

One Application of Duistermaat-Heckman Measure in Quantum Information Theory

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Abstract: While the exact separability probability of $8/33$ for two-qubit states under the Hilbert-Schmidt measure has been reported by Huang and Khoi, detailed derivations remain inaccessible for general audiences. This paper provides a comprehensive, self-contained derivation of this result, elucidating the underlying geometric and probabilistic structures. We achieve this by developing a framework centered on the computation of Hilbert-Schmidt volumes for key components: the quantum state space, relevant flag manifolds, and regular (co)adjoint orbits. Crucially, we establish and leverage the connection between these Hilbert-Schmidt volumes and the symplectic volumes of the corresponding regular co-adjoint orbits, formalized through the Duistermaat-Heckman measure. By meticulously synthesizing these volume computations—specifically, the ratios defining the relevant probability measures—we reconstruct and rigorously verify the $8/33$ separability probability. Our approach offers a transparent pathway to this fundamental constant, detailing the interplay between symplectic geometry, representation theory, and quantum probability.

Keywords: separability probability; Duistermaat-Heckman measure; Hilbert-Schmidt volume; symplectic volume

1. Introduction

Quantum entanglement, a hallmark of quantum mechanics, is a fundamental resource for quantum information processing. Distinguishing entangled states from separable (non-entangled) states is therefore a central problem in quantum information theory. For bipartite systems, particularly the simplest non-trivial case of two-qubits, determining the likelihood of encountering a separable state within the ensemble of all possible states is a question of significant theoretical interest. This is formalized as the *separability probability* [1,2].

Under the *Hilbert-Schmidt measure*—a natural metric induced by the Hilbert-Schmidt distance on the space of density matrices [3]—the exact value of the separability probability for two-qubit states has been a subject of extensive numerical investigation and conjecture. Early numerical studies strongly suggested the fraction $8/33$ as the exact value (with $29/64$ for the real two-qubit case) [4–6]. This conjecture was rigorously confirmed recently by Huang and Khoi [7] (separately, the fraction $29/64$ was analytically proved by Lovas and Andai [8]). However, while their result establishes the exact separability probability, the detailed mathematical derivation presented in [7] remains inaccessible to a significant portion of the community.

This paper aims to bridge this gap in accessibility and understanding. Our primary contribution is to provide a detailed, self-contained, and pedagogically clear derivation of the $8/33$ ¹ separability probability for two-qubit states under the Hilbert-Schmidt measure. We achieve this by leveraging a powerful geometric framework centered on the computation of volumes [9] in the relevant state spaces and their substructures. The core of the used approach lies in the systematic computation of Hilbert-Schmidt/symplectic volumes associated with key geometric objects:

- (1) **Flag manifold:** This is a specific instance constructed by taking the quotient of the unitary group $U(N)$ by its maximal torus.
- (2) **A regular adjoint orbit:** It is an orbit of a fixed diagonal matrix with distinct diagonal entries under the conjugate action of unitary group $U(N)$. Note that this fixed diagonal matrix can be translated to a regular element in the positive Weyl chamber for the unitary group $U(N)$.

¹The derivation of the general value $8/33$ is more complex than that of the specific real two-qubit case $29/64$.

- (3) **The space of all qudit states:** This is the compact manifold of all density matrices acting on the same underlying space $\mathbb{C}^N$.
- (4) **A regular co-adjoint orbit:** It is a orbit of a regular element in the positive Weyl chamber under the co-adjoint action of the unitary group $U(N)$.

A critical insight underpinning the derivation is the profound connection between the Hilbert-Schmidt volumes of adjoint orbits and the symplectic volumes of their corresponding regular co-adjoint orbits. This connection is rigorously established through the *Duistermaat-Heckman (DH) measure* [10], a fundamental tool in symplectic geometry and geometric quantization that intrinsically relates these volume measures. The DH measure's properties are well-documented, with recent generalizations further extending its scope [11,12]. Crucially, its density relative to the Lebesgue measure on the dual Lie algebra is piecewise polynomial, computable via the Boylal-Vergne-Paradan jump formula [13]. Moreover, the DH formalism has proven instrumental in analyzing joint eigenvalue distributions of marginal states in random multipartite quantum systems [14], demonstrating its utility in quantum information theory.

The remainder of this paper is structured as follows: In Section 2, we review necessary background on the Hilbert-Schmidt metric, the geometry of the quantum state space, and separability criteria. We provide details about the computation of the Hilbert-Schmidt volume of the full quantum state space in Section 3 by calculating volume of flag manifolds and (co)adjoint orbit. Section 4 explores the connection to symplectic geometry via co-adjoint orbits and the Duistermaat-Heckman measure, deriving the corresponding symplectic volumes and linking them back to the Hilbert-Schmidt framework. In Section 5, we synthesizes the volume calculations from the previous sections to compute the volume of the separable set and derives the exact separability probability. Finally, in Section 6, we conclude with a discussion of the significance of the result and potential extensions.

2. Preliminaries and Notations

Let the set of all $N \times N$ complex Hermitian matrices be denoted by $\text{Herm}(\mathbb{C}^N)$. Furthermore, $\text{Herm}_s(\mathbb{C}^N)$ denote the set of all elements in $\text{Herm}(\mathbb{C}^N)$ with simple spectrum (i.e., those $N \times N$ complex Hermitian matrices with N distinct eigenvalues).

Proposition 1 ([15,16]). *It holds that the subset $\text{Herm}_s(\mathbb{C}^N)$ is an open and dense of full measure in $\text{Herm}(\mathbb{C}^N)$. That is, the Lebesgue measure of the complement is zero, i.e., the Lebesgue measure of $\text{Herm}(\mathbb{C}^N) \setminus \text{Herm}_s(\mathbb{C}^N)$ is zero.*

Proof. The technical proof is omitted here. $\square$

We will denote by $\mathcal{H}_N = \mathcal{H}_1 \otimes \mathcal{H}_2$ the N -dimensional Hilbert space that describes the composite system of two components $\mathcal{H}_1, \mathcal{H}_2$ with $\dim \mathcal{H}_1 = n, \dim \mathcal{H}_2 = m$ and $N = nm$. The set $D(\mathbb{C}^n \otimes \mathbb{C}^m)$ of bipartite states on $\mathcal{H}_N$ can be represented by complex $N \times N$ positive semi-definite matrices of unit trace. Let

$$D_s(\mathbb{C}^n \otimes \mathbb{C}^m) := D(\mathbb{C}^n \otimes \mathbb{C}^m) \cap \text{Herm}_s(\mathbb{C}^n \otimes \mathbb{C}^m). \quad (1)$$

From Proposition 1, we see easily that $D_s(\mathbb{C}^n \otimes \mathbb{C}^m)$ is also open and dense of full measure in $D(\mathbb{C}^n \otimes \mathbb{C}^m)$. A bipartite state ρ_{AB} in $D(\mathbb{C}^n \otimes \mathbb{C}^m)$ is called *separable* if and only if it can be written as a convex combination of local product states: $\rho_{AB} = \sum_{k=1}^K p_k \rho_{A,k} \otimes \rho_{B,k}$ for some $\rho_{A,k} \in D(\mathbb{C}^n)$, $\rho_{B,k} \in D(\mathbb{C}^m)$, and a probability vector $(p_1, \dots, p_K)$.

A natural parametrization of $\rho \in D(\mathbb{C}^n \otimes \mathbb{C}^m)$ makes use of the traceless Hermitian generators of $SU(n)$ and $SU(m)$:

$$\rho = \frac{1}{nm} \left(\mathbb{1}_{nm} + \sum_{i=1}^{n^2-1} a_i A_i \otimes \mathbb{1}_m + \mathbb{1}_n \otimes \sum_{j=1}^{m^2-1} b_j B_j + \sum_{k=1}^{n^2-1} \sum_{l=1}^{m^2-1} c_{kl} A_k \otimes B_l \right) \quad (2)$$

where

- all a_i, b_j, c_{kl} are in $\mathbb{R}$,
- A_k and B_l are the Hermitian generators of the $SU(n)$ and $SU(m)$, respectively, chosen to satisfy $\langle A_i, A_j \rangle = 2\delta_{ij} = \langle B_i, B_j \rangle$,

- two reduced states are: $\rho_{C^n} = \frac{1}{n} \left(\mathbb{1}_n + \sum_{i=1}^{n^2-1} a_i A_i \right)$ and $\rho_{C^m} = \frac{1}{m} \left(\mathbb{1}_m + \sum_{j=1}^{m^2-1} b_j B_j \right)$,
- The vector

$$\mathbf{x} = (a_1, \dots, a_{n^2-1}, b_1, \dots, b_{m^2-1}, c_{11}, \dots, c_{n^2-1, m^2-1})^\top \in \mathbb{R}^{(nm)^2-1}$$

completely determines the state $\rho \in D(\mathbb{C}^n \otimes \mathbb{C}^m)$ and vice versa. The positivity of density matrices restricts the possible vectors $\mathbf{x}$ to a proper subset $D^{(n \times m)}$ of $\mathbb{R}^{(nm)^2-1}$.

According to the parametrization Eq. (2) of bipartite states, the Hilbert-Schmidt distance $d_{\text{HS}}(\rho', \rho) := \sqrt{\langle \rho' - \rho, \rho' - \rho \rangle_{\text{HS}}}$ (see below Eq. (8)) induces a flat metric on $D^{(n \times m)}$ because

$$d_{\text{HS}}(\rho', \rho) = \frac{1}{nm} \left(2m \sum_{i=1}^{n^2-1} (a'_i - a_i)^2 + 2n \sum_{j=1}^{m^2-1} (b'_j - b_j)^2 + 4 \sum_{k=1}^{n^2-1} \sum_{l=1}^{m^2-1} (c'_{kl} - c_{kl})^2 \right)^{\frac{1}{2}}. \quad (3)$$

In particular, for $n = m = 2$, we get that

$$d_{\text{HS}}(\rho', \rho) = \frac{1}{2} \left(\sum_{i=1}^3 (a'_i - a_i)^2 + \sum_{j=1}^3 (b'_j - b_j)^2 + \sum_{k=1}^3 \sum_{l=1}^3 (c'_{kl} - c_{kl})^2 \right)^{\frac{1}{2}} = \frac{1}{2} d_{\text{Euclid}}(\mathbf{x}', \mathbf{x}), \quad (4)$$

where $\mathbf{x}' = (a', b', c'_{kl})$ and $\mathbf{x} = (a, b, c_{kl})$. In such case, up to an overall constant $\frac{1}{2}$, the following mapping is bijective and isometric:

$$(D(\mathbb{C}^2 \otimes \mathbb{C}^2), d_{\text{HS}}) \cong (D^{(2 \times 2)}, d_{\text{Euclid}}). \quad (5)$$

3. Riemannian Volumes of Manifolds

Recall that an N -dimensional oriented manifold M with a pseudo-Riemannian metric g has a standard volume form ω , known as the Riemannian volume form, whose expression in an oriented chart $(x^1, \dots, x^N)$ is given by

$$\omega = \sqrt{\det(g)} dx^1 \wedge \dots \wedge dx^N, \quad (6)$$

corresponding to the line element (aka, arc length differential or differential of arc length) $ds^2 = \sum_{i,j=1}^N dx^i g_{ij} dx^j$.

If D is a domain of integration on M , then

$$\text{vol}_g(D) := \int_D \omega \quad (7)$$

is called the *Riemannian volume* of D .

In particular, on $D_s(\mathbb{C}^N)$, we have the Hilbert-Schmidt inner product, which is defined by

$$\langle \mathbf{X}, \mathbf{Y} \rangle := \text{Tr}(\mathbf{X}^\dagger \mathbf{Y}). \quad (8)$$

Differentiating this inner product yields a metric on $D_s(\mathbb{C}^N)$ which we denote by g_{HS} . We shall denote by vol_{HS} the Riemannian volume form associated with g_{HS} and refer to the volume measured by vol_{HS} as the HS volume [17].

3.1. Hilbert-Schmidt Volumes of Flag Manifolds

A complex matrix $\mathbf{Z}$ can always be written as $\mathbf{Z} = \mathbf{X} + i\mathbf{Y}$, where $\mathbf{X} = \text{Re}(\mathbf{Z})$ and $\mathbf{Y} = \text{Im}(\mathbf{Z})$ are real matrices. Denote

$$[d\mathbf{Z}] := [d\mathbf{X}][d\mathbf{Y}], \quad (9)$$

where $[d\mathbf{X}]$ means the product of independent differentials in $\mathbf{X}$, and $[d\mathbf{Y}]$ has a similar meaning.

Assume that $\mathbf{U} = (u_{ij}) \in \mathbf{U}(N)$, where $u_{ij} \in \mathbb{C}$. Let $u_{kk} = |u_{kk}|e^{i\theta_k}$ be its polar form, where $\theta_k \in [-\pi, \pi)$. Then

$$\mathbf{U} = (\mathbf{U}\mathbf{T}^{-1})\mathbf{T}, \quad \mathbf{T} = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_N}). \quad (10)$$

This means that any $\mathbf{U} \in \mathbf{U}(N)$ can be factorized into a product of a unitary matrix with diagonal entries being nonnegative and a diagonal unitary matrix. Thus, we have the following diffeomorphism:

$$\mathbf{U}(N) \cong \mathbf{U}(N)/\mathbb{T}^N \times \mathbb{T}^N, \quad (11)$$

where $\mathbb{T}^N := \left\{ \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_N}) : \theta_k \in [-\pi, \pi), k = 1, \dots, N \right\}$, which is the maximal torus of $\mathbf{U}(N)$. For any measurable f over $\mathbf{U}(N)$,

$$\int_{\mathbf{U}(N)} f(\mathbf{U})[\mathbf{U}^\dagger d\mathbf{U}] = \int_{\mathbf{U}(N)/\mathbb{T}^N} \int_{\mathbb{T}^N} f(\mathbf{V}\mathbf{T})[\mathbf{V}^\dagger d\mathbf{V}][\mathbf{T}^\dagger d\mathbf{T}], \quad (12)$$

where $\mathbf{U} = \mathbf{V}\mathbf{T}$ for $\mathbf{V} \in \mathbf{U}(N)/\mathbb{T}^N$ and $\mathbf{T} \in \mathbb{T}^N$, and $[\mathbf{U}^\dagger d\mathbf{U}] = [\mathbf{V}^\dagger d\mathbf{V}][\mathbf{T}^\dagger d\mathbf{T}]$, where $[\mathbf{U}^\dagger d\mathbf{U}]$ is a left-invariant matrix-valued differential form, called Maurer-Cartan form. Since $v_{kk} \in \mathbb{R}_{>0}$, and moreover v_{kk} is not an independent variable, it follows that

$$[\mathbf{V}^\dagger d\mathbf{V}] = \prod_{1 \leq i < j \leq N} \text{Re}(\mathbf{V}^\dagger d\mathbf{V})_{ij} \text{Im}(\mathbf{V}^\dagger d\mathbf{V})_{ij}, \quad (13)$$

$$[\mathbf{U}^\dagger d\mathbf{U}] = \prod_{k=1}^N \text{Im}((\mathbf{U}^\dagger d\mathbf{U})_{kk}) \times \prod_{1 \leq i < j \leq N} \text{Re}(\mathbf{U}^\dagger d\mathbf{U})_{ij} \text{Im}(\mathbf{U}^\dagger d\mathbf{U})_{ij} \quad (14)$$

The Euclid volume of the flag manifold $\mathbf{U}(N)/\mathbb{T}^N$ is given by [15]:

$$\text{vol}_{\text{Euclid}}(\mathbf{U}(N)/\mathbb{T}^N) = \int_{\mathbf{U}(N)/\mathbb{T}^N} [\mathbf{V}^\dagger d\mathbf{V}] = \frac{\pi^{\binom{N}{2}}}{\prod_{k=1}^N \Gamma(k)}. \quad (15)$$

Proposition 2 ([15]). *The Hilbert-Schmidt volume of the flag manifold $\mathbf{U}(N)/\mathbb{T}^N$ is given by*

$$\text{vol}_{\text{HS}}(\mathbf{U}(N)/\mathbb{T}^N) = 2^{\binom{N}{2}} \text{vol}_{\text{Euclid}}(\mathbf{U}(N)/\mathbb{T}^N) = \frac{(2\pi)^{\binom{N}{2}}}{\prod_{k=1}^N \Gamma(k)}, \quad (16)$$

Proof. In fact, the line element for the Hilbert-Schmidt inner metric is given by

$$ds_{\text{HS}}^2 = \langle d\mathbf{V}, d\mathbf{V} \rangle = \left\langle \mathbf{V}^\dagger d\mathbf{V}, \mathbf{V}^\dagger d\mathbf{V} \right\rangle, \quad \mathbf{V} \in \mathbf{U}(N)/\mathbb{T}^N.$$

It is clearly seen that $\mathbf{V}^\dagger d\mathbf{V}$ is skew-Hermitian and its diagonal part is real. Thus the diagonal part of $\mathbf{V}^\dagger d\mathbf{V}$ must be vanished, and

$$\begin{aligned} ds_{\text{HS}}^2 &= \sum_{k=1}^N [(\mathbf{V}^\dagger d\mathbf{V})_{kk}]^2 + 2 \sum_{1 \leq i < j \leq N} \left([\text{Re}(\mathbf{V}^\dagger d\mathbf{V})_{ij}]^2 + [\text{Im}(\mathbf{V}^\dagger d\mathbf{V})_{ij}]^2 \right) \\ &= 2 \sum_{1 \leq i < j \leq N} \left([\text{Re}(\mathbf{V}^\dagger d\mathbf{V})_{ij}]^2 + [\text{Im}(\mathbf{V}^\dagger d\mathbf{V})_{ij}]^2 \right). \end{aligned}$$

The corresponding HS volume element is given by

$$\begin{aligned} dV_{\text{HS}} &= \prod_{1 \leq i < j \leq N} \left(\sqrt{2} \text{Re}(\mathbf{V}^\dagger d\mathbf{V})_{ij} \sqrt{2} \text{Im}(\mathbf{V}^\dagger d\mathbf{V})_{ij} \right) \\ &= 2^{\binom{N}{2}} \prod_{1 \leq i < j \leq N} \left(\text{Re}(\mathbf{V}^\dagger d\mathbf{V})_{ij} \text{Im}(\mathbf{V}^\dagger d\mathbf{V})_{ij} \right) = 2^{\binom{N}{2}} [\mathbf{V}^\dagger d\mathbf{V}], \end{aligned}$$

which leads to the conclusion

$$\begin{aligned}
\text{vol}_{\text{HS}}(\text{U}(N)/\mathbb{T}^N) &= \int_{\text{U}(N)/\mathbb{T}^N} dV_{\text{HS}} = 2^{\binom{N}{2}} \int_{\text{U}(N)/\mathbb{T}^N} [\mathbf{V}^\dagger d\mathbf{V}] \\
&= 2^{\binom{N}{2}} \text{vol}_{\text{Euclid}}(\text{U}(N)/\mathbb{T}^N) \\
&= 2^{\binom{N}{2}} \frac{\pi^{\binom{N}{2}}}{\prod_{k=1}^N \Gamma(k)} = \frac{(2\pi)^{\binom{N}{2}}}{\prod_{k=1}^N \Gamma(k)}.
\end{aligned}$$

This completes the proof. $\square$

3.2. Hilbert-Schmidt Volumes of Regular Adjoint Orbits

Denote the adjoint orbit $\mathcal{U}_{\mathbf{x}}$ for $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$, where $x_1 > \dots > x_N$, as

$$\mathcal{U}_{\mathbf{x}} := \left\{ \mathbf{U} \text{diag}(x_1, \dots, x_N) \mathbf{U}^\dagger : \mathbf{U} \in \text{U}(N) \right\}. \quad (17)$$

Let $V_N(\mathbf{x}) := \prod_{1 \leq i < j \leq N} (x_i - x_j)$. The stablizer group of $\mathbf{x}$ is just the flag manifold $\text{U}(N)/\mathbb{T}^N$. Apparently, $\mathcal{U}_{\mathbf{x}} \cong \{\mathbf{x}\} \times \text{U}(N)/\mathbb{T}^N$, where $\mathbb{T}^N = \text{U}(1)^{\times N}$ is N -torus. Suppose that the unitary group $\text{U}(N)$ is equipped with Haar probability measure μ_{Haar} . Consider the following map

$$\begin{aligned}
\pi : (\text{U}(N), \mu_{\text{Haar}}) &\longmapsto (\mathcal{U}_{\mathbf{x}}, \nu := \pi_* \mu_{\text{Haar}}), \\
\mathbf{U} &\mapsto \mathbf{U} \text{diag}(x_1, \dots, x_N) \mathbf{U}^\dagger.
\end{aligned}$$

The push-forward measure of μ_{Haar} along the map π is $\nu := \pi_* \mu_{\text{Haar}}$, called the *orbital measure* [18]. So the integral along the orbit is given by the following formula

$$\begin{aligned}
\int_{\mathcal{U}_{\mathbf{x}}} f d\nu &= \langle \nu, f \rangle = \langle \pi_* \mu_{\text{Haar}}, f \rangle = \langle \mu_{\text{Haar}}, \pi^* f \rangle \\
&= \int_{\text{U}(N)} f(\mathbf{U} \text{diag}(x_1, \dots, x_N) \mathbf{U}^\dagger) d\mu_{\text{Haar}}(\mathbf{U}).
\end{aligned}$$

The following result [19] is mentioned without proof. Here we provide detailed proof for the reference.

