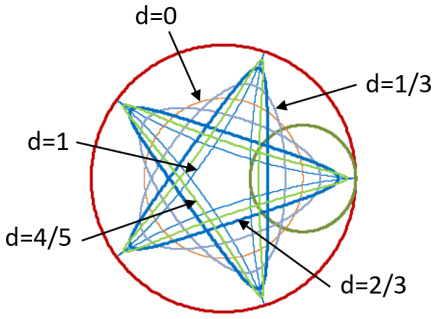
Some of these hypocycloids would look identical. For a given fixed circle, if we denote a hypocycloid of n points connected circularly in the order $[1,\ 1+i,\ 1+2i,\ \ldots]$ as HyC(n, i), then HyC(n, i) and HyC(n, $n-i$) would look identical, differing only in having been traced in opposite directions, and are hence called congruent. Other hypocycloids

in the set would fundamentally differ from each other in form by possessing different connectivity of vertices (see Figure 4).

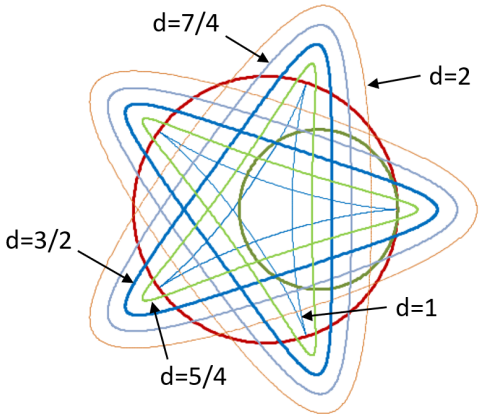
It has been seen (Figure 5c, d) how HyC(n, 1) and HyC(n, $n-1$) would change into rounded polygons by changing d from 1 to $1/(\alpha-1)$. This section looks at how the other hypocycloids HyC(n, i), ($1 < i < n-1$), would respond to various values of d.

For HyC(5, 2), the value of d was changed from 1 downward, on the same lines as Figures 5c and 5d, to see the effect of this upon curvature of the flanks. It was noticed that minimum curvature and maximum flatness was achieved at $d = 2/3$. And the figure produced resembled the 5-pointed star or the pentagram (with rounded tips) — of geometric simplicity and elegance similar to the pentagon! This rounded pentagram was fully inside the fixed circle and had sides of long lengths of apparent flatness and zero curvature at their centre. This transition of HyC(5, 2) to a pentagram (with rounded tips) is captured in Figure 12a.

For HyC(5, 3) minimum curvature and maximum flatness were achieved at $d = 3/2$. The trochoid produced was — again — a pentagram-like figure with rounded tips, but larger this time and spilling over the fixed circle, not fully inside it. Since d was larger than 1, this rounded pentagram was a prolate hypotrochoid as opposed to the curtate one earlier (with $d = 2/3$). Figure 12b shows the transition of HyC(5, 3) to a rounded pentagram.



(a) $\alpha = 2/5$, $d = 2/3$



(b) $\alpha = 3/5$, $d = 3/2$

Figure 12: Pentagrams obtained from Spirograph. (a) can physically be drawn, being curtate, but (b) cannot be drawn physically since the output figure, being prolate, intersects the fixed disk itself.

The exercise was repeated for other hypocycloids, of different cusp counts, and a pattern that emerged was that these hypo™cycloids generated the full family of rounded polygrams possible for the cusp count. Hypocycloid(n, i) generated Polygram(n, i), using

standard nomenclature of polygram classification [Wikd] And it was with delight that the author realised that the same formula — $d = 1/(\alpha - 1)$ — that generated flat-sided rounded poly*gon*s generated flat-sided rounded poly*gram*s also! This is expressed in Rule 6.

> The spirograph Sp(I, α, d) (where $\alpha = n/i$; $i \in \mathbb{Z}$, $1 < i < n/2$ and i, n are co-prime; $d = 1/(\alpha - 1)$) gives curtate polygrams(n, i) with rounded tips contained wholly inside the base circle. The spirograph Sp(I, α, d) (where $\alpha = n/(n - i)$; $i \in \mathbb{Z}$, $1 < i < n/2$ and i, n are co-prime; $d = 1/(\alpha - 1)$) gives prolate polygrams(n, i) with rounded tips lying partially or wholly outside the base circle. : Rule 6

The polygrams obtained are the singly-connected ones, *i.e.*, those possible to draw without lifting the pencil, as one must in a Spirograph anyway. This filters out polygrams that are not possible to draw in one stroke, spirograph or no spirograph. For example, there is no singly-connected hexagram, *i.e.*, a six-pointed star that can be drawn without lifting the pen — the 6-sided Star of David (✡) has to be drawn as two separate overlapping triangles.