Proposition 3. *The Hilbert-Schmidt volume of the adjoint orbit $\mathcal{U}_{\mathbf{x}} \cong \text{U}(N)/\mathbb{T}^N$ is given by*

$$\text{vol}_{\text{HS}}(\mathcal{U}_{\mathbf{x}}) = V_N(\mathbf{x})^2 \text{vol}_{\text{HS}}(\text{U}(N)/\mathbb{T}^N). \quad (18)$$

Proof. Indeed, for $\mathbf{X} \in \mathcal{U}_{\mathbf{x}}$, i.e., $\mathbf{X} = \mathbf{V} \text{diag}(x_1, \dots, x_N) \mathbf{V}^\dagger$, where $\mathbf{V} \in \text{U}(N)/\mathbb{T}^N$. Then

$$d\mathbf{X} = d\mathbf{V} \text{diag}(x_1, \dots, x_N) \mathbf{V}^\dagger + \mathbf{V} \text{diag}(x_1, \dots, x_N) d\mathbf{V}^\dagger,$$

which implies that

$$\mathbf{V}^\dagger (d\mathbf{X}) \mathbf{V} = \mathbf{V}^\dagger d\mathbf{V} \text{diag}(x_1, \dots, x_N) + \text{diag}(x_1, \dots, x_N) d\mathbf{V}^\dagger \mathbf{V}.$$

Due to the fact that $\mathbf{V}^\dagger \mathbf{V} = \mathbb{1}_N$, we infer that

$$(d\mathbf{V}^\dagger) \mathbf{V} + \mathbf{V}^\dagger d\mathbf{V} = 0 \iff (d\mathbf{V}^\dagger) \mathbf{V} = -\mathbf{V}^\dagger d\mathbf{V}.$$

This indicates that $\mathbf{V}^\dagger (d\mathbf{X}) \mathbf{V} = [\mathbf{V}^\dagger d\mathbf{V}, \text{diag}(x_1, \dots, x_N)]$. Thus

$$\begin{aligned}
[d\mathbf{X}] &= [\mathbf{V}^\dagger d\mathbf{X} \mathbf{V}] = \prod_{i < j} [\mathbf{V}^\dagger d\mathbf{V}, \text{diag}(x_1, \dots, x_N)]_{ij} \\
&= \prod_{i < j} (x_j - x_i) \text{Re}(\mathbf{V}^\dagger d\mathbf{V})_{ij} (x_j - x_i) \text{Im}(\mathbf{V}^\dagger d\mathbf{V})_{ij} \\
&= V_N(\mathbf{x})^2 [\mathbf{V}^\dagger d\mathbf{V}],
\end{aligned}$$

where $[\mathbf{V}^\dagger d\mathbf{V}] = \prod_{1 \leq i < j \leq N} \operatorname{Re}(\mathbf{V}^\dagger d\mathbf{V})_{ij} \operatorname{Im}(\mathbf{V}^\dagger d\mathbf{V})_{ij}$ is the Euclid volume element on the flag manifold $\mathbf{U}(N)/\mathbb{T}^N$. We can derive that $[d\mathbf{X}] = V_N(\mathbf{x})^2 [\mathbf{V}^\dagger d\mathbf{V}]$. Now

$$\begin{aligned} \operatorname{vol}_{\text{Euclid}}(\mathcal{U}_x) &= \int_{\mathcal{U}_x} [d\mathbf{X}] = V_N(\mathbf{x})^2 \int_{\mathbf{U}(N)/\mathbb{T}^N} [\mathbf{V}^\dagger d\mathbf{V}] \\ &= V_N(\mathbf{x})^2 \operatorname{vol}_{\text{Euclid}}(\mathbf{U}(N)/\mathbb{T}^N). \end{aligned}$$

The differential of arc length is given by

$$\begin{aligned} ds_{\text{HS}}^2 &= \langle d\mathbf{X}, d\mathbf{X} \rangle = \langle [\mathbf{V}^\dagger d\mathbf{V}, \operatorname{diag}(x_1, \dots, x_N)], [\mathbf{V}^\dagger d\mathbf{V}, \operatorname{diag}(x_1, \dots, x_N)] \rangle \\ &= \sum_{i,j=1}^N \left| [\mathbf{V}^\dagger d\mathbf{V}, \operatorname{diag}(x_1, \dots, x_N)]_{ij} \right|^2 = \sum_{i,j=1}^N \left| (x_j - x_i) (\mathbf{V}^\dagger d\mathbf{V})_{ij} \right|^2 \\ &= \sum_{i \neq j} (x_j - x_i)^2 \left| (\mathbf{V}^\dagger d\mathbf{V})_{ij} \right|^2 = 2 \sum_{1 \leq i < j \leq N} (x_j - x_i)^2 \left| (\mathbf{V}^\dagger d\mathbf{V})_{ij} \right|^2 \\ &= \sum_{1 \leq i < j \leq N} \left(\sqrt{2}(x_j - x_i) \operatorname{Re}(\mathbf{V}^\dagger d\mathbf{V})_{ij} \right)^2 + \sum_{1 \leq i < j \leq N} \left(\sqrt{2}(x_j - x_i) \operatorname{Im}(\mathbf{V}^\dagger d\mathbf{V})_{ij} \right)^2. \end{aligned}$$

The corresponding HS volume is given by $dV_{\text{HS}} = 2^{\binom{N}{2}} V_N(\mathbf{x})^2 [\mathbf{V}^\dagger d\mathbf{V}]$. Thus

$$\begin{aligned} \operatorname{vol}_{\text{HS}}(\mathcal{U}_x) &= 2^{\binom{N}{2}} \operatorname{vol}_{\text{Euclid}}(\mathcal{U}_x) = 2^{\binom{N}{2}} V_N(\mathbf{x})^2 \operatorname{vol}_{\text{Euclid}}(\mathbf{U}(N)/\mathbb{T}^N) \\ &= 2^{\binom{N}{2}} V_N(\mathbf{x})^2 \frac{\pi^{\binom{N}{2}}}{\prod_{k=1}^N \Gamma(k)} = V_N(\mathbf{x})^2 \frac{(2\pi)^{\binom{N}{2}}}{\prod_{k=1}^N \Gamma(k)}, \end{aligned}$$

implying that the HS volume of adjoint orbit $\mathcal{U}_x \cong \mathbf{U}(N)/\mathbb{T}^N$ is given by the following formula:

$$\operatorname{vol}_{\text{HS}}(\mathcal{U}_x) = V_N(\mathbf{x})^2 \frac{(2\pi)^{\binom{N}{2}}}{\prod_{k=1}^N \Gamma(k)} = V_N(\mathbf{x})^2 \operatorname{vol}_{\text{HS}}(\mathbf{U}(N)/\mathbb{T}^N).$$

We are done. $\square$

3.3. Hilbert-Schmidt Volumes of Quantum State Spaces

Let

$$\mathcal{C}_N := \left\{ (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 > \dots > x_N \right\}, \quad (19)$$

$$\Delta_{N-1} := \left\{ (x_1, \dots, x_N) \in \mathbb{R}_{\geq 0}^N : \sum_{k=1}^N x_k = 1 \right\}. \quad (20)$$

The natural projection $\mathbf{U}(N) \rightarrow \mathbf{U}(N)/\mathbb{T}^N$, where $\mathbb{T}^N = \{\operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_N}) : \theta_1, \dots, \theta_N \in \mathbb{R}\}$ such that:

$$\mathbf{U} \mapsto [\mathbf{U}] \equiv \mathbf{U}\mathbb{T}^N \in \mathbf{U}(N)/\mathbb{T}^N.$$

Define the map $\tau : \operatorname{Herm}_s(\mathbb{C}^N) \rightarrow \mathcal{C}_N \times \mathbf{U}(N)/\mathbb{T}^N$.

Theorem 1 ([16]). *It holds that the above map τ is a diffeomorphism:*

$$\operatorname{Herm}_s(\mathbb{C}^N) \stackrel{\tau}{\cong} \mathcal{C}_N \times (\mathbf{U}(N)/\mathbb{T}^N). \quad (21)$$

Moreover, the map τ induces the following two diffeomorphisms:

$$\mathcal{D}_s(\mathbb{C}^N) \cong (\Delta_{N-1} \cap \mathcal{C}_N) \times (\mathbf{U}(N)/\mathbb{T}^N), \quad (22)$$

$$\mathcal{U}_x \cong \{x\} \times (\mathbf{U}(N)/\mathbb{T}^N) \cong \mathbf{U}(N)/\mathbb{T}^N. \quad (23)$$

Proof. The proof is omitted here. $\square$

Although the HS volume of the set of all density matrices acting on $\mathbb{C}^N$ is derived already by Życzkowski, we still include the proof here for completeness.

Theorem 2 ([3]). *It holds that*

$$\text{vol}_{\text{HS}}(\mathcal{D}(\mathbb{C}^N)) = \sqrt{N}(2\pi)^{\binom{N}{2}} \frac{\prod_{k=1}^N \Gamma(k)}{\Gamma(N^2)}. \quad (24)$$

Proof. The infinitesimal distance takes a particularly simple form $ds_{\text{HS}}^2 = \langle d\rho, d\rho \rangle_{\text{HS}}$ valid for any dimension N . Making use of the diagonal form $\rho = \mathbf{U}\Lambda\mathbf{U}^\dagger$, we may write

$$d\rho = \mathbf{V} \left(d\Lambda + [\mathbf{V}^\dagger d\mathbf{V}, \Lambda] \right) \mathbf{V}^\dagger,$$

where $\mathbf{V} \in \text{U}(N)/\mathbb{T}^N$. Thus the differential of arc length can be rewritten as

$$ds_{\text{HS}}^2 = \sum_{j=1}^N d\lambda_j^2 + 2 \sum_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 \left| \langle i | \mathbf{V}^\dagger d\mathbf{V} | j \rangle \right|^2.$$

Apparently, $\sum_{j=1}^N d\lambda_j = 0$ since $\sum_{j=1}^N \lambda_j = 1$. Thus

$$ds_{\text{HS}}^2 = \sum_{i,j=1}^{N-1} d\lambda_i g_{ij} d\lambda_j + 2 \sum_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 \left| \langle i | \mathbf{V}^\dagger d\mathbf{V} | j \rangle \right|^2.$$

The HS volume element is given by

$$dV_{\text{HS}} = \sqrt{N} 2^{\binom{N}{2}} V_N(\lambda)^2 \prod_{k=1}^{N-1} d\lambda_k \prod_{1 \leq i < j \leq N} \text{Re}(\mathbf{V}^\dagger d\mathbf{V})_{ij} \text{Im}(\mathbf{V}^\dagger d\mathbf{V})_{ij}.$$

The corresponding volume element gains a factor $\sqrt{\det(g)} = \sqrt{N}$, where $g = (g_{ij})_{(N-1) \times (N-1)}$ is the metric in the $(N-1)$ -dimensional simplex Δ_{N-1} of eigenvalues. Note that $g = \mathbb{1}_{N-1} + |\mathbf{e}\rangle\langle\mathbf{e}|$ for $(N-1)$ -dimensional vector $\mathbf{e} := (1, \dots, 1)^\top$. Therefore

$$\begin{aligned} \text{vol}_{\text{HS}}(\mathcal{D}(\mathbb{C}^N)) &= \int_{\mathcal{D}(\mathbb{C}^N)} dV_{\text{HS}} = \int_{\Delta_{N-1} \cap \mathcal{C}_N} V_N(\lambda)^2 \sqrt{N} \prod_{k=1}^{N-1} d\lambda_k \times \int_{\text{U}(N)/\mathbb{T}^N} 2^{\binom{N}{2}} [\mathbf{V}^\dagger d\mathbf{V}] \\ &= \text{vol}_{\text{HS}}(\text{U}(N)/\mathbb{T}^N) \int_{\Delta_{N-1} \cap \mathcal{C}_N} V_N(\lambda)^2 \sqrt{N} \prod_{k=1}^{N-1} d\lambda_k, \end{aligned}$$

where

$$\int_{\Delta_{N-1} \cap \mathcal{C}_N} V_N(\lambda)^2 \prod_{k=1}^{N-1} d\lambda_k = \frac{1}{N!} \int_{\mathbb{R}_{\geq 0}^N} \delta\left(1 - \sum_{k=1}^N \lambda_k\right) V_N(\lambda)^2 \prod_{k=1}^N d\lambda_k = \frac{\left(\prod_{k=1}^N \Gamma(k)\right)^2}{\Gamma(N^2)}.$$

Substituting the last result into the above, we get the desired result. $\square$

Remark 1. We can partition the set $\mathcal{D}(\mathbb{C}^N)$ of all density matrices acting on $\mathbb{C}^N$ according to the spectrum $\Lambda = (\lambda_1, \dots, \lambda_N) \in \Delta_{N-1} \cap \mathcal{C}_N$:

$$\mathcal{D}(\mathbb{C}^N) = \bigsqcup_{\Lambda \in \Delta_{N-1} \cap \mathcal{C}_N} \mathcal{U}_\Lambda. \quad (25)$$

From this observation and previous discussion, we get that

$$\begin{aligned}
\text{vol}_{\text{HS}}(D(\mathbb{C}^N)) &= \sqrt{N} \frac{\left(\prod_{k=1}^N \Gamma(k)\right)^2}{\Gamma(N^2)} \cdot \text{vol}_{\text{HS}}(\text{U}(N)/\mathbb{T}^N) \\
&= \sqrt{N} \int_{\Delta_{N-1} \cap \mathcal{C}_N} V_N(\Lambda)^2 [d\Lambda] \cdot \text{vol}_{\text{HS}}(\text{U}(N)/\mathbb{T}^N) \\
&= \sqrt{N} \int_{\Delta_{N-1} \cap \mathcal{C}_N} \text{vol}_{\text{HS}}(\mathcal{U}_\Lambda) [d\Lambda].
\end{aligned}$$

Based on it, we infer that every adjoint orbit $\mathcal{U}_\Lambda$ is subject to the distribution

$$dm(\Lambda) = \sqrt{N} \delta(1 - \text{Tr}(\Lambda)) [d\Lambda]. \quad (26)$$

It holds that

$$\text{vol}_{\text{HS}}(D(\mathbb{C}^N)) = \int_{\mathcal{C}_N \cap \Delta_{N-1}} \text{vol}_{\text{HS}}(\mathcal{U}_\Lambda) dm(\Lambda). \quad (27)$$

4. Symplectic Volumes of Regular Co-Adjoint Orbits

Consider a compact connected Lie group K with its Lie algebra $\mathfrak{k}$. Let T be the maximal torus of K with its Lie algebra $\mathfrak{t}$, namely, the Cartan subalgebra. Choose an K -invariant inner product on $\mathfrak{k}$. By restricting such K -invariant inner product on $\mathfrak{t}$, we can identify $\mathfrak{t}^*$, the dual space of $\mathfrak{t}$, with $\mathfrak{t}$.

Let R^+ be the set of all positive roots of K . Then R^+ determines the positive Weyl chamber $\mathfrak{t}_{\geq 0}$ via the way

$$\mathfrak{t}_{\geq 0} = \{X \in \mathfrak{t}_{\geq 0} : \alpha(X) \geq 0, \forall \alpha \in R^+\}. \quad (28)$$

Analogously, we also identify $\mathfrak{t}_{\geq 0}^*$ with $\mathfrak{t}_{\geq 0}$.

Consider an element $\hat{\lambda} \in \mathfrak{it}_{>0}^*$, the interior of positive Weyl chamber $\mathfrak{it}_{\geq 0}^*$. Let

$$\mathcal{O}_{\hat{\lambda}} := K \cdot \hat{\lambda} = \left\{ \text{Ad}_g^* \hat{\lambda} : g \in K \right\} \quad (29)$$

be the co-adjoint orbit, identified by co-adjoint action of K on $\mathfrak{it}_{>0}^* \cong \mathfrak{it}_{>0}$. There is a K -equivariantly diffeomorphism

$$\mathcal{O}_{\hat{\lambda}} \cong K/T.$$

Moreover, the coadjoint orbit $\mathcal{O}_{\hat{\lambda}}$ carry the so-called standard symplectic form, i.e., Kirillov-Kostant-Souriau form ω_{KKS} [20]. Now we get a symplectic form $(\mathcal{O}_{\hat{\lambda}}, \omega_{\text{KKS}})$. The top degree of ω_{KKS} will identify a volume form $\frac{\omega_{\text{KKS}}^d}{d!}$, where $d := \frac{1}{2} \dim(\mathcal{O}_{\hat{\lambda}}) = \frac{1}{2} \dim(K/T)$.

Definition 1. The Liouville measure on $\mathcal{O}_{\hat{\lambda}}$ is defined by

$$\mu_{\mathcal{O}_{\hat{\lambda}}}(B) := \int_B \frac{\omega_{\text{KKS}}^d}{d!}, \quad (30)$$

where B is a Borel subset of $\mathcal{O}_{\hat{\lambda}}$. The symplectic volume of $\mathcal{O}_{\hat{\lambda}}$ is just $\text{vol}_{\text{symp}}(\mathcal{O}_{\hat{\lambda}}) = \mu_{\mathcal{O}_{\hat{\lambda}}}(\mathcal{O}_{\hat{\lambda}})$.

Proposition 4 ([21]). Let T be a maximal torus of a compact connected Lie group K . Let $\hat{\lambda} \in \mathfrak{it}_{>0}^*$ for which $\mathcal{O}_{\hat{\lambda}}$ is the coadjoint orbit through $\hat{\lambda}$. The symplectic volume of $\mathcal{O}_{\hat{\lambda}}$, with respect to the KKS form ω_{KKS} , is given by

$$\text{vol}_{\text{symp}}(\mathcal{O}_{\hat{\lambda}}) = \left(\prod_{\alpha \in R^+} \langle \hat{\lambda}, \alpha \rangle \right) \left(\prod_{\alpha \in R^+} \frac{2\pi}{\langle \rho, \alpha \rangle} \right), \quad (31)$$

where $\rho := \frac{1}{2} \sum_{\alpha \in R^+} \alpha$.

Remark 2. According to Harish-Chandra's volume formula [22],

$$\text{vol}_{\text{HS}}(K/T) = \prod_{\alpha \in R^+} \frac{2\pi}{\langle \rho, \alpha \rangle}. \quad (32)$$

Based on this formula, we get that

$$\text{vol}_{\text{symp}}(\mathcal{O}_{\hat{\lambda}}) = \left(\prod_{\alpha \in R^+} \langle \hat{\lambda}, \alpha \rangle \right) \text{vol}_{\text{HS}}(K/T). \quad (33)$$

Example 1. Let $K = \text{U}(N)$. The maximal torus of $\text{U}(N)$ is $T = \mathbb{T}^N$, and its Lie algebra $\mathfrak{t}$ such that $\mathfrak{it}^* \cong \mathfrak{it} \cong \mathbb{R}^N$. Moreover $\mathfrak{it}_{>0}^* \cong \mathcal{C}_N$. Then the set of all positive roots R^+ , where

$$R^+ = \left\{ \alpha_i - \alpha_j : 1 \leq i < j \leq N \right\}, \quad (34)$$

where α_k is a linear functional taking the k -th position's imaginary part. Moreover $d = |R^+| = \binom{N}{2}$. The symplectic volume of the coadjoint orbit $\mathcal{O}_{\hat{\lambda}}$ is

$$\text{vol}_{\text{symp}}(\mathcal{O}_{\hat{\lambda}}) = V_N(\hat{\lambda}) \text{vol}_{\text{HS}}(\text{U}(N)/\mathbb{T}^N). \quad (35)$$

When we consider both HS volume and symplectic volume for the same manifold $\mathcal{O}_{\hat{\lambda}}$, we will get the following relationship between such volumes.

Proposition 5. Let $\hat{\lambda} \in \mathfrak{it}_{>0}^* \cong \mathfrak{it}_{>0}$ for the unitary group $\text{U}(N)$. The HS volume and the symplectic volume of $\mathcal{O}_{\hat{\lambda}}$ are connected via the formula:

$$\text{vol}_{\text{HS}}(\mathcal{O}_{\hat{\lambda}}) = V_N(\hat{\lambda}) \times \text{vol}_{\text{symp}}(\mathcal{O}_{\hat{\lambda}}). \quad (36)$$

Proof. According to Proposition 3,

$$\begin{aligned} \text{vol}_{\text{HS}}(\mathcal{O}_{\hat{\lambda}}) &= V_N(\hat{\lambda})^2 \text{vol}_{\text{HS}}(\text{U}(N)/\mathbb{T}^N) = V_N(\hat{\lambda}) \left[V_N(\hat{\lambda}) \text{vol}_{\text{HS}}(\text{U}(N)/\mathbb{T}^N) \right] \\ &= V_N(\hat{\lambda}) \text{vol}_{\text{symp}}(\mathcal{O}_{\hat{\lambda}}), \end{aligned}$$

where the last equality is obtained from Eq. (35). $\square$

5. Duistermaat-Heckman Measure

The Duistermaat-Heckman (DH) Theorem, introduced in [10], revolutionized symplectic geometry and mathematical physics by uncovering profound connections between Hamiltonian dynamics, localization, and topology. Its key conclusions — including the exact stationary phase approximation, the convexity of images of moment map, the polynomiality of reduced symplectic volumes, relations to equivariant cohomology and localization [23], non-Abelian generalizations, applications in Physics, and combinatorial and algebraic consequences — have each had significant impact.

Central to these developments is the powerful Duistermaat-Heckman (DH) measure [24]. This measure provides a crucial tool across symplectic geometry, representation theory, and mathematical physics. Its strength lies in transforming complex global geometric problems into manageable local computations concentrated at fixed points, thereby effectively bridging symplectic geometry with representation theory, combinatorics, and physics.

Definition 2 (Push-forward of a measure). Given measurable spaces $(X, \mathcal{F})$ and $(Y, \mathcal{G})$, a measurable mapping $\Phi : X \rightarrow Y$ and a measure $\mu : \mathcal{F} \rightarrow [0, +\infty)$, the push-forward of μ is defined to be the measure $\Phi_*\mu : \mathcal{G} \rightarrow [0, +\infty)$ given by

$$(\Phi_*\mu)(B) := \mu\left(\Phi^{-1}(B)\right) \quad \text{for } B \in \mathcal{G}.$$

Proposition 6 (Change of variables formula). *A measurable function f on Y is integrable with respect to the push-forward measure $\Phi_*\mu$ if and only if the composition $\Phi^*f = f \circ \Phi$ is integrable with respect to the measure μ . In that case, the integrals coincide*

$$\int_Y f d(\Phi_*\mu) = \int_X f \circ \Phi d\mu = \int_X \Phi^*f d\mu.$$

In the sense of distribution, $\langle \Phi_\mu, f \rangle = \langle \mu, \Phi^*f \rangle$.*

Definition 3 (Liouville measure). *Let (M, ω) be a symplectic manifold of $2n$ dimension. A Borel set in M is a set generated from compact subsets of M under countable union and complementation. Given a Borel set B in M , the Liouville measure of B is defined as*

$$\mu(B) = \int_B \frac{\omega^n}{n!}.$$

The symplectic volume of M is defined by

$$\text{vol}_{\text{symp}}(M) = \mu(M) = \int_M \frac{\omega^n}{n!}.$$

In order to define Duistermaat-Heckman measure, we need the following notion of moment map (or momentum map) is a fundamental concept in symplectic geometry and mathematical physics, providing a bridge between symplectic group actions and conserved quantities. It generalizes the idea of conserved momenta in Hamiltonian mechanics to more abstract geometric settings.

Definition 4 (Moment map). *Given a smooth manifold M equipped with a symplectic form ω (a closed, non-degenerate 2-form). A Lie group K acts on M via symplectomorphisms (symmetry transformations preserving ω). The so-called moment map is a map $\Phi : M \rightarrow \mathfrak{t}^*$ satisfying:*

- (i) *For every $X \in \mathfrak{k}$, the function $\Phi^X(m) := \langle \Phi(m), X \rangle$ is a Hamiltonian function for the vector field X_M on M generated by X :*

$$d\Phi^X = \iota_{X_M}\omega.$$

- (ii) *Φ is K -equivariant with respect to the coadjoint action Ad^* on $\mathfrak{t}^*$:*

$$\Phi(g \cdot m) = (\text{Ad}_g^*\Phi)(m), \quad \forall g \in K.$$

In such a case, this action of K on M is called Hamiltonian action. At this time, (M, K, ω) is called Hamiltonian K -manifold.

Definition 5 (Duistermaat-Heckman measure). *Let (M, K, ω) (here ω is a symplectic form) be a compact, connected Hamiltonian K -manifold of dimension $2n$ and a choice of moment map $\Phi : M \rightarrow \mathfrak{t}^*$.*

- (i) *The non-Abelian Duistermaat-Heckman measure DH_M^K is defined as follows:*

$$\text{DH}_M^K = \frac{1}{p_K} (\tau_K)_* \Phi_* \mu \quad (37)$$

where $\tau_K : \mathfrak{t}^ \rightarrow \mathfrak{t}_{\geq 0}^*$ is defined as $\tau_K(\mathcal{O}_{\hat{\lambda}}) = \{\hat{\lambda}\}$ for $\hat{\lambda} \in \mathfrak{t}_{\geq 0}^*$ and $p_K(\hat{\lambda}) = \text{vol}_{\text{symp}}(\mathcal{O}_{\hat{\lambda}})$, which is known from Eq. (31).*

- (ii) *The Abelian Duistermaat-Heckman measure is defined as:*

$$\text{DH}_M^T = \tau_* \Phi_* \mu, \quad (38)$$

where $\tau : \mathfrak{t}^ \rightarrow \mathfrak{t}^*$ is the projection dual to the inclusion map $\mathfrak{t} \hookrightarrow \mathfrak{k}$.*

The computation of the density of the pushforward measure with respect to the Lebesgue measure on the positive Weyl chamber is governed by the Duistermaat-Heckman (DH) theorem [10] and its generalizations. But in a special

case, for instance, where the underlying symplectic manifold is just a coadjoint orbit through a regular element in $\mathfrak{it}_{>0}^*$, there is an explicit formula for computing the density. In the next subsection, we describe it explicitly.

5.1. Harish-Chandra Formula and Derivative Principle

The following result gives the Fourier transform formula of Abelian Duistermaat-Heckman measure for the coadjoint orbit $\mathcal{O}_{\hat{\lambda}}$. By this, we can identify the analytical expression of Abelian Duistermaat-Heckman measure.

Theorem 3 (Harish-Chandra, [22]). *The Fourier transform of abelian Duistermaat-Heckman $\text{DH}_{\mathcal{O}_{\hat{\lambda}}}^T$ measure is given by*

$$\left\langle \text{DH}_{\mathcal{O}_{\hat{\lambda}}}^T, e^{i\langle -, X \rangle} \right\rangle = \sum_{w \in W} (-1)^{l(w)} e^{i\langle w\hat{\lambda}, X \rangle} \prod_{\alpha > 0} \frac{1}{i\langle \alpha, X \rangle}. \quad (39)$$

for every X in the Lie algebra of T which is not orthogonal to any root. That is,

$$\text{DH}_{\mathcal{O}_{\hat{\lambda}}}^T = \sum_{w \in W} (-1)^{l(w)} \delta_{w\hat{\lambda}} \star H_{-\alpha_1} \star \cdots \star H_{-\alpha_R}. \quad (40)$$

Here $l(w)$ is the length of the Weyl group element $w \in W$, and δ_{α} for the Dirac measure at α ; $\star$ means the convolution.

Subsequently, the so-called derivative principle is provided in which the non-Abelian Duistermaat-Heckman measure is obtained from Abelian Duistermaat-Heckman measure.

Theorem 4 (Derivative principle, [14]). *It holds that*

$$\text{DH}_{\mathcal{O}_{\hat{\lambda}}}^K \Big|_{\mathfrak{it}_{\geq 0}^*} = \left(\prod_{\alpha > 0} \partial_{-\alpha} \right) \text{DH}_{\mathcal{O}_{\hat{\lambda}}}^T \Big|_{\mathfrak{it}_{\geq 0}^*}.$$

Consider more generally the action of $\tilde{T}$ on a co-adjoint K -orbit $\mathcal{O}_{\hat{\lambda}}$ induced by a group homomorphism $\tilde{T} \rightarrow T \subset K$. Note that it follows directly that

$$\text{DH}_{\mathcal{O}_{\hat{\lambda}}}^{\tilde{T}} = \pi_* \text{DH}_{\mathcal{O}_{\hat{\lambda}}}^T$$

where $\pi : \mathfrak{it}^* \rightarrow \mathfrak{it}^*$ is the restriction map from the dual Lie algebra $\mathfrak{t}^*$ of T to that $\mathfrak{t}^*$ of $\tilde{T}$. Thus we get that

$$\text{DH}_{\mathcal{O}_{\hat{\lambda}}}^{\tilde{T}} = \sum_{w \in W} (-1)^{l(w)} \delta_{\pi(w\hat{\lambda})} \star H_{-\pi(\alpha_1)} \star \cdots \star H_{-\pi(\alpha_R)}.$$

In what follows, we will now use the non-Abelian Heckman algorithm to treat the case of random two-qubit states with fixed, non-degenerate global eigenvalue spectrum. That is, we consider the action of $\tilde{K} = \text{SU}(2) \times \text{SU}(2)$ on a coadjoint $K = \text{SU}(4)$ -orbit through a point $\hat{\lambda}$ in the interior of positive Weyl chamber of K .

- Let $\tilde{T}$ be the maximal torus of $\tilde{K}$.
- Symmetric group $W = S_4$ is the Weyl group of $K = \text{SU}(4)$.
- $\text{DH}_{\mathcal{O}_{\hat{\lambda}}}^{\tilde{T}} = \sum_{w \in S_4} \text{sign}(w) \delta_{\pi(w\hat{\lambda})} \star H_{-\pi(\alpha_1)} \star \cdots \star H_{-\pi(\alpha_6)}$, where $\{\alpha_k : k = 1, \dots, 6\}$ is the set of positive roots of $K = \text{SU}(4)$.
- The dual Lie algebra $\mathfrak{t}^*$ of $\tilde{T}$ can be identified with $\mathfrak{it}^* \cong \mathbb{R}^2$.
- The maps π is given by

$$\pi(\hat{\lambda}_1, \dots, \hat{\lambda}_4) = 2i(\hat{\lambda}_1 + \hat{\lambda}_2, \hat{\lambda}_1 + \hat{\lambda}_3).$$

- One computes readily that the $-\pi(\alpha_k)$ are precisely the weights

$$\{(-2, 2), (-2, 0)_{(2)}, (-2, -2), (0, -2)_{(2)}\}.$$

In particular, two negative roots of $\tilde{K}$ are contained in this list (each of them is in fact contained twice).

The following result appears in [14] without proof. We provide a detailed proof for completeness.

Proposition 7. *The non-Abelian Duistermaat-Heckman measure for the action of $\tilde{K} = \mathrm{SU}(2) \times \mathrm{SU}(2)$ on a coadjoint K -orbit $\mathcal{O}_{\hat{\lambda}} = K \cdot \hat{\lambda}$ with $\hat{\lambda}$, where $K = \mathrm{SU}(4)$, in the interior of positive Weyl chamber of T is given by*

$$\mathrm{DH}_{\mathcal{O}_{\hat{\lambda}}}^{\tilde{K}} \Big|_{\mathfrak{i}\tilde{\mathfrak{t}}_{\geq 0}^*} = \left(\sum_{w \in S_4} \mathrm{sign}(w) \delta_{\pi(w\hat{\lambda})} \right) \star H_{(-2,2)} \star H_{(-2,0)} \star H_{(-2,-2)} \star H_{(0,-2)} \Big|_{\mathfrak{i}\tilde{\mathfrak{t}}_{\geq 0}^*}, \quad (41)$$

where $\mathfrak{i}\tilde{\mathfrak{t}}_{\geq 0}^* \cong \mathbb{R}_{\geq 0}^2$ is the positive Weyl chamber of $\tilde{T}$, the maximal torus of $\tilde{K}$.

Proof. Recall that the Weyl group of $K = \mathrm{SU}(4)$ is the symmetric group $W = S_4$, with $(-1)^{l(w)}$ equal to the signum of a permutation $w \in S_4$. Then

$$\mathrm{DH}_{\mathcal{O}_{\hat{\lambda}}}^{\tilde{T}} = \pi_* \mathrm{DH}_{\mathcal{O}_{\hat{\lambda}}}^T = \sum_{w \in S_4} \mathrm{sign}(w) \delta_{\pi(w\hat{\lambda})} \star H_{-\pi(\alpha_1)} \star \cdots \star H_{-\pi(\alpha_6)},$$

where T is the maximal torus of $K = \mathrm{SU}(4)$, and $\tilde{T}$ is the maximal torus of $\tilde{K} = \mathrm{SU}(2) \times \mathrm{SU}(2)$, and $\{\alpha_k : k = 1, \dots, 6\}$ are the positive roots of $K = \mathrm{SU}(4)$, i.e.,

$$\begin{aligned} \alpha_1 &= (1, -1, 0, 0), \alpha_2 = (1, 0, -1, 0), \alpha_3 = (1, 0, 0, -1), \\ \alpha_4 &= (0, 1, -1, 0), \alpha_5 = (0, 1, 0, -1), \alpha_6 = (0, 0, 1, -1). \end{aligned}$$

Since $\hat{\lambda}_1 > \cdots > \hat{\lambda}_4$, it follows that this π maps each $\hat{\lambda}$ to $2i(\hat{\lambda}_1 + \hat{\lambda}_2, \hat{\lambda}_1 + \hat{\lambda}_3)$, where $2(\hat{\lambda}_1 + \hat{\lambda}_2)$ and $2(\hat{\lambda}_1 + \hat{\lambda}_3)$ are the maximal eigenvalues of two reduced density matrices determined by a global density matrix Λ , where $\Lambda = \mathrm{diag}(\hat{\lambda}_1 + \frac{1}{4}, \dots, \hat{\lambda}_4 + \frac{1}{4})$. One computes readily that the $-\pi(\alpha_k)$ are precisely the weights

$$\{(-2, 2), (-2, 0)_{(2)}, (-2, -2), (0, -2)_{(2)}\}.$$

Indeed,

$$\begin{aligned} -\pi(\alpha_1) &= -\pi(\alpha_6) = (0, -2), -\pi(\alpha_3) = (-2, -2), \\ -\pi(\alpha_2) &= -\pi(\alpha_5) = (-2, 0), -\pi(\alpha_4) = (-2, 2). \end{aligned}$$

That is,

$$\{-\pi(\alpha_k) : k \in [6]\} = \{(-2, 2), (-2, 0)_{(2)}, (-2, -2), (0, -2)_{(2)}\}.$$

In particular, the two negative roots $\{(-2, 0), (0, -2)\}$ of $\tilde{K} = \mathrm{SU}(2) \times \mathrm{SU}(2)$ are contained in this list $\mathfrak{t}$ (each of them is in fact contained twice). From the above discussion, we see that

$$\mathrm{DH}_{\mathcal{O}_{\hat{\lambda}}}^{\tilde{T}} = \left(\sum_{w \in S_4} \mathrm{sign}(w) \delta_{\pi(w\hat{\lambda})} \right) \star H_{(0,-2)} \star H_{(-2,0)} \star H_{(-2,-2)} \star H_{(-2,2)} \star H_{(-2,0)} \star H_{(0,-2)}.$$

Therefore, we arrived at the following formula:

$$\begin{aligned} \mathrm{DH}_{\mathcal{O}_{\hat{\lambda}}}^{\tilde{K}} \Big|_{\mathfrak{i}\tilde{\mathfrak{t}}_{\geq 0}^*} &= \left(\prod_{\alpha > 0} \partial_{-\alpha} \right) \mathrm{DH}_{\mathcal{O}_{\hat{\lambda}}}^{\tilde{T}} \Big|_{\mathfrak{i}\tilde{\mathfrak{t}}_{\geq 0}^*} \\ &= \left(\sum_{w \in S_4} \mathrm{sign}(w) \delta_{\pi(w\hat{\lambda})} \right) \star H_{(-2,-2)} \star H_{(-2,2)} \star H_{(-2,0)} \star H_{(0,-2)} \Big|_{\mathfrak{i}\tilde{\mathfrak{t}}_{\geq 0}^*}. \end{aligned}$$

We are done. $\square$

We have already known the fact that a density matrix of a generic two-qubit state is represented by the 2×2 block matrix

$$\rho_{12} = \begin{pmatrix} A & C \\ C^\dagger & B \end{pmatrix},$$

where A, B are 2×2 positive semi-definite complex matrices. Now we present the two reduced marginal states, respectively:

$$\rho_1 = \begin{pmatrix} \text{Tr}(A) & \text{Tr}(C) \\ \text{Tr}(C^\dagger) & \text{Tr}(B) \end{pmatrix}, \quad \rho_2 = A + B.$$

From this, we see that for $\rho_{12} = \Lambda$ with $\Lambda_1 \geq \dots \geq \Lambda_4 \geq 0$ and $\sum_{j=1}^4 \Lambda_j = 1$, it can be rewritten as $\rho - \frac{\mathbb{1}_4}{4} = \text{diag}(\hat{\lambda}_1, \dots, \hat{\lambda}_4)$, where $\hat{\lambda}_1 \geq \dots \geq \hat{\lambda}_4$ and $\sum_{j=1}^4 \hat{\lambda}_j = 0$. Thus $\hat{\lambda}$ is in the positive Weyl chamber $\mathfrak{t}_{\geq 0}$ of T .