Even though connectivity-wise an abstract Polygram(n, q) is the same as (or congruent to) Polygram(n, $n - q$), in their physical realisation through a Spirograph they are *not* identical but come in different sizes and layouts and require different-sized moving gears to produce, as was seen in the example of the pentagram above.

The list of polygrams possible upto $n = 9$ are given in Table 3 and depicted in Figure 13 (the curtate polygrams) and Figure 14 (the prolate polygrams).

Internal polygon (curtate)		**Rounded Polygon traced**	**External polygon (prolate)**	
α **(R/r)**	**d**		α **(R/r)**	**d**
5/2	2/3	Pentagram	5/3	3/2
7/2	2/5	Septagram	7/4	4/3
7/3	3/4		7/5	5/2
8/3	3/5	Octagram	8/5	5/3
9/2	2/7	Nonagram	9/5	5/4
9/4	4/5		9/7	7/2
		...		
n/i $1 < i < n/2$	$i/(n - i)$	n-pointed polygram	n/i $n/2 < i < n$	$i/(n - i)$

Table 3: Polygram possibilities with hypotrochoids.

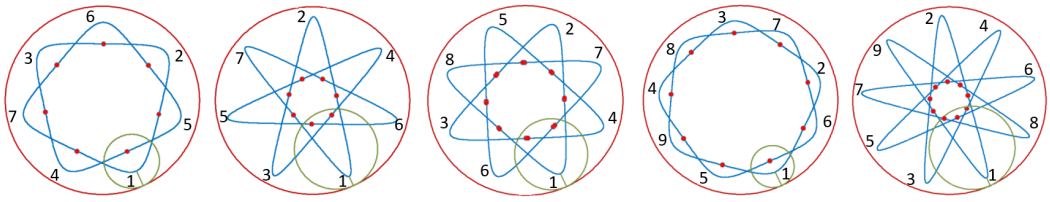


Figure 13: Some curtate polygram (with rounded tips) output of the Spirograph, all lying inside the static circle. The polygram vertex numbering is the order in which that vertex is drawn by the Spirograph.

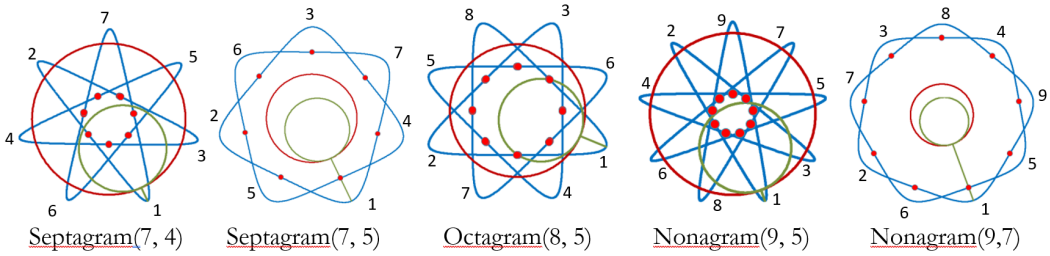


Figure 14: Some prolate polygrams spilling over or lying completely outside the static circle. Notice the counterintuitive order of drawing vertices for the same motions as the previous case (Figure 13).

From the observation that the pentagram had 2 possibilities (curtate and prolate combined) and the septagram and nonagram 4 each, more polygrams were investigated to find out the number of their geometric possibilities. The results are given in the table below for values of n (the number of vertices of the polygram) upto 17 :

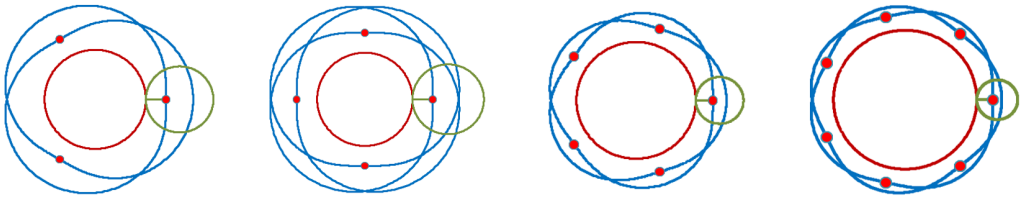
Number n of vertices in polygram	5	7	8	9	10	11	12	13	14	15	16	17
Distinct polygrams possible $\Pi(n)$	2	4	2	4	2	8	2	10	4	6	6	14