Now the map $\rho \mapsto (\rho_1, \rho_2)$, where $\rho_1 = \text{Tr}_2(\rho)$ and $\rho_2 = \text{Tr}_1(\rho)$, can be equivalently represented as

$$\begin{aligned} \pi(\hat{\lambda}) &= \left(\begin{pmatrix} \hat{\lambda}_1 + \hat{\lambda}_2 & 0 \\ 0 & \hat{\lambda}_3 + \hat{\lambda}_4 \end{pmatrix}, \begin{pmatrix} \hat{\lambda}_1 + \hat{\lambda}_3 & 0 \\ 0 & \hat{\lambda}_2 + \hat{\lambda}_4 \end{pmatrix} \right) \\ &\stackrel{\text{def}}{=} (2(\hat{\lambda}_1 + \hat{\lambda}_2), 2(\hat{\lambda}_1 + \hat{\lambda}_3)), \end{aligned}$$

where

$$\begin{pmatrix} \hat{\lambda}_1 + \hat{\lambda}_2 & 0 \\ 0 & \hat{\lambda}_3 + \hat{\lambda}_4 \end{pmatrix} \stackrel{\text{def}}{=} \hat{\lambda}_1 + \hat{\lambda}_2 - (\hat{\lambda}_3 + \hat{\lambda}_4) = 2(\hat{\lambda}_1 + \hat{\lambda}_2)$$

and

$$\begin{pmatrix} \hat{\lambda}_1 + \hat{\lambda}_3 & 0 \\ 0 & \hat{\lambda}_2 + \hat{\lambda}_4 \end{pmatrix} \stackrel{\text{def}}{=} \hat{\lambda}_1 + \hat{\lambda}_3 - (\hat{\lambda}_2 + \hat{\lambda}_4) = 2(\hat{\lambda}_1 + \hat{\lambda}_3).$$

Denote $\mathbf{h} = \text{diag}(1, -1)$. Then $\text{diag}(\hat{\lambda}_1 + \hat{\lambda}_2, \hat{\lambda}_3 + \hat{\lambda}_4) = (\hat{\lambda}_1 + \hat{\lambda}_2) \cdot \mathbf{h}$ and $\text{diag}(\hat{\lambda}_1 + \hat{\lambda}_3, \hat{\lambda}_2 + \hat{\lambda}_4) = (\hat{\lambda}_1 + \hat{\lambda}_3) \cdot \mathbf{h}$. Since $\hat{\lambda}_1 + \hat{\lambda}_3 \geq \hat{\lambda}_2 + \hat{\lambda}_4$ and $(\hat{\lambda}_1 + \hat{\lambda}_3) + (\hat{\lambda}_2 + \hat{\lambda}_4) = 0$, thus $\hat{\lambda}_1 + \hat{\lambda}_3 \geq 0$, a fortiori $\hat{\lambda}_1 + \hat{\lambda}_2 \geq \hat{\lambda}_1 + \hat{\lambda}_3 \geq 0$. Note that $\hat{\lambda}_1 + \hat{\lambda}_4$ is not non-negative in general.

5.2. Boysal-Vergne-Paradan Jump Formula

In 2009, Boysal and Vergne published a paper [13] in which they investigated the push-forward of Lebesgue measures on the cone $\mathbb{R}_{\geq 0}^N$ along a linear map. Later in his PhD thesis, Walter found that the density of Abelian Duistermaat-Heckman measure with respect to the Lebesgue measure on the affine hull of the Abelian moment polytope is given by the volume of a parametrized polytope [25]. In view of this result, Boysal and Vergne's result is connected with Abelian Duistermaat-Heckman measure, and can be adapted to calculate the density. To this end, we need firstly introduce some notions involved in question.

Definition 6 ([13]). Let $e \in V$ be a primitive vector. It defines a hyperplane in V^* (the dual space of V):

$$W = \{\alpha \in V^* : \langle \alpha, e \rangle = 0\} \stackrel{\text{def}}{=} e^\perp.$$

Let P be a polynomial function on V^* and let Ψ be a sequence of vectors not belong to W , i.e., $\Psi \cap W = \emptyset$. We define, for $\alpha \in V^*$,

$$\text{Pol}(P, \Psi, e)(\alpha) = \text{Res}_{z=0} \left[\left(P(\partial_x) \frac{e^{\langle \alpha, x + ze \rangle}}{\prod_{\psi \in \Psi} \langle \psi, x + ze \rangle} \right)_{x=0} \right]. \quad (42)$$

Remark 3. Note that the function $\text{Pol}(P, \Psi, e)$ depends only on the restriction p of P to W . If p is a polynomial function on $W = e^\perp$. We define

$$\text{Pol}(p, \Psi, e) := \text{Pol}(P, \Psi, e),$$

where P is any polynomial on V^* extending p .

Definition 7 (Wall). Let $\Psi = [\psi_1, \dots, \psi_N]$ be a sequence of non-zero, not necessarily distinct, linear forms on V , i.e., $\psi_k \in V^*$, lying in an open half-space. If Ψ spans the whole space V^* with $\dim(V) = n$, then a wall of Ψ is a (real) hyperplane generated by $n - 1$ linearly independent elements of Ψ .

Note that V^* is separated into two open half-spaces by the wall $W := \{\phi \in V^* : \langle \phi, e \rangle = 0\}$ in V^* . Let $V_{\pm}^*$ denote the corresponding open half-spaces, that is,

$$V_+^* := \{\phi \in V^* : \langle \phi, e \rangle > 0\} \text{ and } V_-^* := \{\phi \in V^* : \langle \phi, e \rangle < 0\}.$$

Let $C_1 \subset V_+^*$ and $C_2 \subset V_-^*$ be two chambers on two sides of W and adjacent. We choose the measures dx on V and $d\phi$ on V^* , respectively, and choose Lebesgue measure dt on $\mathbb{R}^N$. We also choose the measure dw on W such that $d\phi = dw dt$ with $t = \langle \phi, e \rangle$ for $\phi \in V^*$. Based on this, we can write Ψ as

$$\Psi = [\Psi_0, \Psi^+, \Psi^-],$$

where $\Psi^{\pm} = \Psi \cap V_{\pm}^*$, and $\Psi_0 = \Psi \cap W$.

Theorem 5 (Boysal-Vergne-Paradan jump formula, [13]). *Let W be a wall, determined by a vector $e \in V$, Ψ be a sequence of vectors spanning the whole space V^* . Denote $\Psi_0 := \Psi \cap W$. Let $v_{12} = v(\Psi_0, dw, C_{12})$ be the polynomial function on W associated to the chamber C_{12} of Ψ_0 . Then, if $\langle C_1, e \rangle > 0$ and V_{12} is any extension of v_{12} on V^* ,*

$$v(\Psi, dx, C_1) - v(\Psi, dx, C_2) = \text{Pol}(v_{12}, \Psi \setminus \Psi_0, e). \quad (43)$$

Although the following result is obtained in [14], the details of proof is not provided. For reader's convenience, we present it here using Boysal-Vergne-Paradan jump formula.

Proposition 8. *The measure $H_{(-2,2)} \star H_{(-2,0)} \star H_{(-2,-2)} \star H_{(0,-2)}$ has Lebesgue density*

$$p(r, s) = \begin{cases} p_1(r, s) \equiv \frac{(r+s)^2}{64}, & \text{if } (r, s) \in C_1 = \{(r, s) : 0 \leq s \leq -r\}; \\ p_2(r, s) \equiv \frac{r^2 + 2rs - s^2}{64}, & \text{if } (r, s) \in C_2 = \{(r, s) : r \leq s \leq 0\}; \\ p_3(r, s) \equiv \frac{r^2}{32}, & \text{if } (r, s) \in C_3 = \{(r, s) : s \leq r \leq 0\}, \\ p_0(r, s) \equiv 0, & \text{if } (r, s) \in C_0 = \mathbb{R}^2 \setminus (C_1 \cup C_2 \cup C_3). \end{cases}$$

The density function $p(r, s)$ (its graph has already appeared in [14]) and its support that is decomposed into three chambers can be visualized in the following Figure 1.

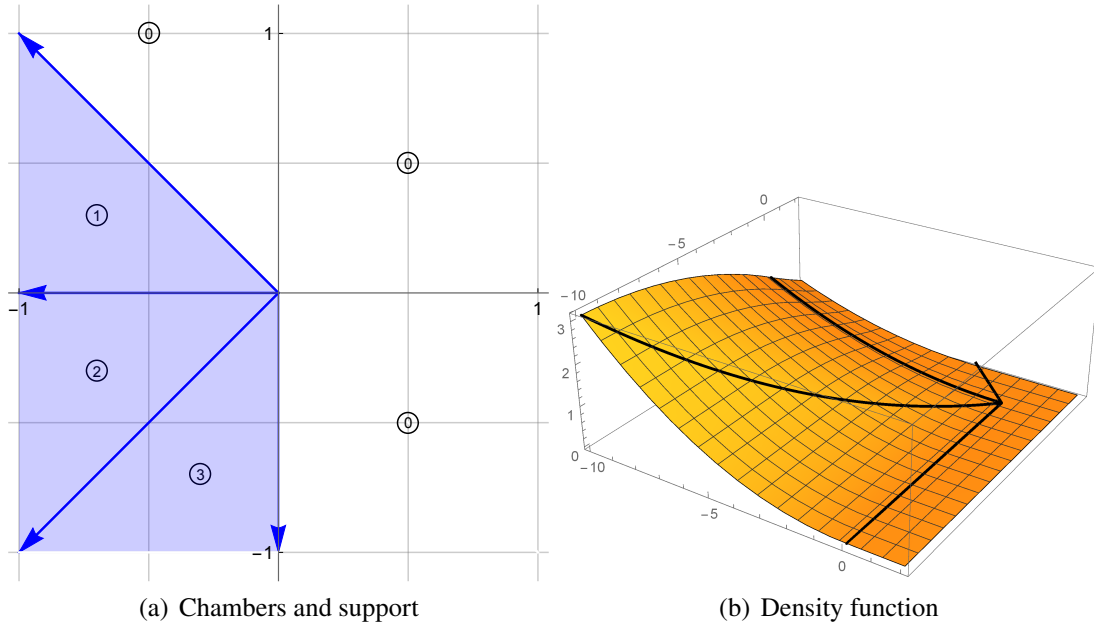


Figure 1. The support of the iterated convolution and its density over the support.

Proof. The measure $H_{(-2,2)} \star H_{(-2,0)} \star H_{(-2,-2)} \star H_{(0,-2)}$ is, in fact, the non-Abelian Duistermaat-Heckman measure that is on the closures of the regular chambers containing the vertex $(0,0)$ given by the above convolution:

$$\delta_{(0,0)} \star H_{(-2,2)} \star H_{(-2,0)} \star H_{(-2,-2)} \star H_{(0,-2)}.$$

Then its density is denote by $p(r,s)$. Denote $O(0,0), P_1(-2,2), P_2(-2,0), P_3(-2,-2), P_4(0,-2)$. Then

$$|OP_1P_2| := C_1, \quad |OP_2P_3| := C_2, \quad |OP_3P_4| := C_3.$$

Denote $\mathbb{R}^2 - C_1 - C_2 - C_3 := C_0$. Thus

$$\mathbb{R}^2 = C_0 \cup C_1 \cup C_2 \cup C_3.$$

(i) Clearly $p \equiv 0$ on C_0 .

- The wall W_{01} separating C_0 and C_1 is given by the equation: $r + s = 0$. Its normal vector $\xi = (-1, -1)$. Just only one weights $\omega_1 = (-2, 2)$ lies on the linear hyperplane spanned by W_{01} (other weights are outside of W_{01} : $\omega_2 = (-2, 0), \omega_3 = (-2, -2), \omega_4 = (0, -2)$).
- Consider the push-forward of Lebesgue measure on $\mathbb{R}_{\geq 0}^1$ along the linear map $P_{W_{01}} : u \mapsto u\omega_1$. Its density with respect to dw is given by a single homogeneous polynomial on the wall W_{01} .
- Denote by $p_{W_{01}}$ any polynomial function extending it to all of the dual Lie algebra of $\tilde{T}$.
- Clearly $p_{W_{01}} = \frac{1}{2}$. Indeed,

$$p_{W_{01}}^{-1} = d\lambda(\omega_1, \xi / \|\xi\|^2) = \left| \det \begin{pmatrix} -2 & 2 \\ -1/2 & -1/2 \end{pmatrix} \right| = 2$$

$$\iff p_{W_{01}} = \frac{1}{2}.$$

Note that, during the proof, we abuse notation by using identical symbols for a differential form and its induced measure. Hence $\mu = (r, s)$

$$p(\mu) = \frac{1}{2} \text{Res}_{z=0} \left(\frac{e^{\langle \mu, x+z\xi \rangle}}{\prod_{k=2}^4 \langle \omega_k, x+z\xi \rangle} \right)_{x=0},$$

where

$$\frac{e^{\langle \mu, x+z\xi \rangle}}{\prod_{k=2}^4 \langle \omega_k, x+z\xi \rangle} = \frac{\exp(r(x_1 - z) + s(x_2 - z))}{[-2(x_1 - z)][-2(x_1 - z) - 2(x_2 - z)][-2(x_2 - z)]}.$$

Thus

$$\begin{aligned} p(\mu) &= \frac{1}{2} \times \left(-\frac{1}{8} \right) \text{Res}_{z=0} \left(\frac{\exp(r(x_1 - z) + s(x_2 - z))}{(x_1 - z)(x_1 + x_2 - 2z)(x_2 - z)} \right)_{x=0} \\ &= \frac{1}{32} \text{Res}_{z=0} \left(\frac{e^{-(r+s)z}}{z^3} \right) = \frac{1}{64} (r+s)^2. \end{aligned}$$

(ii) The wall W_{12} separating C_1 and C_2 is given by the equation: $s = 0$. Its normal vector $\xi = (0, -1)$. Just only one weights $\omega_1 = (-2, 0)$ lies on the linear hyperplane spanned by W_{12} (other weights are outside of W_{12} : $\omega_2 = (-2, 2), \omega_3 = (-2, -2), \omega_4 = (0, -2)$).

- Consider the push-forward of Lebesgue measure on $\mathbb{R}_{\geq 0}^1$ along the linear map $P_{W_{12}} : u \mapsto u\omega_1$. Its density with respect to dw is given by a single homogeneous polynomial on the wall W_{12} .
- Denote by $p_{W_{12}}$ any polynomial function extending it to all of the dual Lie algebra of $\tilde{T}$. Clearly $p_{W_{12}} = \frac{1}{2}$. Indeed,

$$p_{W_{12}}^{-1} = d\lambda(\omega_1, \xi / \|\xi\|^2) = \left| \det \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \right| = 2 \iff p_{W_{12}} = \frac{1}{2}.$$

Hence

$$p(\mu) - \frac{1}{64}(r+s)^2 = \frac{1}{2} \text{Res}_{z=0} \left(\frac{e^{\langle \mu, \mathbf{x} + z\tilde{\xi} \rangle}}{\prod_{k=2}^4 \langle \omega_k, \mathbf{x} + z\tilde{\xi} \rangle} \right)_{\mathbf{x}=\mathbf{0}},$$

where

$$\frac{e^{\langle \mu, \mathbf{x} + z\tilde{\xi} \rangle}}{\prod_{k=2}^4 \langle \omega_k, \mathbf{x} + z\tilde{\xi} \rangle} = \frac{e^{rx_1 + s(x_2 - z)}}{[-2x_1 + 2(x_2 - z)][-2x_1 - 2(x_2 - z)][-2(x_2 - z)]}.$$

Thus

$$\begin{aligned} p(\mu) - \frac{(r+s)^2}{64} &= \frac{1}{2} \times -\frac{1}{8} \text{Res}_{z=0} \left(\frac{e^{rx_1 + s(x_2 - z)}}{(x_1 - x_2 + z)(x_1 + x_2 - z)(x_2 - z)} \right)_{\mathbf{x}=\mathbf{0}} \\ &= -\frac{1}{16} \text{Res}_{z=0} \left(\frac{e^{-sz}}{z^3} \right) = -\frac{s^2}{32}. \end{aligned}$$

Therefore, on the chamber C_2

$$p(\mu) = p(r, s) = \frac{(r+s)^2}{64} - \frac{s^2}{32} = \frac{r^2 + 2rs - s^2}{64}.$$

(iii) The wall W_{23} separating C_2 and C_3 is given by the equation: $r - s = 0$. Its normal vector $\tilde{\xi} = (1, -1)$. Just only one weights $\omega_1 = (-2, -2)$ lies on the linear hyperplane spanned by W_{23} (other weights are outside of W_{23} : $\omega_2 = (-2, 2), \omega_3 = (-2, 0), \omega_4 = (0, -2)$).

- Consider the push-forward of Lebesgue measure on $\mathbb{R}_{\geq 0}^1$ along the linear map $P_{W_{23}} : u \mapsto u\omega_1$. Its density with respect to dw is given by a single homogeneous polynomial on the wall W_{23} .
- Denote by $p_{W_{23}}$ any polynomial function extending it to all of the dual Lie algebra of $\tilde{T}$. Clearly $p_{W_{23}} = \frac{1}{2}$.

Indeed,

$$p_{W_{23}}^{-1} = d\lambda(\omega_1, \tilde{\xi} / \|\tilde{\xi}\|^2) = \left| \det \begin{pmatrix} -2 & -2 \\ 1/2 & -1/2 \end{pmatrix} \right| = 2 \iff p_{W_{23}} = \frac{1}{2}.$$

Hence

$$p(\mu) - \frac{r^2 + 2rs - s^2}{64} = \frac{1}{2} \text{Res}_{z=0} \left(\frac{e^{\langle \mu, \mathbf{x} + z\tilde{\xi} \rangle}}{\prod_{k=2}^4 \langle \omega_k, \mathbf{x} + z\tilde{\xi} \rangle} \right)_{\mathbf{x}=\mathbf{0}},$$

where

$$\frac{e^{\langle \mu, \mathbf{x} + z\tilde{\xi} \rangle}}{\prod_{k=2}^4 \langle \omega_k, \mathbf{x} + z\tilde{\xi} \rangle} = \frac{e^{r(x_1 + z) + s(x_2 - z)}}{[-2(x_1 + z) + 2(x_2 - z)][-2(x_1 + z)][-2(x_2 - z)]}.$$

Thus

$$\begin{aligned} p(\mu) - \frac{r^2 + 2rs - s^2}{64} &= \frac{1}{2} \times -\frac{1}{8} \text{Res}_{z=0} \left(\frac{e^{r(x_1 + z) + s(x_2 - z)}}{(x_1 - x_2 + 2z)(x_1 + z)(x_2 - z)} \right)_{\mathbf{x}=\mathbf{0}} \\ &= \frac{1}{32} \text{Res}_{z=0} \left(\frac{e^{(r-s)z}}{z^3} \right) = \frac{(r-s)^2}{64}. \end{aligned}$$

Therefore, on the chamber C_3

$$p(\mu) = p(r, s) = \frac{r^2 + 2rs - s^2}{64} + \frac{(r-s)^2}{64} = \frac{r^2}{32}.$$

This completes the proof. $\square$

6. Applications of Duistermaat-Heckman Measure

For $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}_{\geq 0}^3$ with $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$ and $\sum_{k=1}^4 \lambda_k = 1$, denote the $\mathrm{SU}(4)$ -adjoint orbit of $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ by

$$\mathcal{U}_\Lambda := \left\{ \mathbf{u} \Lambda \mathbf{u}^\dagger : \mathbf{u} \in \mathrm{SU}(4) \right\}.$$

Let

$$\hat{\Lambda} := \Lambda - \frac{\mathbb{1}_4}{4} = \mathrm{diag}(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4).$$

where $\hat{\lambda}_k = \lambda_k - \frac{1}{4}$. Denote by $\hat{\lambda} = (\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4)$. Apparently $\hat{\lambda}_1 > \hat{\lambda}_2 > \hat{\lambda}_3 > \hat{\lambda}_4$ and $\sum_{k=0}^4 \hat{\lambda}_k = 0$. It is easily seen that

$$\hat{\Lambda} \in \mathfrak{ih},$$

where $\mathfrak{t}$ is the Cartan subalgebra of $\mathfrak{su}(4)$, the Lie algebra of $\mathrm{SU}(4)$. By choosing Ad-invariant inner product on $\mathfrak{t}$, we can identify $\mathfrak{t}^*$ with $\mathfrak{t}$. Let $\tilde{\mathfrak{t}}$ be the Lie algebra of maximal torus $\tilde{T}$ of $\tilde{K} = \mathrm{SU}(2) \times \mathrm{SU}(2)$.

Denote

$$c_3 = 2(\hat{\lambda}_1 + \hat{\lambda}_2), \quad c_2 = 2(\hat{\lambda}_1 + \hat{\lambda}_3), \quad c_1 = 2|\hat{\lambda}_1 + \hat{\lambda}_4|. \quad (44)$$

From the above definition, we infer that

$$c_3 > c_2 > c_1 > 0. \quad (45)$$

Proposition 9. Consider the Lie group $\tilde{K} = \mathrm{SU}(2) \times \mathrm{SU}(2)$ with its maximal torus $\tilde{T}$ whose Lie algebra is given by $\tilde{\mathfrak{t}}$. The positive Weyl chamber $\mathfrak{it}_{\geq 0}^*$ can be identified with $\mathbb{R}_{\geq 0}^2$.

Proof. The Lie group $\mathrm{SU}(2)$ has a rank one Lie algebra, $\mathfrak{su}(2)$, with Cartan subalgebra $\mathfrak{h}$ spanned by $\mathbf{H} = |0\rangle\langle 0| - |1\rangle\langle 1|$. The root system is of a single positive root α satisfying $\alpha(\mathbf{H}) = 2$. The positive Weyl chamber is a subset of the Cartan subalgebra $\mathfrak{h}$ defined by the condition that all positive roots take nonnegative values. Any element $h \in \mathfrak{h}$ can be expressed as $h = z\mathbf{H}$ for some $z \in \mathbb{R}$. Then $\alpha(h) = z\alpha(\mathbf{H}) = 2z \geq 0$. Geometrically, $\mathfrak{h} \cong \mathbb{R}$. Thus, the positive Weyl chamber for $\mathrm{SU}(2)$ is the set of all nonnegative multipliers of $\mathbf{H}$, i.e.,

$$\mathfrak{h}_{\geq 0} = \{z\mathbf{H} : z \geq 0\} \cong \mathbb{R}_{\geq 0}.$$

The Lie group $\tilde{K} = \mathrm{SU}(2) \times \mathrm{SU}(2)$ has rank two. Its Lie algebra is $\tilde{\mathfrak{t}} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, and the Cartan subalgebra $\tilde{\mathfrak{t}}$ is spanned by the generators $\mathbf{H}_1$ from the first $\mathfrak{su}(2)$ factor and $\mathbf{H}_2$ from the second $\mathfrak{su}(2)$ factor. Using the standard representation, the generators are

$$\mathbf{H}_1 = \mathbf{H} \otimes |0\rangle\langle 0| \text{ and } \mathbf{H}_2 = \mathbf{H} \otimes |1\rangle\langle 1|.$$

The root system of $\tilde{K} = \mathrm{SU}(2) \times \mathrm{SU}(2)$ is the disjoint union of the root systems of each $\mathrm{SU}(2)$ factor. The positive roots are α_1 for the first factor and α_2 for the second factor, defined

$$\alpha_1(\mathbf{H}_1) = 2, \alpha_1(\mathbf{H}_2) = 0, \alpha_2(\mathbf{H}_1) = 0, \alpha_2(\mathbf{H}_2) = 2.$$

The positive Weyl chamber is the subset of the Cartan subalgebra where all positive roots take non-negative values. For an element $h = z_1\mathbf{H}_1 + z_2\mathbf{H}_2 \in \mathfrak{it}$, the conditions are

$$\begin{aligned} \alpha_1(h) &= \alpha_1(z_1\mathbf{H}_1 + z_2\mathbf{H}_2) = 2z_1 + 0z_2 = 2z_1 \geq 0, \\ \alpha_2(h) &= \alpha_2(z_1\mathbf{H}_1 + z_2\mathbf{H}_2) = 0z_1 + 2z_2 = 2z_2 \geq 0. \end{aligned}$$

Thus, the positive Weyl chamber is identified with

$$\mathfrak{it}_{\geq 0} = \{z_1\mathbf{H}_1 + z_2\mathbf{H}_2 : z_1 \geq 0, z_2 \geq 0\} \cong \mathbb{R}_{\geq 0}^2.$$

We are done. $\square$

With $i\tilde{t}_{\geq 0}^* \cong \mathbb{R}_{\geq 0}^2$, we have already seen that

$$\text{DH}_{\mathcal{O}_{\hat{\lambda}}}^{\tilde{K}} \Big|_{i\tilde{t}_{\geq 0}^*} = \sum_{w \in S_4} \text{sign}(w) \delta_{\pi(w\hat{\lambda})} \star H_{(-2,0)} \star H_{(-2,-2)} \star H_{(0,-2)} \star H_{(-2,2)} \Big|_{i\tilde{t}_{\geq 0}^*} \quad (46)$$

is supported on $i\tilde{t}_{\geq 0}^*$, which amounts to be also supported on $\mathbb{R}_{\geq 0}^2$. We can write it in the following form:

Proposition 10. *For $\tilde{K} = \text{SU}(2) \times \text{SU}(2)$, it holds that*

$$\text{DH}_{\mathcal{O}_{\hat{\lambda}}}^{\tilde{K}} \Big|_{\mathbb{R}_{\geq 0}^2} = \sum_{w \in S_4} \text{sign}(w) \delta_{\pi(w\hat{\lambda})} \star H_{(-2,0)} \star H_{(-2,-2)} \star H_{(0,-2)} \star H_{(-2,2)} \Big|_{\mathbb{R}_{\geq 0}^2} \quad (47)$$

with its density function being given by

$$\begin{aligned} & \sum_{w \in S_4} \text{sign}(w) \delta_{\pi(w\hat{\lambda})} \star p(x, y) \Big|_{\mathbb{R}_{\geq 0}^2} \\ &= \left(\delta_{(c_2, \pm c_1)} - \delta_{(c_1, c_2)} + \delta_{(c_1, c_3)} - \delta_{(c_3, \pm c_1)} + \delta_{(c_3, \pm c_2)} - \delta_{(c_2, c_3)} \right) \star p(x, y) \Big|_{\mathbb{R}_{\geq 0}^2}, \end{aligned} \quad (48)$$

where the expression of $p(x, y)$ is from Proposition 8.

Proof. Sketch of proof can be given for Eq. (48): Indeed, we have already known the support of $p(x, y)$ is $\cup_{k=1}^3 C_k$. Then the support of $(\delta_{\pi(w\hat{\lambda})} \star p)(x, y)$ is the shifted version of $\cup_{k=1}^3 C_k$ to the point $\pi(w\hat{\lambda})$. However, points whose corresponding shifted supports do not intersect with $\mathbb{R}_{\geq 0}^2$ do not contribute to the density. In other words, by calculation, we find that it remains only 9 points can contribute to the density. $\square$

Remark 4. *Moreover, the non-Abelian moment polytope for the action of $\tilde{K} = \text{SU}(2) \times \text{SU}(2)$ on a generic coadjoint $K = \text{SU}(4)$ -orbit can be identified as*

$$\Delta_{\tilde{K}}(\mathcal{O}_{\hat{\lambda}}) := \left\{ (x, y) \in [0, c_3]^2 : x + y \leq c_2 + c_3, |x - y| \leq c_3 - c_1 \right\}, \quad (49)$$

where c_k 's are from Eq. (44). This is just the support of the density in Eq. (48). Based on the constraint Eq. (49), we can re-derive the compatibility solution for two-qubit quantum marginal problem. Let me describe it below: Let $\rho_{AB} \in \text{D}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ with two reduced states (ρ_A, ρ_B) , where the eigenvalues of ρ_{AB} are ordered non-increasingly as $\lambda_1 \geq \dots \geq \lambda_4 \geq 0$, and the minimal eigenvalue of $\rho_X (X = A, B)$ is denoted by $\lambda_{\min}(\rho_X)$. Via $\hat{\lambda}_j = \lambda_j - \frac{1}{4}$ for $j \in \{1, 2, 3, 4\}$ and $x = 1 - 2\lambda_{\min}(\rho_1)$ and $y = 1 - 2\lambda_{\min}(\rho_2)$, we see that the condition $x \in [0, c_3]$ can be rewritten as

$$\min\{\lambda_{\min}(\rho_A), \lambda_{\min}(\rho_B)\} \geq \lambda_3(\rho_{AB}) + \lambda_4(\rho_{AB}); \quad (50)$$

and the condition $x + y \leq c_2 + c_3$ can be rewritten as

$$\lambda_{\min}(\rho_A) + \lambda_{\min}(\rho_B) \geq \lambda_2(\rho_{AB}) + \lambda_3(\rho_{AB}) + 2\lambda_4(\rho_{AB}); \quad (51)$$

and the condition $|x - y| \leq c_3 - c_1$ can be rewritten as

$$|\lambda_{\min}(\rho_A) - \lambda_{\min}(\rho_B)| \geq \min\{\lambda_1(\rho_{AB}) - \lambda_3(\rho_{AB}), \lambda_2(\rho_{AB}) - \lambda_4(\rho_{AB})\}. \quad (52)$$

In summary, for any given 2-tuple (ρ_A, ρ_B) of qubit states, there exists a global two-qubit states ρ_{AB} such that $\rho_A = \text{Tr}_A(\rho_{AB})$ and $\rho_B = \text{Tr}_B(\rho_{AB})$ if and only if the three compatibility conditions Eqs. (50)–(52) hold. This result is obtained already by Bravyi [26].

Consider the pushforward measure $F_*(\mu_{\mathcal{O}_\lambda})$ of Liouville measure $\mu_{\mathcal{O}_\lambda}$ along the map $F : \mathcal{O}_\lambda \rightarrow \mathfrak{ih}_{\geq 0}^* \cong \mathbb{R}_{\geq 0}$, where $\mathfrak{h}_{\geq 0}^*$ is the dual Cartan subalgebra of $\mathfrak{su}(2)$ and $F = q_1 \circ \pi \circ \Phi$,

$$F : \mathcal{O}_\lambda(\subset \mathfrak{k}^*) \xrightarrow{\Phi} \mathfrak{k}^* \xrightarrow{\pi} \mathfrak{ih}_{\geq 0}^* = \mathfrak{ih}_{\geq 0}^* \oplus \mathfrak{ih}_{\geq 0}^* \xrightarrow{q_1} \mathfrak{ih}_{\geq 0}^*. \quad (53)$$

Proposition 11. *The pushforward measure $F_*\mu_{\mathcal{O}_\lambda}$ is just*

$$F_*(\mu_{\mathcal{O}_\lambda}) = (q_1)_*\pi_*\Phi_*(\mu_{\mathcal{O}_\lambda}) = (q_1)_*\left(p_{\tilde{K}}\mathrm{DH}_{\mathcal{O}_\lambda}^{\tilde{K}}\right) \quad (54)$$

whose density function is given by $(2\pi)^6 \mathcal{I}(x|\hat{\lambda})$, where

$$\mathcal{I}(x|\hat{\lambda}) := \int_{\mathbb{R}_{\geq 0}^2} xy \left[\left(\delta_{(c_2, \pm c_1)} - \delta_{(c_1, c_2)} + \delta_{(c_1, c_3)} - \delta_{(c_3, \pm c_1)} + \delta_{(c_3, \pm c_2)} - \delta_{(c_2, c_3)} \right) \star p \right](x, y) dx dy,$$

where the expression of $p(x, y)$ is from Proposition 8. Moreover, it holds that

$$\mathcal{I}(x|\hat{\lambda}) = \mathcal{I}_1(x|\hat{\lambda}) + \mathcal{I}_2(x|\hat{\lambda}) + \mathcal{I}_3(x|\hat{\lambda}), \quad (55)$$

where

$$\mathcal{I}_1(x|\hat{\lambda}) = \frac{c_3^2 - c_2^2}{64} x(x - c_1)^2 \chi_{[0, c_1]}(x), \quad (56)$$

$$\mathcal{I}_2(x|\hat{\lambda}) = \frac{c_1^2 - c_3^2}{64} x(x - c_2)^2 \chi_{[0, c_2]}(x), \quad (57)$$

$$\mathcal{I}_3(x|\hat{\lambda}) = \frac{c_2^2 - c_1^2}{64} x(x - c_3)^2 \chi_{[0, c_3]}(x), \quad (58)$$

where c_k 's are from Eq. (44), and $\chi_A(x)$ is the indicator of a set A .

Proof. Note that $p_{\tilde{K}}(\frac{x}{2}\mathbf{h}, \frac{y}{2}\mathbf{h}) = p_{\mathrm{SU}(2)}(\frac{x}{2}\mathbf{h})p_{\mathrm{SU}(2)}(\frac{y}{2}\mathbf{h}) = xy$. We consider the following integral

$$\begin{aligned} \mathcal{I}(x|\hat{\lambda}) &:= \int_{\mathbb{R}_{\geq 0}^2} xy \left[\left(\delta_{(c_2, \pm c_1)} - \delta_{(c_1, c_2)} + \delta_{(c_1, c_3)} - \delta_{(c_3, \pm c_1)} + \delta_{(c_3, \pm c_2)} - \delta_{(c_2, c_3)} \right) \star p \right](x, y) dx dy \\ &= \int_{\mathbb{R}_{\geq 0}^2} xy \left[\left(\delta_{(c_1, c_3)} - \delta_{(c_1, c_2)} \right) \star p \right](x, y) dx dy \\ &\quad + \int_{\mathbb{R}_{\geq 0}^2} xy \left[\left(-\delta_{(c_2, c_3)} + \delta_{(c_2, c_1)} + \delta_{(c_2, -c_1)} \right) \star p \right](x, y) dx dy \\ &\quad + \int_{\mathbb{R}_{\geq 0}^2} xy \left[\left(\delta_{(c_3, c_2)} + \delta_{(c_3, -c_2)} - \delta_{(c_3, c_1)} - \delta_{(c_3, -c_1)} \right) \star p \right](x, y) dx dy \end{aligned}$$

which is the density of the Duistermaat-Heckman measure with respect to the Lebesgue measure on the positive Weyl chamber (which is identified with $\mathbb{R}_{\geq 0}^2$) of $\mathrm{SU}(2) \times \mathrm{SU}(2)$. Denote by

$$\mathcal{I}_1(x|\hat{\lambda}) = \int_{\mathbb{R}_{\geq 0}^2} xy \left[\left(\delta_{(c_1, c_3)} - \delta_{(c_1, c_2)} \right) \star p \right](x, y) dx dy, \quad (59)$$

$$\mathcal{I}_2(x|\hat{\lambda}) = \int_{\mathbb{R}_{\geq 0}^2} xy \left[\left(-\delta_{(c_2, c_3)} + \delta_{(c_2, c_1)} + \delta_{(c_2, -c_1)} \right) \star p \right](x, y) dx dy, \quad (60)$$

$$\mathcal{I}_3(x|\hat{\lambda}) = \int_{\mathbb{R}_{\geq 0}^2} xy \left[\left(\delta_{(c_3, c_2)} + \delta_{(c_3, -c_2)} - \delta_{(c_3, c_1)} - \delta_{(c_3, -c_1)} \right) \star p \right](x, y) dx dy. \quad (61)$$

Note that $c_1 = 2|\hat{\lambda}_1 + \hat{\lambda}_4| = 2|\hat{\lambda}_2 + \hat{\lambda}_3|$ because $\hat{\lambda}_1 + \hat{\lambda}_4 = -(\hat{\lambda}_2 + \hat{\lambda}_3)$. Thus

$$c_1 = \max\{\hat{\lambda}_1 + \hat{\lambda}_4, \hat{\lambda}_2 + \hat{\lambda}_3\}.$$

- (a) The range $[0, c_3]$ of parameter x can be decomposed into two parts $[0, c_3] = [0, c_1] \cup [c_1, c_3]$. In order to calculate $\mathcal{I}_1$, note that $\mathcal{I}_1 \equiv 0$ if $x \in [c_1, c_3]$, it suffices to calculate it for $x \in [0, c_1]$. Thus

$$\begin{aligned}\mathcal{I}_1(x|\hat{\lambda}) &= \int_0^{x+c_3-c_1} xy \cdot p_3(x-c_1, y-c_3) dy + \int_{x+c_3-c_1}^{c_3} xy \cdot p_2(x-c_1, y-c_3) dy \\ &\quad + \int_{c_3}^{-x+c_3+c_1} xy \cdot p_1(x-c_1, y-c_3) dy - \int_0^{x+c_2-c_1} xy \cdot p_3(x-c_1, y-c_2) dy \\ &\quad - \int_{x+c_2-c_1}^{c_2} xy \cdot p_2(x-c_1, y-c_2) dy - \int_{c_2}^{-x+c_1+c_2} xy \cdot p_1(x-c_1, y-c_2) dy \\ &= \frac{c_3^2 - c_2^2}{64} x(x-c_1)^2,\end{aligned}$$

which leads to

$$\mathcal{I}_1(x|\hat{\lambda}) = \begin{cases} \frac{c_3^2 - c_2^2}{64} x(x-c_1)^2, & \text{if } x \in [0, c_1], \\ 0, & \text{if } x \in [c_1, c_3]. \end{cases} \quad (62)$$

With the help of the following Figure 2, the above calculation is easily obtained.

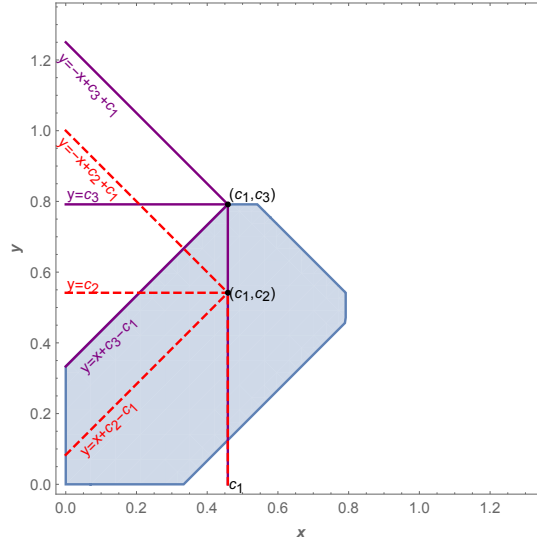


Figure 2. The calculation of $\mathcal{I}_1$ over the non-Abelian moment polytope.

- (b) The range $[0, c_3]$ of parameter x can be decomposed into three parts: $[0, c_3] = [0, c_2 - c_1] \cup [c_2 - c_1, c_2] \cup [c_2, c_3]$. Note that $\mathcal{I}_2 \equiv 0$ when $x \in [c_2, c_3]$. It suffices to calculate $\mathcal{I}_2$ on $[0, c_2 - c_1] \cup [c_2 - c_1, c_2]$. See the following Figure 3.