The pattern that emerges from this table is that $\Pi(n)$ is equal to the number of integers between 1 and n that are co-prime with n, excluding 1 and $n-1$ since those would contribute to forming polygons and not polygrams. This is captured very elegantly by using Euler's Totient function $\Phi(n)$ [Wikb] as follows:

> The number Π of distinct polygrams (with rounded tips) possible to make by the spirograph Sp(I, α, d) (where $\alpha = n/i$; $i \in \mathbb{Z}$, $1 < i < n$; $d = 1/(\alpha - 1)$) is given by the formula: $\Pi(n) = \Phi(n) - 2$: Rule 7

The relevance of prime numbers in the treatment of Spirograph polygrams is that the polygrams obtained for values of i that are divisors of n would be identical to the polygrams obtained for the irreducible value of n/i. and would therefore not contribute to the count of distinct polygrams. Also, as mentioned above, for any n each curtate polygram(n, i) would have a prolate counterpart polygram(n, n-i) that is congruent but different in size.

Unlike with Hypocycloids, no success was met in obtaining polygrams with epicycloids, although the magic formula $d = 1/(\alpha + 1)$ did yield flat-flanked figures such as shown in Figure 15 from non-integer values of α.



(a) $\alpha = 3/2$, $d = 2/5$ (b) $\alpha = 4/3$, $d = 3/7$ (c) $\alpha = 5/2$, $d = 2/7$ (d) $\alpha = 7/2$, $d = 2/9$

Figure 15: Some combinations of α and $d = 1/(\alpha + 1)$ in epitrochoids. All these have flat extents but are otherwise far from being possible to be called polygrams.

4.2.1 Arbitrary rational values of α

Up to now, we have only considered gear ratios α with values expressible as simple rational numbers $n/(n - i)$, $i = 1, \ldots, n - 1$, (*i.e.*, α from $n/(n - 1)$ to n).

The question might arise about what shape might be obtained for a value of α expressed as an arbitrary decimal, say 5.32. The answer is to express α in fraction format N/D where N and D are co-prime. So $\alpha = 5.32$ would be written as the fraction 532/100. Reducing this to a coprime form, we get 133/25. Rule 1.1 tells us that the hypotrochoid would have 133 cusps (if d is taken as 1) with each cusp connected to the twenty-fifth one after it. Rule 2 tells us that at $d = 1/(5.32\text{-}1) = 25/108 = 0.23\underline{148}$[4] this figure would have flat sides. Since a trochoid of 133 sides would become a 133-sided rounded polygon for α values of 133 (by Rule 3) and 133/132 (by Rule 4) and a polygram for other denominators 2—131 that are prime with 133 (by Rule 6), the [α, d] combination of [5.32, 0.23$\underline{148}$] would give us Polygram(133, 25). Since d is less than 1, the polygram would be fully internal to the base circle. Figure 16 shows the cycloid Tr(I, 5.32, 1) (only three initial loops are shown, the data was too much to generate more).

Thus for any rational value of $\alpha = N/D$ (N, D co-prime), the output of the spirograph can be described precisely as a trochoid of N loops with each loop connected to the Dth

[4*] Recurring part of the decimal underlined.

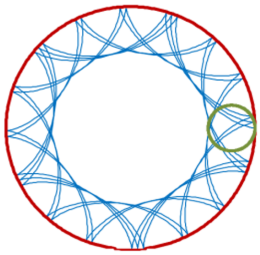


Figure 16: Three loops of the epicycloid Tr(I, 5.32, 1) shown to illustrate that even random values for α, like 5.32, map to a predictable, specific geometry of closed-loop trochoids.

one after it. The larger D is, the larger the number of loops within one circle, thus making the trochoid more 'crowded'. However crowded the spirograph pattern is, though, the number of loops is finite; the spirograph pattern closes in upon itself and is not endless. In these cases of rational α, the trochoid is an algebraic curve, meaning that in addition to the parametric equations of sections 4.1.1 and 4.1.3, it also bears a polynomial relation between x and y. Only for irrational values for α (such as Π), which have no finite fractional representation, would the spirograph pattern be truly endless, never closing to where it started, and crowding the pattern to a fill.

5 The Design question

Aside from all these what-if questions of analysis, one could put the knowledge obtained to answer how-to questions of synthesis as well. Here are two such questions and the answers to them:

Q: 'How do I obtain a 17-sided rounded polygon from the Spirograph?'

Answer: Use i) $\alpha = 17$ and $d = 1/(17-1) = 1/16 = 0.0625$, with internal rolling, or ii) $\alpha = 17/16$ and $d = 1/(17/16-1) = 16$, with internal rolling, or iii) $\alpha = 17$ and $d = 1/(17+1) = 1/18$, with external rolling.