- (b1) For $x \in [0, c_2 - c_1]$, we see that

$$\begin{aligned}\mathcal{I}_2(x|\hat{\lambda}) &= - \int_0^{x+c_3-c_2} xyp_3(x-c_2, y-c_3) dy - \int_{x+c_3-c_2}^{c_3} xyp_2(x-c_2, y-c_3) dy \\ &\quad - \int_{c_3}^{-x+c_3+c_2} xyp_1(x-c_2, y-c_3) dy + \int_0^{c_1} xyp_2(x-c_2, y-c_1) dy \\ &\quad + \int_{c_1}^{-x+c_2+c_1} xyp_3(x-c_2, y-c_1) dy + \int_0^{-x+c_2-c_1} xyp_3(x-c_2, y+c_1) dy \\ &= \frac{c_1^2 - c_3^2}{64} x(x-c_2)^2.\end{aligned}$$

(b2) For $x \in [c_2 - c_1, c_2]$,

$$\begin{aligned}
\mathcal{I}_2(x|\hat{\lambda}) &= - \int_0^{x+c_3-c_2} xyp_3(x-c_2, y-c_3)dy - \int_{x+c_3-c_2}^{c_3} xyp_2(x-c_2, y-c_3)dy \\
&\quad - \int_{c_3}^{-x+c_3+c_2} xyp_1(x-c_2, y-c_3)dy + \int_0^{x+c_1-c_2} xyp_3(x-c_2, y-c_1)dy \\
&\quad + \int_{x+c_1-c_2}^{c_1} xyp_2(x-c_2, y-c_1)dy + \int_{c_1}^{-x+c_2+c_1} xyp_1(x-c_2, y-c_1)dy \\
&= \frac{c_1^2 - c_3^2}{64} x(x-c_2)^2.
\end{aligned}$$

This leads to

$$\mathcal{I}_2(x|\hat{\lambda}) = \begin{cases} \frac{c_1^2 - c_3^2}{64} x(x-c_2)^2, & \text{if } x \in [0, c_2], \\ 0, & \text{if } x \in [c_2, c_3]. \end{cases} \quad (63)$$

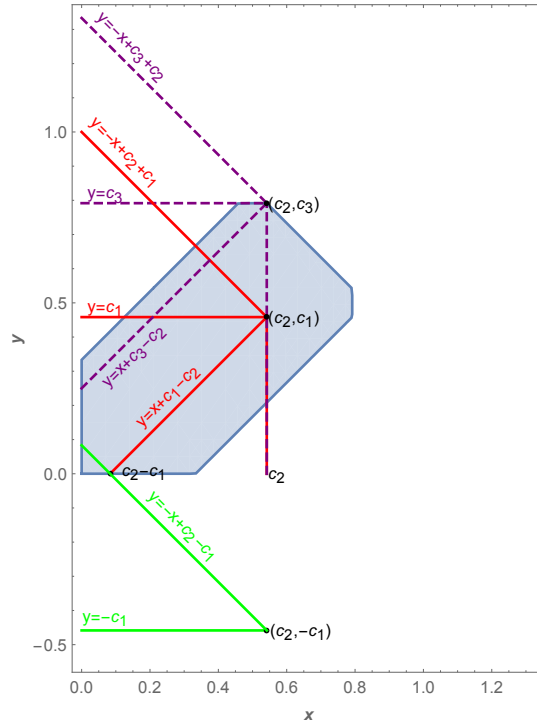


Figure 3. The calculation of $\mathcal{I}_2$ over the non-Abelian moment polytope.

- (c) The range $[0, c_3]$ of parameter x can be decomposed into three parts: $[0, c_3] = [0, c_3 - c_2] \cup [c_3 - c_2, c_3 - c_1] \cup [c_3 - c_1, c_3]$. It suffices to calculate $\mathcal{I}_2$ on $[0, c_2 - c_1] \cup [c_2 - c_1, c_2]$. See the following Figure 4.
- (c1) For $x \in [0, c_3 - c_2]$, we see that

$$\begin{aligned}
\mathcal{I}_3(x|\hat{\lambda}) &= \int_0^{c_2} xyp_2(x-c_3, y-c_2)dx dy + \int_{c_2}^{-x+c_3+c_2} xyp_1(x-c_3, y-c_2)dx dy \\
&\quad + \int_0^{-x+c_3-c_2} xyp_1(x-c_3, y+c_2)dx dy - \int_0^{c_1} xyp_2(x-c_3, y-c_1)dx dy \\
&\quad - \int_{c_1}^{-x+c_3+c_1} xyp_1(x-c_3, y-c_1)dx dy - \int_0^{-x+c_3-c_1} xyp_1(x-c_3, y+c_1)dx dy \\
&= \frac{c_2^2 - c_1^2}{64} x(x-c_3)^2.
\end{aligned}$$

(c2) For $x \in [c_3 - c_2, c_3 - c_1]$,

$$\begin{aligned}
\mathcal{I}_3(x|\hat{\lambda}) &= \int_0^{x+c_2-c_3} xyp_3(x-c_3, y-c_2)dx dy + \int_{x+c_2-c_3}^{c_2} xyp_2(x-c_3, y-c_2)dx dy \\
&+ \int_{c_2}^{-x+c_3+c_2} xyp_1(x-c_3, y-c_2)dx dy - \int_0^{c_1} xyp_2(x-c_3, y-c_1)dx dy \\
&- \int_{c_1}^{-x+c_3+c_1} xyp_1(x-c_3, y-c_1)dx dy - \int_0^{-x+c_3-c_1} xyp_1(x-c_3, y+c_1)dx dy \\
&= \frac{c_2^2 - c_1^2}{64} x(x-c_3)^2.
\end{aligned}$$

(c3) For $x \in [c_3 - c_1, c_3]$,

$$\begin{aligned}
\mathcal{I}_3(x|\hat{\lambda}) &= \int_0^{x+c_2-c_3} xyp_3(x-c_3, y-c_2)dx dy + \int_{x+c_2-c_3}^{c_2} xyp_2(x-c_3, y-c_2)dx dy \\
&+ \int_{c_2}^{-x+c_2+c_3} xyp_1(x-c_3, y-c_2)dx dy - \int_0^{x+c_1-c_3} xyp_3(x-c_3, y-c_1)dx dy \\
&- \int_{x+c_1-c_3}^{c_1} xyp_2(x-c_3, y-c_1)dx dy - \int_{c_1}^{-x+c_3+c_1} xyp_1(x-c_3, y-c_1)dx dy \\
&= \frac{c_2^2 - c_1^2}{64} x(x-c_3)^2.
\end{aligned}$$

These leads to

$$\mathcal{I}_3(x|\hat{\lambda}) = \frac{c_2^2 - c_1^2}{64} x(x-c_3)^2, \quad (x \in [0, c_3]). \quad (64)$$

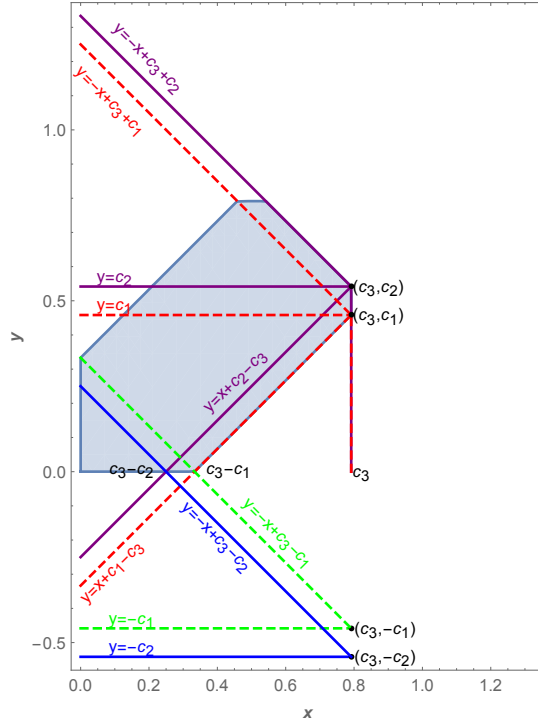


Figure 4. The calculation of $\mathcal{I}_3$ over the non-Abelian moment polytope.

This completes the proof. $\square$

6.1. Separability Probability in the Conditioned State Space

For a given qudit state $\eta \in \mathcal{D}(\mathbb{C}^n)$, Milz and Strunz considered the conditioned state space [27]

$$\mathcal{D}^\eta(\mathbb{C}^n \otimes \mathbb{C}^m) := \{\rho \in \mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m) : \text{Tr}_{\mathbb{C}^m}(\rho) = \eta \in \mathcal{D}(\mathbb{C}^n)\} \quad (65)$$

and the set of separable states in $D^\eta(\mathbb{C}^n \otimes \mathbb{C}^m)$, denoted by $D_{\text{sep}}^\eta(\mathbb{C}^n \otimes \mathbb{C}^m)$. They studied the separability probability in the conditioned state space

$$P_{\text{sep}}^{(n \times m)}(\eta) := \frac{\text{vol}_{\text{HS}}(D_{\text{sep}}^\eta(\mathbb{C}^n \otimes \mathbb{C}^m))}{\text{vol}_{\text{HS}}(D^\eta(\mathbb{C}^n \otimes \mathbb{C}^m))}. \quad (66)$$

All further considerations will be simplified by the observation that both the Hilbert-Schmidt measure in the set $D^\eta(\mathbb{C}^n \otimes \mathbb{C}^m)$ and the separability of a state $\rho \in D^\eta(\mathbb{C}^n \otimes \mathbb{C}^m)$ are invariant under a transformation $\mathbf{U} \otimes \mathbb{1}_m$, where $\mathbf{U} \in \text{SU}(n)$. Indeed,

$$D^{\mathbf{U}\eta\mathbf{U}^\dagger}(\mathbb{C}^n \otimes \mathbb{C}^m) = (\mathbf{U} \otimes \mathbb{1}_m) D^\eta(\mathbb{C}^n \otimes \mathbb{C}^m) (\mathbf{U} \otimes \mathbb{1}_m)^\dagger. \quad (67)$$

For $n = 2$, the qubit state $\eta \in D(\mathbb{C}^2)$ can be represented by Bloch vector $\eta = \frac{1}{2}(\mathbb{1}_2 + \mathbf{a} \cdot \boldsymbol{\sigma})$. We denote by $\eta \approx \mathbf{a}$ simply. Thus we make the following identifications:

$$\begin{aligned} D^{\mathbf{a}}(\mathbb{C}^2 \otimes \mathbb{C}^m) &= D^\eta(\mathbb{C}^2 \otimes \mathbb{C}^m), \\ D_{\text{sep}}^{\mathbf{a}}(\mathbb{C}^2 \otimes \mathbb{C}^m) &= D_{\text{sep}}^\eta(\mathbb{C}^2 \otimes \mathbb{C}^m), \\ P_{\text{sep}}^{(2 \times m)}(\mathbf{a}) &= P_{\text{sep}}^{(2 \times m)}(\eta). \end{aligned}$$

Thus Eq. (66), can be represented as

$$P_{\text{sep}}^{(2 \times m)}(\mathbf{a}) := \frac{\text{vol}_{\text{HS}}(D_{\text{sep}}^{\mathbf{a}}(\mathbb{C}^2 \otimes \mathbb{C}^m))}{\text{vol}_{\text{HS}}(D^{\mathbf{a}}(\mathbb{C}^2 \otimes \mathbb{C}^m))}. \quad (68)$$

We have already known the fact that $\text{SU}(2)$ is the double cover of $\text{SO}(3)$ from Lie Theory, which leads to the following $\mathbf{U}\eta\mathbf{U}^\dagger = \frac{1}{2}(\mathbb{1}_2 + (\mathbf{O}\mathbf{a}) \cdot \boldsymbol{\sigma})$, where both $\mathbf{U} \in \text{SU}(2)$ and $\mathbf{O} = (o_{ij}) \in \text{SO}(3)$ are connected via $o_{ij} = \frac{1}{2} \text{Tr}(\sigma_i \mathbf{U} \sigma_j \mathbf{U}^\dagger)$.

Proposition 12. *It holds that*

$$P_{\text{sep}}^{(2 \times m)}(\mathbf{O}\mathbf{a}) = P_{\text{sep}}^{(2 \times m)}(\mathbf{a}) \quad (69)$$

for any $\mathbf{O} \in \text{SO}(3)$ and $\mathbf{a} \in \mathbb{R}^3$ with $|\mathbf{a}| \leq 1$.

Proof. In fact, it is easily seen that

$$\begin{aligned} \text{vol}_{\text{HS}}((\mathbf{U} \otimes \mathbb{1}_m) D^\eta(\mathbb{C}^n \otimes \mathbb{C}^m) (\mathbf{U} \otimes \mathbb{1}_m)^\dagger) &= \text{vol}_{\text{HS}}(D^\eta(\mathbb{C}^n \otimes \mathbb{C}^m)) \\ \text{vol}_{\text{HS}}((\mathbf{U} \otimes \mathbb{1}_m) D_{\text{sep}}^\eta(\mathbb{C}^n \otimes \mathbb{C}^m) (\mathbf{U} \otimes \mathbb{1}_m)^\dagger) &= \text{vol}_{\text{HS}}(D_{\text{sep}}^\eta(\mathbb{C}^n \otimes \mathbb{C}^m)) \end{aligned}$$

holds for all $\mathbf{U} \in \text{SU}(2)$, which means that $P_{\text{sep}}^{(2 \times m)}(\mathbf{U}\eta\mathbf{U}^\dagger) = P_{\text{sep}}^{(2 \times m)}(\eta)$. The desired result is obtained by translating it into the Bloch vector form. $\square$

Let $a = |\mathbf{a}| \in [0, 1]$. From the above Proposition 12, we see that $P_{\text{sep}}^{(2 \times m)}(\mathbf{a})$ is constant on the sphere $S_a := \{\mathbf{a} \in \mathbb{R}^3 : |\mathbf{a}| = a\}$. In view of this reason, we let $\mathbf{a} = (0, 0, a)$ and

$$P_{\text{sep}}^{(2 \times m)}(a) := \frac{\text{vol}_{\text{HS}}(D_{\text{sep}}^{\mathbf{a}}(\mathbb{C}^2 \otimes \mathbb{C}^m))}{\text{vol}_{\text{HS}}(D^{\mathbf{a}}(\mathbb{C}^2 \otimes \mathbb{C}^m))}. \quad (70)$$

where

$$D^{\mathbf{a}}(\mathbb{C}^2 \otimes \mathbb{C}^m) := \left\{ \rho \in D(\mathbb{C}^2 \otimes \mathbb{C}^m) : \text{Tr}_{\mathbb{C}^m}(\rho) = \frac{1}{2}(\mathbb{1}_2 + a\sigma_3) \right\}. \quad (71)$$

Denote the unit ball by $B_1 := \{\mathbf{a} \in \mathbb{R}^3 : |\mathbf{a}| \leq 1\}$. Then $B_1 = \cup_{a \in [0,1]} S_a$, which leads to the following identifications:

$$D(\mathbb{C}^2 \otimes \mathbb{C}^2) = \bigcup_{\eta \in D(\mathbb{C}^2)} D^\eta(\mathbb{C}^2 \otimes \mathbb{C}^2) = \bigcup_{a \in B_1} D^a(\mathbb{C}^2 \otimes \mathbb{C}^2) \quad (72)$$

$$= \bigcup_{a \in [0,1]} \bigcup_{\mathbf{a} \in S_a} D^a(\mathbb{C}^2 \otimes \mathbb{C}^2). \quad (73)$$

Thus we get that

Proposition 13. *It holds that*

$$\text{vol}_{\text{HS}}(D(\mathbb{C}^2 \otimes \mathbb{C}^2)) = \frac{1}{8} \int_{B_1} \text{vol}_{\text{HS}}(D^a(\mathbb{C}^2 \otimes \mathbb{C}^2)) [d\mathbf{a}] \quad (74)$$

$$= \frac{\pi}{2} \int_0^1 a^2 \text{vol}_{\text{HS}}(D^a(\mathbb{C}^2 \otimes \mathbb{C}^2)) da. \quad (75)$$

Proof. Recall that $d_{\text{HS}}(\rho', \rho) = \frac{1}{2} d_{\text{Euclid}}(\mathbf{x}', \mathbf{x})$, where $\mathbf{x}' = (a', b', c'_{kl})$ and $\mathbf{x} = (a, b, c_{kl})$. Thus three parameters in $\mathbf{a}$ determine Euclid volume $[d\mathbf{a}]$, and thus contribute a scaling factor $\left(\frac{1}{2}\right)^3$ multiple of Euclid volume $[d\mathbf{a}]$, i.e., in the HS volume element for $D(\mathbb{C}^2 \otimes \mathbb{C}^2)$. Therefore we get that

$$\begin{aligned} \text{vol}_{\text{HS}}(D(\mathbb{C}^2 \otimes \mathbb{C}^2)) &= \int_{B_1} \text{vol}_{\text{HS}}(D^a(\mathbb{C}^2 \otimes \mathbb{C}^2)) (2^{-3} [d\mathbf{a}]) \\ &= \frac{1}{8} \int_0^1 da \int_{S_a} \text{vol}_{\text{HS}}(D^a(\mathbb{C}^2 \otimes \mathbb{C}^2)) [d\mathbf{a}] \\ &= \frac{1}{8} \int_0^1 da \int \delta(a - |\mathbf{a}|) \text{vol}_{\text{HS}}(D^a(\mathbb{C}^2 \otimes \mathbb{C}^2)) [d\mathbf{a}] \\ &= \frac{1}{8} \int_0^1 da \text{vol}_{\text{HS}}(D^a(\mathbb{C}^2 \otimes \mathbb{C}^2)) \int \delta(a - |\mathbf{a}|) [d\mathbf{a}] \\ &= \frac{1}{8} \int_0^1 (4\pi a^2) \text{vol}_{\text{HS}}(D^a(\mathbb{C}^2 \otimes \mathbb{C}^2)) da \\ &= \frac{\pi}{2} \int_0^1 a^2 \text{vol}_{\text{HS}}(D^a(\mathbb{C}^2 \otimes \mathbb{C}^2)) da. \end{aligned}$$

We have done it. $\square$

Next, we calculate the HS volume $\text{vol}_{\text{HS}}(D^a(\mathbb{C}^2 \otimes \mathbb{C}^2))$. Indeed,

Proposition 14 ([8]). *It holds that*

$$\text{vol}_{\text{HS}}(D^a(\mathbb{C}^2 \otimes \mathbb{C}^2)) = \text{vol}_{\text{HS}}(D^0(\mathbb{C}^2 \otimes \mathbb{C}^2)) (1 - a^2)^6, \quad (76)$$

where $a \in [0, 1)$. Moreover,

$$\text{vol}_{\text{HS}}(D^0(\mathbb{C}^2 \otimes \mathbb{C}^2)) = \frac{\pi^5}{9676800}. \quad (77)$$

Proof. This expression in Eq. (76) is conjectured by Milz and Strunz [27], and proved by Lovas and Andai [8]. From the above formula, we get that

$$\text{vol}_{\text{HS}}\left(D(\mathbb{C}^2 \otimes \mathbb{C}^2)\right) = \text{vol}_{\text{HS}}\left(D^0(\mathbb{C}^2 \otimes \mathbb{C}^2)\right) \times \frac{\pi}{2} \int_0^1 a^2(1-a^2)^6 da \quad (78)$$

$$= \frac{2^9 \pi}{45045} \text{vol}_{\text{HS}}\left(D^0(\mathbb{C}^2 \otimes \mathbb{C}^2)\right). \quad (79)$$

Note that, from Theorem 2, we can see that

$$\text{vol}_{\text{HS}}\left(D(\mathbb{C}^2 \otimes \mathbb{C}^2)\right) = \text{vol}_{\text{HS}}\left(D(\mathbb{C}^4)\right) = \sqrt{4}(2\pi)^{\binom{4}{2}} \frac{\prod_{k=1}^4 \Gamma(k)}{\Gamma(4^2)} \quad (80)$$

$$= \frac{2(2\pi)^6(2!)(3!)}{15!}. \quad (81)$$

By combining the above formulas, we can derive that $\text{vol}_{\text{HS}}(D^0(\mathbb{C}^2 \otimes \mathbb{C}^2)) = \frac{\pi^5}{9676800}$. $\square$

The following result was conjectured in [27], and proven in [8], which is described as a theorem without proof:

Theorem 6 ([8]). *It holds that $P_{\text{sep}}^{(2 \times 2)}(a)$ is constant for all $a \in [0, 1)$.*

Proof. Please see the proof in [8, Corollary 2]. $\square$

6.2. Separability Probability of Two-Qubit States

The separability probability is given by

$$P_{\text{sep}}^{(n \times m)} := \frac{\text{vol}_{\text{HS}}(D_{\text{sep}}(\mathbb{C}^n \otimes \mathbb{C}^m))}{\text{vol}_{\text{HS}}(D(\mathbb{C}^n \otimes \mathbb{C}^m))}. \quad (82)$$

Here $D_{\text{sep}}(\mathbb{C}^n \otimes \mathbb{C}^m)$ is the set of separable states in $D(\mathbb{C}^n \otimes \mathbb{C}^m)$. In what follows, we focus on the case where $n = m = 2$.

For $\rho \in D(\mathbb{C}^2 \otimes \mathbb{C}^2)$, we can view ρ as a 4×4 matrix acting on $\mathbb{C}^4$ and then its partial transpose w.r.t. 1st subsystem, are respectively:

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} \text{ and } \rho^\Gamma = \begin{pmatrix} \rho_{11} & \rho_{21} & \rho_{13} & \rho_{23} \\ \rho_{12} & \rho_{22} & \rho_{14} & \rho_{24} \\ \rho_{31} & \rho_{41} & \rho_{33} & \rho_{43} \\ \rho_{32} & \rho_{42} & \rho_{34} & \rho_{44} \end{pmatrix}. \quad (83)$$

Let $E_n(i, j)$ be the elementary matrix, obtained by changing the i -th and j -th rows/columns of identity matrix $\mathbb{1}_n$. Now

$$E_4(2, 3)\rho E_4(2, 3) = \begin{pmatrix} \rho_{11} & \rho_{13} & \rho_{12} & \rho_{14} \\ \rho_{31} & \rho_{33} & \rho_{32} & \rho_{34} \\ \rho_{21} & \rho_{23} & \rho_{22} & \rho_{24} \\ \rho_{41} & \rho_{43} & \rho_{42} & \rho_{44} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (84)$$

and

$$E_4(2, 3)\rho^\Gamma E_4(2, 3) = \begin{pmatrix} \rho_{11} & \rho_{13} & \rho_{21} & \rho_{23} \\ \rho_{31} & \rho_{33} & \rho_{41} & \rho_{43} \\ \rho_{12} & \rho_{14} & \rho_{22} & \rho_{24} \\ \rho_{32} & \rho_{34} & \rho_{42} & \rho_{44} \end{pmatrix} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}. \quad (85)$$

Denote

$$D_{\text{ss}}(\mathbb{C}^2 \otimes \mathbb{C}^2) := \left\{ D(\mathbb{C}^2 \otimes \mathbb{C}^2) : \lambda_{\max}(\rho) \leq \frac{1}{2} \right\}. \quad (86)$$

Proposition 15 ([7]). *Let $\rho \in D^0(\mathbb{C}^2 \otimes \mathbb{C}^2)$. It holds that ρ is separable if and only if $\rho \in D_{ss}(\mathbb{C}^2 \otimes \mathbb{C}^2)$. In other words,*

$$D_{sep}^0(\mathbb{C}^2 \otimes \mathbb{C}^2) = D^0(\mathbb{C}^2 \otimes \mathbb{C}^2) \cap D_{ss}(\mathbb{C}^2 \otimes \mathbb{C}^2). \quad (87)$$

Proof. Note that the characteristic polynomial of ρ is given by $f_\rho(\lambda) = \det(\rho - \lambda \mathbb{1}_4)$. Then

$$\begin{aligned} f_\rho(\lambda) &= \det(E_4(2, 3)\rho E_4(2, 3) - \lambda \mathbb{1}_4) = \det \begin{pmatrix} A - \lambda \mathbb{1}_2 & B \\ C & D - \lambda \mathbb{1}_2 \end{pmatrix} \\ &= \det(A - \lambda \mathbb{1}_2) \det((D - \lambda \mathbb{1}_2) - C(A - \lambda \mathbb{1}_2)^{-1}B). \end{aligned}$$

Similarly, the characteristic polynomial of ρ^Γ is given by

$$\begin{aligned} f_{\rho^\Gamma}(\lambda) &= \det(E_4(2, 3)\rho^\Gamma E_4(2, 3) - \lambda \mathbb{1}_4) = \det \begin{pmatrix} A - \lambda \mathbb{1}_2 & C \\ B & D - \lambda \mathbb{1}_2 \end{pmatrix} \\ &= \det(A - \lambda \mathbb{1}_2) \det((D - \lambda \mathbb{1}_2) - B(A - \lambda \mathbb{1}_2)^{-1}C) \\ &= \det(D - \lambda \mathbb{1}_2) \det((A - \lambda \mathbb{1}_2) - C(D - \lambda \mathbb{1}_2)^{-1}B). \end{aligned}$$

Due to the fact that $\rho \in D^0(\mathbb{C}^2 \otimes \mathbb{C}^2)$, we see that $\text{Tr}_2(\rho) = \frac{1}{2}\mathbb{1}_2$, which is equivalent to $A + D = \frac{1}{2}\mathbb{1}_2$. Now we have shown that

$$f_{\rho^\Gamma}(\lambda) = \det(D - \lambda \mathbb{1}_2) \det((A - \lambda \mathbb{1}_2) - C(D - \lambda \mathbb{1}_2)^{-1}B). \quad (88)$$

Thus replacing the 2-tuple (λ, D) by $(\frac{1}{2} - \lambda, \frac{1}{2}\mathbb{1}_2 - A)$ in the first factor of Eq. (88), we get that the first factor $\det(D - \lambda \mathbb{1}_2)$ is transformed into

$$\det\left(\left(\frac{1}{2}\mathbb{1}_2 - A\right) - \left(\frac{1}{2} - \lambda\right)\mathbb{1}_2\right) = \det(\lambda \mathbb{1}_2 - A) = \det(A - \lambda \mathbb{1}_2).$$

Thus replacing λ by $\frac{1}{2} - \lambda$ and using $A + D = \frac{1}{2}\mathbb{1}_2$ in the second factor of Eq. (88), we get that the second factor $\det((A - \lambda \mathbb{1}_2) - C(D - \lambda \mathbb{1}_2)^{-1}B)$ is transformed into

$$\det\left(\left(\left(\frac{1}{2}\mathbb{1}_2 - D\right) - \left(\frac{1}{2} - \lambda\right)\mathbb{1}_2\right) - C\left(\left(\frac{1}{2}\mathbb{1}_2 - A\right) - \left(\frac{1}{2} - \lambda\right)\mathbb{1}_2\right)^{-1}B\right).$$

That is, the last expression is reduced to the following:

$$\det\left((-D + \lambda \mathbb{1}_2) - C(-A + \lambda \mathbb{1}_2)^{-1}B\right) = \det\left((D - \lambda \mathbb{1}_2) - C(A - \lambda \mathbb{1}_2)^{-1}B\right).$$

Thus

$$f_{\rho^\Gamma}\left(\frac{1}{2} - \lambda\right) = \det(A - \lambda \mathbb{1}_2) \det\left((D - \lambda \mathbb{1}_2) - C(A - \lambda \mathbb{1}_2)^{-1}B\right) = f_\rho(\lambda).$$

In summary, we get that $f_{\rho^\Gamma}\left(\frac{1}{2} - \lambda\right) = f_\rho(\lambda)$. With these preparations, we can now present the proof below.

($\implies$) Now suppose that $\rho \in D_{sep}^0(\mathbb{C}^2 \otimes \mathbb{C}^2)$. That is, $\rho \in D^0(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is separable. Choose any eigenvalue $t \geq 0$ of the separable state ρ , i.e., $f_\rho(t) = 0$, then $f_{\rho^\Gamma}(\frac{1}{2} - t) = f_\rho(t) = 0$, namely, $\frac{1}{2} - t$ satisfying the eigen-equation $f_{\rho^\Gamma}(\lambda) = 0$, thus $\frac{1}{2} - t \geq 0$ is an eigenvalue of $\rho^\Gamma \geq 0$ by Peres-Horodecki criterion. Therefore $t \leq \frac{1}{2}$, or $\rho \in D_{ss}(\mathbb{C}^2 \otimes \mathbb{C}^2)$.

($\impliedby$) If $\rho \in D^0(\mathbb{C}^2 \otimes \mathbb{C}^2) \cap D_{ss}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, i.e., any generic eigenvalue t of ρ satisfying $0 \leq t \leq \frac{1}{2}$, and moreover $f_{\rho^\Gamma}(\frac{1}{2} - t) = f_\rho(t) = 0$, which means that $\frac{1}{2} - t \geq 0$ is an eigenvalue of ρ^Γ . All eigenvalues of ρ^Γ are nonnegative, that is, $\rho^\Gamma \geq 0$. By Peres-Horodecki criterion, ρ is separable.

This completes the proof. $\square$

Denote

$$f(a) := \text{vol}_{\text{HS}}\left(D^a(\mathbb{C}^2 \otimes \mathbb{C}^2) \cap D_{\text{ss}}(\mathbb{C}^2 \otimes \mathbb{C}^2)\right), \quad (89)$$

where $a \in [0, 1)$. Since the partial trace is a linear map, the function $f(a)$ is continuous. Therefore, we can find $f(0) = \lim_{a \rightarrow 0^+} f(a)$. Clearly $f(0) = \text{vol}_{\text{HS}}\left(D_{\text{sep}}^0(\mathbb{C}^2 \otimes \mathbb{C}^2)\right)$ by Proposition 15. So we firstly calculate $f(a)$ in a small neighborhood of 0. Denote

$$\mathcal{C}_4 = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 > x_2 > x_3 > x_4 \right\}, \quad (90)$$

$$\hat{\mathcal{C}} = \left[-\frac{1}{4}, \frac{1}{4}\right]^4 \cap \mathcal{C}_4. \quad (91)$$

In fact, we have the following result:

Proposition 16 ([7]). *It holds that*

$$f(a) = \frac{\pi^5}{319334400} (1-a)^9 (33a^3 + 162a^2 + 72a + 8), \quad a \in \left(0, \frac{1}{3}\right). \quad (92)$$

Proof. For all $t \in (0, 1)$, denote $B_t := \{a \in \mathbb{R}^3 : |a| \leq t\}$. Denote

$$D^{B_t}(\mathbb{C}^2 \otimes \mathbb{C}^2) := \bigcup_{a \in B_t} D^a(\mathbb{C}^2 \otimes \mathbb{C}^2)$$

Note that

$$\begin{aligned} \text{vol}_{\text{HS}}\left(D^{B_t}(\mathbb{C}^2 \otimes \mathbb{C}^2) \cap D_{\text{ss}}(\mathbb{C}^2 \otimes \mathbb{C}^2)\right) &= \text{vol}_{\text{HS}}\left(\bigcup_{a \in B_t} D^a(\mathbb{C}^2 \otimes \mathbb{C}^2) \cap D_{\text{ss}}(\mathbb{C}^2 \otimes \mathbb{C}^2)\right) \\ &= \int_0^t da \text{vol}_{\text{HS}}\left(\bigcup_{a \in S_a} D^a(\mathbb{C}^2 \otimes \mathbb{C}^2) \cap D_{\text{ss}}(\mathbb{C}^2 \otimes \mathbb{C}^2)\right) \\ &= \int_0^t da \left[\frac{1}{8} \int_{S_a} [da] \text{vol}_{\text{HS}}\left(D^a(\mathbb{C}^2 \otimes \mathbb{C}^2) \cap D_{\text{ss}}(\mathbb{C}^2 \otimes \mathbb{C}^2)\right) \right] \\ &= \int_0^t da \left[\frac{1}{8} \int [da] \delta(a - |a|) \text{vol}_{\text{HS}}\left(D^a(\mathbb{C}^2 \otimes \mathbb{C}^2) \cap D_{\text{ss}}(\mathbb{C}^2 \otimes \mathbb{C}^2)\right) \right] \\ &= \frac{\pi}{8} \int_0^t da \text{vol}_{\text{HS}}\left(D^a(\mathbb{C}^2 \otimes \mathbb{C}^2) \cap D_{\text{ss}}(\mathbb{C}^2 \otimes \mathbb{C}^2)\right) \int \delta(a - |a|) [da]. \end{aligned}$$

Since $f(a) = \text{vol}_{\text{HS}}\left(D^a(\mathbb{C}^2 \otimes \mathbb{C}^2) \cap D_{\text{ss}}(\mathbb{C}^2 \otimes \mathbb{C}^2)\right)$, it follows that

$$\frac{\pi}{2} \int_0^t a^2 f(a) da = \text{vol}_{\text{HS}}\left(D^{B_t}(\mathbb{C}^2 \otimes \mathbb{C}^2) \cap D_{\text{ss}}(\mathbb{C}^2 \otimes \mathbb{C}^2)\right). \quad (93)$$

Now we partition the set $D_{\text{ss}}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ by adjoint $\text{SU}(4)$ -orbits as

$$D_{\text{ss}}(\mathbb{C}^2 \otimes \mathbb{C}^2) = \bigcup_{\lambda \in [0, \frac{1}{2}]^4 \cap \mathcal{C}_4} \mathcal{U}_\lambda, \quad (94)$$

where $\mathcal{U}_\lambda := \{U\Lambda U^\dagger : U \in \text{SU}(4)\}$ for $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_4)$, obtained from $\lambda = (\lambda_1, \dots, \lambda_4)$. It is easily seen that the adjoint orbit $\mathcal{U}_\lambda$ can be identified with the co-adjoint orbit $\mathcal{O}_{\hat{\lambda}}$, where $\hat{\lambda} = \lambda - (\frac{1}{4}, \dots, \frac{1}{4})$. That is,

$$\mathcal{U}_\lambda \cong \mathcal{O}_{\hat{\lambda}},$$

where $\lambda \in [0, \frac{1}{2}]^4 \cap \mathcal{C}_4$ iff $\hat{\lambda} \in [-\frac{1}{4}, \frac{1}{4}]^4 \cap \mathcal{C}_4$. Under the above identification, $\mathbf{X} = \rho - \frac{1}{4}\mathbb{1}_4$, we get that

$$\begin{aligned} \mathbf{D}^{B_t}(\mathbb{C}^2 \otimes \mathbb{C}^2) &= \left\{ \rho \in \mathbf{D}(\mathbb{C}^2 \otimes \mathbb{C}^2) : \text{Tr}_2(\rho) = \frac{1}{2}(\mathbb{1}_2 + \mathbf{a} \cdot \boldsymbol{\sigma}), |\mathbf{a}| \leq t \right\} \\ &\cong \left\{ \mathbf{X} \in \text{Herm}(\mathbb{C}^4) : \text{Tr}_2(\mathbf{X}) = \frac{\mathbf{a}}{2} \cdot \boldsymbol{\sigma}, |\mathbf{a}| \leq t \right\} \\ &= \left\{ \mathbf{X} \in \text{Herm}(\mathbb{C}^4) : \text{Tr}(\mathbf{X}) = 0, |\lambda_{\pm}(\text{Tr}_2(\mathbf{X}))| \leq \frac{t}{2} \right\} := \mathcal{E}_t. \end{aligned}$$

That is, the set $\mathbf{D}^{B_t}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is translated into the set $\mathcal{E}_t$. Thus the HS volume is invariant under the translation and

$$\begin{aligned} \text{vol}_{\text{HS}}(\mathbf{D}^{B_t}(\mathbb{C}^2 \otimes \mathbb{C}^2) \cap \mathbf{D}_{\text{ss}}(\mathbb{C}^2 \otimes \mathbb{C}^2)) &= \text{vol}_{\text{HS}}\left(\bigcup_{\lambda \in [0, \frac{1}{2}]^4 \cap \mathcal{C}_4} \mathbf{D}^{B_t}(\mathbb{C}^2 \otimes \mathbb{C}^2) \cap \mathcal{U}_{\lambda}\right) \\ &= \text{vol}_{\text{HS}}\left(\bigcup_{\hat{\lambda} \in \hat{\mathcal{C}}} \mathcal{E}_t \cap \mathcal{O}_{\hat{\lambda}}\right) = \int_{\hat{\lambda} \in \hat{\mathcal{C}}} \text{vol}_{\text{HS}}(\mathcal{O}_{\hat{\lambda}} \cap \mathcal{E}_t) dm(\hat{\lambda}), \end{aligned}$$

where we used the facts in Eq. (26) and Eq. (27). Up to now, we have already obtained that

$$\frac{\pi}{2} \int_0^t a^2 f(a) da = \int_{\hat{\lambda} \in \hat{\mathcal{C}}} \text{vol}_{\text{HS}}(\mathcal{O}_{\hat{\lambda}} \cap \mathcal{E}_t) dm(\hat{\lambda}) \quad (95)$$

$$= \int_{\hat{\lambda} \in \hat{\mathcal{C}}} V_4(\hat{\lambda}) \text{vol}_{\text{symp}}(\mathcal{O}_{\hat{\lambda}} \cap \mathcal{E}_t) dm(\hat{\lambda}), \quad (96)$$

where we used the relationship (36) between HS volume of adjoint orbit and symplectic volume of co-adjoint orbit in the last equality. Consider the map F in Eq. (53),

$$F : (\mathcal{O}_{\hat{\lambda}}, \mu_{\mathcal{O}_{\hat{\lambda}}}) \rightarrow (\mathbb{R}_{>0}, da),$$

where $F(\mathbf{X}) = \lambda_+(\text{Tr}_2(\mathbf{X})) - \lambda_-(\text{Tr}_2(\mathbf{X})) = 2\lambda_+(\text{Tr}_2(\mathbf{X}))$ for any $\mathbf{X} \in \mathcal{O}_{\hat{\lambda}}$, i.e., twice the positive eigenvalue of $\text{Tr}_2(\mathbf{X})$. By the definition of F , we see that

$$F^{-1}([0, t]) = \mathcal{O}_{\hat{\lambda}} \cap \mathcal{E}_t,$$

implying that

$$\text{vol}_{\text{symp}}(\mathcal{E}_t \cap \mathcal{O}_{\hat{\lambda}}) = \text{vol}_{\text{symp}}(F^{-1}([0, t])) = \mu_{\mathcal{O}_{\hat{\lambda}}}(F^{-1}([0, t])) \quad (97)$$

$$= (F_* \mu_{\mathcal{O}_{\hat{\lambda}}})([0, t]) = \int_0^t \frac{dF_* \mu_{\mathcal{O}_{\hat{\lambda}}}}{da} da, \quad (98)$$

where $\frac{dF_* \mu_{\mathcal{O}_{\hat{\lambda}}}}{da} := (2\pi)^6 \mathcal{I}(a|\hat{\lambda})$ is the Radon-Nikodym derivative of push-forward measure $F_* \mu_{\mathcal{O}_{\hat{\lambda}}}$ with respect to Lebesgue measure on the ray $\mathbb{R}_{>0}$. Moreover, the analytical expression of such Radon-Nikodym derivative $\frac{dF_* \mu_{\mathcal{O}_{\hat{\lambda}}}}{da} = (2\pi)^6 \mathcal{I}(a|\hat{\lambda})$ is given by Eq. (55). Once again, we have obtained that

$$\begin{aligned} \frac{\pi}{2} \int_0^t a^2 f(a) da &= \int_{\hat{\lambda} \in \hat{\mathcal{C}}} V_4(\hat{\lambda}) \text{vol}_{\text{symp}}(\mathcal{E}_t \cap \mathcal{O}_{\hat{\lambda}}) dm(\hat{\lambda}) \\ &= \int_{\hat{\lambda} \in \hat{\mathcal{C}}} V_4(\hat{\lambda}) \left(\int_0^t \frac{dF_* \mu_{\mathcal{O}_{\hat{\lambda}}}}{da} da \right) dm(\hat{\lambda}) = \int_{\hat{\lambda} \in \hat{\mathcal{C}}} V_4(\hat{\lambda}) \left(\int_0^t (2\pi)^6 \mathcal{I}(a|\hat{\lambda}) da \right) dm(\hat{\lambda}) \\ &= (2\pi)^6 \int_0^t \left(\int_{\hat{\lambda} \in \hat{\mathcal{C}}} V_4(\hat{\lambda}) \mathcal{I}(a|\hat{\lambda}) dm(\hat{\lambda}) \right) da, \end{aligned}$$

implying that

$$\frac{\pi}{2} \int_0^t a^2 f(a) da = (2\pi)^6 \int_0^t \left(\int_{\hat{\lambda} \in \hat{\mathcal{C}}} V_4(\hat{\lambda}) \mathcal{I}(a|\hat{\lambda}) dm(\hat{\lambda}) \right) da. \quad (99)$$