Q: 'How can I obtain a polygram of 43 points with connectivity 6 from a Spirograph?'

Answer: Use $\alpha = 43/6$ and $d = 1/(43/6-1) = 6/37 = 0.162$.

6 Consolidation

The graph in Figure 17 shows the mapping of rounded polygons and polygrams as combinations of d vs. r/R (it was more convenient to use r/R instead of R/r because

of the ascending order between 0 and 1). The triangles represent the polygons and the circles the polygrams. The polygons to the left of the (1/2, 1) data point are the ones internal to the base circle and the ones to the right, external. Likewise, the polygrams to the left of the (1/2, 1) data point are the ones fully internal to the base circle and the ones to the right, overlapping or external. The curve is the locus of all points which would exhibit extended local flatness on the trochoid. Points intermediate to the ones shown would certainly exist and would represent polygrams with more loops.

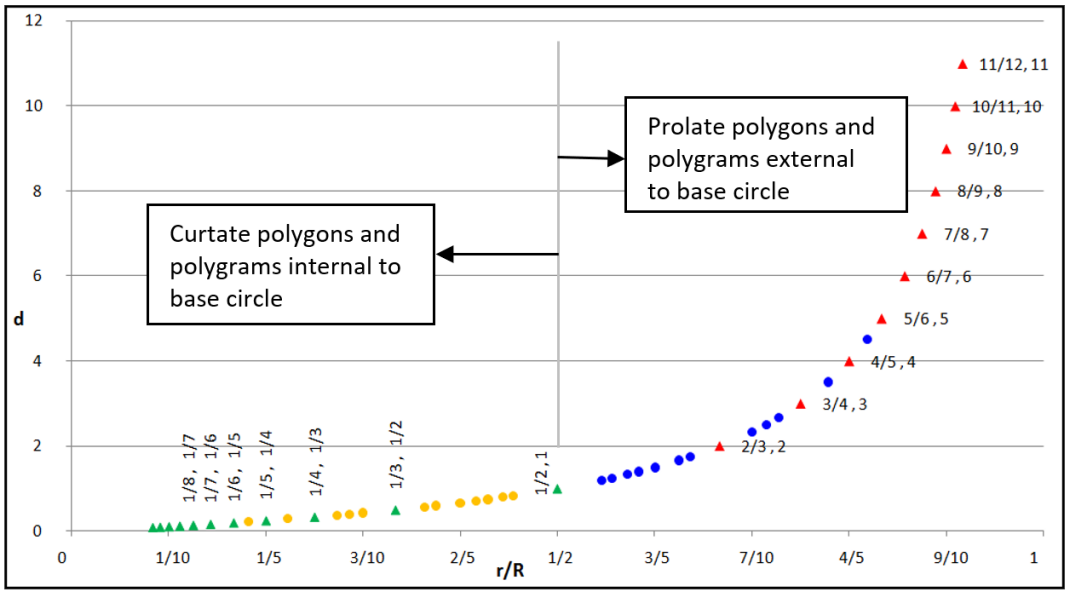


Figure 17: Mapping of $(r/R,\ d)$ combinations that give rounded polygons or polygrams.

7 Scope for further work

The author intends to take this forward to a usable design tool in which one can enter the parameters and dimensions of the rounded polygons she wants (*e.g.*, number of vertices, side length) and the tool would compute the possible relative sizes of the Spirograph elements (gears) and the placement of tracing holes on it and the configuration of rolling (internal or external).

Another agenda for future work is to report the minimum radius one is getting at the corners in this exercise. That would be the radius of the smallest osculating circle at the corner.

8 Conclusion

This paper reports on the exploration of the output of the classic geometric toy called the Spirograph, relating its structural geometry to the visual output it produces. The impact of structural descriptors like tracing point location and gear ratio upon geometric output parameters like cusp count, cusp connectivity, and curvature were studied. The inputs were initially random, driven by trial-and-error, but once interesting and unexpected shapes resembling polygons and polygrams were found to get generated, the pattern behind the inputs was sought. This led to the discovery of patterns (seen in all the tables) in the mathematical relationships between the structural parameters that were as strikingly simple and elegant as the shapes they produced. The most satisfying to the author was the discovery that it was the chosen way of representing the radial distance of the tracing point (as $d.r$) that led to these elegant hyperbolic relationship patterns $d = 1/(\alpha \pm 1)$. It is hoped that because of this treatment the reader would appreciate mathematics in general and geometry in particular a little more.

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