By taking the derivative with respect to t , and then replacing t by a , we get that

$$\frac{\pi}{2}a^2f(a) = (2\pi)^6 \int_{\hat{\lambda} \in \hat{C}} V_4(\hat{\lambda}) \mathcal{I}(a|\hat{\lambda}) d\mathbf{m}(\hat{\lambda}) \implies \quad (100)$$

$$f(a) = \frac{2}{\pi}a^{-2}(2\pi)^6 \int_{\hat{\lambda} \in \hat{C}} V_4(\hat{\lambda}) \mathcal{I}(a|\hat{\lambda}) d\mathbf{m}(\hat{\lambda}), \quad (101)$$

where $a \in (0, \frac{1}{3})$. Now we compute the following integral

$$\int_{\hat{\lambda} \in \hat{C}} V_4(\hat{\lambda}) \mathcal{I}(a|\hat{\lambda}) d\mathbf{m}(\hat{\lambda}) = 2 \int_{[-\frac{1}{4}, \frac{1}{4}]^4 \cap \mathcal{C}_4} \prod_{1 \leq i < j \leq 4} (\hat{\lambda}_i - \hat{\lambda}_j) \mathcal{I}(a|\hat{\lambda}) \delta\left(\sum_{k=1}^4 \hat{\lambda}_k\right) \prod_{l=1}^4 d\hat{\lambda}_l \quad (102)$$

Let the change of variables be given below:

$$\begin{cases} \hat{\lambda}_1 = t_1 + \frac{t_2}{2} + \frac{t_3}{3} + \frac{t_4}{4} - \frac{1}{4}, \\ \hat{\lambda}_2 = \frac{t_2}{2} + \frac{t_3}{3} + \frac{t_4}{4} - \frac{1}{4}, \\ \hat{\lambda}_3 = \frac{t_3}{3} + \frac{t_4}{4} - \frac{1}{4}, \\ \hat{\lambda}_4 = \frac{t_4}{4} - \frac{1}{4}. \end{cases} \quad (103)$$

Its Jacobian is given by

$$J = \det\left(\frac{\partial(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3)}{\partial(t_1, t_2, t_3)}\right) = \begin{vmatrix} \frac{3}{4} & \frac{1}{4} & \frac{1}{12} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{12} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{12} \end{vmatrix} = \frac{1}{4!}$$

We note that

- (i) $t_k \geq 0$ ($k = 1, \dots, 4$) due to the fact that $\hat{\lambda}_1 > \hat{\lambda}_2 > \hat{\lambda}_3 > \hat{\lambda}_4$.
- (ii) $t_1 + t_2 + t_3 \leq 1$ because $t_4 \geq 0$ and $\sum_{k=1}^4 t_k = 1$ due to the fact that $\sum_{k=1}^4 \hat{\lambda}_k = 0$.
- (iii) $3t_1 + t_2 + \frac{t_3}{3} \leq 1$ because $t_1 + \frac{t_2}{2} + \frac{t_3}{3} + \frac{t_4}{4} - \frac{1}{4} \leq \frac{1}{4}$ and $\sum_{k=1}^4 t_k = 1$.

In summary, $\hat{C} = [-\frac{1}{4}, \frac{1}{4}]^4 \cap \mathcal{C}_4$ is transformed into the following form:

$$R = \left\{ (t_1, t_2, t_3) \in \mathbb{R}_{\geq 0}^3 : t_1 + t_2 + t_3 \leq 1, 3t_1 + t_2 + \frac{t_3}{3} \leq 1 \right\}. \quad (104)$$

According to such change of variables, we get that

$$\begin{cases} c_3 = 2(\hat{\lambda}_1 + \hat{\lambda}_2) = t_1 + t_2 + \frac{t_3}{3} \\ c_2 = 2(\hat{\lambda}_1 + \hat{\lambda}_3) = t_1 + \frac{t_3}{3} \\ c_1 = 2|\hat{\lambda}_1 + \hat{\lambda}_4| = \left| t_1 - \frac{t_3}{3} \right| \\ V_4(\hat{\lambda}) = \frac{t_1 t_2 t_3 (2t_1 + t_2)(3t_2 + 2t_3)(6t_1 + 3t_2 + 2t_3)}{432} =: V(t_1, t_2, t_3) \end{cases} \quad (105)$$

Therefore we get that via change of variables

$$\tilde{\mathcal{I}}_1(x|\mathbf{t}) = \frac{t_2(t_1 + \frac{t_2}{2} + \frac{t_3}{3})}{32} \left(\left| t_1 - \frac{t_3}{3} \right| - x \right)^2 x, \quad \left| t_1 - \frac{t_3}{3} \right| \geq x \quad (106)$$

$$\tilde{\mathcal{I}}_2(x|\mathbf{t}) = \frac{(t_1 + \frac{t_2}{2})(\frac{t_2}{2} + \frac{t_3}{3})}{16} \left(t_1 + \frac{t_3}{3} - x \right)^2 x, \quad t_1 + \frac{t_3}{3} \geq x \quad (107)$$

$$\tilde{\mathcal{I}}_3(x|\mathbf{t}) = \frac{t_1 t_3}{48} \left(t_1 + t_2 + \frac{t_3}{3} - x \right)^2 x, \quad t_1 + t_2 + \frac{t_3}{3} \geq x \quad (108)$$

With these preparations, we can transform the integral in Eq. (102) into the following form:

$$\begin{aligned}
\int_{\hat{\lambda} \in \hat{C}} V_4(\hat{\lambda}) \mathcal{I}(x|\hat{\lambda}) dm(\hat{\lambda}) &= \frac{2}{4!} \int_R V(t_1, t_2, t_3) (\tilde{\mathcal{I}}_1(x|t) + \tilde{\mathcal{I}}_2(x|t) + \tilde{\mathcal{I}}_3(x|t)) dt_1 dt_2 dt_3 \\
&= \frac{2}{4!} \int_R V(t_1, t_2, t_3) \tilde{\mathcal{I}}_1(x|t) dt_1 dt_2 dt_3 \\
&\quad + \frac{2}{4!} \int_R V(t_1, t_2, t_3) \tilde{\mathcal{I}}_2(x|t) dt_1 dt_2 dt_3 \\
&\quad + \frac{2}{4!} \int_R V(t_1, t_2, t_3) \tilde{\mathcal{I}}_3(x|t) dt_1 dt_2 dt_3 \\
&= M_1 + M_2 + M_3.
\end{aligned}$$

- Note that the following three planes $\{t_1 + t_2 + t_3 = 1, 3t_1 + t_2 + \frac{t_3}{3} = 1, t_1 - \frac{t_3}{3} = 0\}$ intersects at the line segment in $\mathbb{R}_{\geq 0}^3$:

$$\begin{cases} t_1 = t \\ t_2 = 1 - 4t \\ t_3 = 3t \end{cases} \quad \left(t \in [0, \frac{1}{4}] \right). \quad (109)$$

- The following two planes $\{3t_1 + t_2 + \frac{t_3}{3} = 1, t_1 - \frac{t_3}{3} = x\}$ intersects at the line segment in $\mathbb{R}_{\geq 0}^3$:

$$\begin{cases} t_1 = t \\ t_2 = 1 + x - 4t \\ t_3 = -3x + 3t \end{cases} \quad \left(t \in [x, \frac{1+x}{4}] \right). \quad (110)$$

- The following two planes $\{t_1 + t_2 + t_3 = 1, t_1 - \frac{t_3}{3} = -x\}$ intersects at the line segment in $\mathbb{R}_{\geq 0}^3$:

$$\begin{cases} t_1 = t \\ t_2 = 1 - 3x - 4t \\ t_3 = 3x + 3t \end{cases} \quad \left(t \in [0, \frac{1-3x}{4}] \right). \quad (111)$$

In summary, the plane $|t_1 - \frac{t_3}{3}| = x$ intersects with R iff $x \in [0, \frac{1}{3}]$. For $m \geq 2$, denote

$$\Delta_m := \{(p_1, \dots, p_m) \in \mathbb{R}_{\geq 0}^m : p_1 + \dots + p_m \leq 1\}.$$

Now for $x \in [0, \frac{1}{3}]$, we get that

- (1) For the domain of the first integral for a given x , we get that

$$\begin{aligned}
R_1(x) &:= \left\{ (t_1, t_2, t_3) \in \mathbb{R}_{\geq 0}^3 : t_1 + t_2 + t_3 \leq 1, 3t_1 + t_2 + \frac{t_3}{3} \leq 1, |t_1 - \frac{t_3}{3}| \geq x \right\} \\
&= R_{1,a}(x) \cup R_{1,b}(x),
\end{aligned}$$

where

$$\begin{aligned}
R_{1,a}(x) &:= \left\{ (t_1, t_2, t_3) \in \mathbb{R}_{\geq 0}^3 : t_1 + t_2 + t_3 \leq 1, 3t_1 + t_2 + \frac{t_3}{3} \leq 1, t_1 - \frac{t_3}{3} \geq x \right\} \\
&= \left\{ (t_1, t_2, t_3) \in \mathbb{R}_{\geq 0}^3 : 0 \leq t_3 \leq \frac{3(1-3x)}{4}, 0 \leq t_2 \leq 1 - 3x - \frac{4t_3}{3}, x + \frac{t_3}{3} \leq t_1 \leq \frac{1}{3} - \frac{t_2}{3} - \frac{t_3}{9} \right\} \\
R_{1,b}(x) &:= \left\{ (t_1, t_2, t_3) \in \mathbb{R}_{\geq 0}^3 : t_1 + t_2 + t_3 \leq 1, 3t_1 + t_2 + \frac{t_3}{3} \leq 1, t_1 - \frac{t_3}{3} \leq -x \right\} \\
&= \left\{ (t_1, t_2, t_3) \in \mathbb{R}_{\geq 0}^3 : 0 \leq t_2 \leq 1 - 3x, 0 \leq t_1 \leq \frac{1-3x-t_2}{4}, 3(x+t_1) \leq t_3 \leq 1 - t_1 - t_2 \right\}.
\end{aligned}$$

In such case, we can evaluate M_1 as follows: For $x \in (0, \frac{1}{3})$,

$$\begin{aligned}
M_1 &= \frac{2}{4!} \int_R V(t_1, t_2, t_3) \tilde{\mathcal{I}}_1(x|t) dt_1 dt_2 dt_3 = \frac{2}{4!} \int_{R_1(x)} V(t_1, t_2, t_3) \tilde{\mathcal{I}}_1(x|t) dt_1 dt_2 dt_3 \\
&= \frac{2}{4!} \int_{R_{1,a}(x)} V(t_1, t_2, t_3) \tilde{\mathcal{I}}_1(x|t) dt_1 dt_2 dt_3 + \frac{2}{4!} \int_{R_{1,b}(x)} V(t_1, t_2, t_3) \tilde{\mathcal{I}}_1(x|t) dt_1 dt_2 dt_3 \\
&= \frac{x(3x-1)^9(567x^4 - 3564x^3 + 5526x^2 - 3152x - 1345)}{774741019852800} \\
&\quad + \frac{x(3x-1)^9(567x^4 - 3564x^3 + 5526x^2 - 3152x - 1345)}{774741019852800} \\
&= \frac{x(3x-1)^9(567x^4 - 3564x^3 + 5526x^2 - 3152x - 1345)}{387370509926400},
\end{aligned}$$

thus

$$M_1 = \frac{x(3x-1)^9(567x^4 - 3564x^3 + 5526x^2 - 3152x - 1345)}{387370509926400}.$$

(2) For the domain of the second integral for a given x , we get that

$$\begin{aligned}
R_2(x) &:= \left\{ (t_1, t_2, t_3) \in \mathbb{R}_{\geq 0}^3 : t_1 + t_2 + t_3 \leq 1, 3t_1 + t_2 + \frac{t_3}{3} \leq 1, t_1 + \frac{t_3}{3} \geq x \right\} \\
&= (\Delta_3 \cup R_{2,ab}(x)) \setminus (R_{2,a}(x) \cup R_{2,b}(x)),
\end{aligned}$$

where

$$\begin{cases}
R_{2,a}(x) &:= \left\{ (t_1, t_2, t_3) \in \mathbb{R}_{\geq 0}^3 : t_1 + t_2 + t_3 \leq 1, 3t_1 + t_2 + \frac{t_3}{3} > 1 \right\} \\
R_{2,b}(x) &:= \left\{ (t_1, t_2, t_3) \in \mathbb{R}_{\geq 0}^3 : t_1 + t_2 + t_3 \leq 1, t_1 + \frac{t_3}{3} < x \right\} \\
R_{2,ab}(x) &:= R_{2,a}(x) \cap R_{2,b}(x).
\end{cases}$$

Note that

$$\begin{aligned}
R_{2,a}(x) &= \left\{ (t_1, t_2, t_3) \in \mathbb{R}^3 : 0 \leq t_3 < \frac{3}{4}, 0 \leq t_2 < 1 - \frac{4t_3}{3}, \frac{1}{3} - \frac{t_2}{3} - \frac{t_3}{9} < t_1 \leq 1 - t_2 - t_3 \right\}, \\
R_{2,b}(x) &= \left\{ (t_1, t_2, t_3) \in \mathbb{R}^3 : 0 \leq t_3 < 3x, 0 \leq t_1 < x - \frac{t_3}{3}, 0 \leq t_2 \leq 1 - t_1 - t_3 \right\}, \\
R_{2,ab}(x) &= \left\{ (t_1, t_2, t_3) \in \mathbb{R}^3 : 0 \leq t_3 < \frac{3x}{2}, \frac{t_3}{3} < t_1 < x - \frac{t_3}{3}, 1 - 3t_1 - \frac{t_3}{3} < t_2 \leq 1 - t_1 - t_3 \right\}.
\end{aligned}$$

In such case, we can evaluate M_2 as follows: For $x \in (0, \frac{1}{3})$,

$$\begin{aligned}
M_2 &= \frac{2}{4!} \int_R V(t_1, t_2, t_3) \tilde{\mathcal{I}}_2(x|t) dt_1 dt_2 dt_3 = \frac{2}{4!} \int_{R_2(x)} V(t_1, t_2, t_3) \tilde{\mathcal{I}}_2(x|t) dt_1 dt_2 dt_3 \\
&= \frac{2}{4!} \int_{\Delta_3} V(t_1, t_2, t_3) \tilde{\mathcal{I}}_2(x|t) dt_1 dt_2 dt_3 + \frac{2}{4!} \int_{R_{2,ab}(x)} V(t_1, t_2, t_3) \tilde{\mathcal{I}}_2(x|t) dt_1 dt_2 dt_3 \\
&\quad - \frac{2}{4!} \int_{R_{2,a}(x)} V(t_1, t_2, t_3) \tilde{\mathcal{I}}_2(x|t) dt_1 dt_2 dt_3 - \frac{2}{4!} \int_{R_{2,b}(x)} V(t_1, t_2, t_3) \tilde{\mathcal{I}}_2(x|t) dt_1 dt_2 dt_3 \\
&= -\frac{x(725985x^2 - 560326x + 123355)}{96842627481600} \\
&\quad - \frac{x^8(2894x^6 - 24570x^5 + 81900x^4 - 150150x^3 + 160875x^2 - 96525x + 25740)}{177124147200} \\
&\quad + \frac{x(637485030x^2 - 525965687x + 120181250)}{99166850541158400} \\
&\quad - \frac{x^7(1572x^7 - 17550x^6 + 62712x^5 - 120835x^4 + 141570x^3 - 106821x^2 + 51480x - 12870)}{132843110400},
\end{aligned}$$

implying that

$$M_2 = -\frac{499x^{14}}{17712414720} + \frac{41x^{13}}{151388160} - \frac{1061x^{12}}{1135411200} + \frac{653x^{11}}{371589120} - \frac{163x^{10}}{82575360} + \frac{557x^9}{412876800} \\ - \frac{11x^8}{20643840} + \frac{x^7}{10321920} - \frac{90533x^3}{84757991915520} + \frac{3677549x^2}{7628219272396800} - \frac{613427x}{9916685054115840}.$$

(3) For the domain of the third integral for a given x , we get that

$$R_3(x) := \left\{ (t_1, t_2, t_3) \in \mathbb{R}_{\geq 0}^3 : t_1 + t_2 + t_3 \leq 1, 3t_1 + t_2 + \frac{t_3}{3} \leq 1, t_1 + t_2 + \frac{t_3}{3} \geq x \right\} \\ = (\Delta_3 \cup R_{3,ab}(x)) \setminus (R_{3,a}(x) \cup R_{3,b}(x)),$$

where

$$\begin{cases} R_{3,a}(x) &:= \left\{ (t_1, t_2, t_3) \in \mathbb{R}_{\geq 0}^3 : t_1 + t_2 + t_3 \leq 1, 3t_1 + t_2 + \frac{t_3}{3} > 1 \right\}, \\ R_{3,b}(x) &:= \left\{ (t_1, t_2, t_3) \in \mathbb{R}_{\geq 0}^3 : t_1 + t_2 + \frac{t_3}{3} < x \right\}, \\ R_{3,ab}(x) &:= R_{3,a}(x) \cap R_{3,b}(x), \end{cases}$$

Note that

$$R_{3,a}(x) = \left\{ (t_1, t_2, t_3) \in \mathbb{R}^3 : 0 \leq t_3 < \frac{3}{4}, 0 \leq t_2 < 1 - \frac{4t_3}{3}, \frac{1}{3} - \frac{t_2}{3} - \frac{t_3}{9} < t_1 \leq 1 - t_2 - t_3 \right\}, \\ R_{3,b}(x) = \left\{ (t_1, t_2, t_3) \in \mathbb{R}^3 : 0 \leq t_3 < 3x, 0 \leq t_2 < x - \frac{t_3}{3}, 0 \leq t_1 \leq x - t_2 - \frac{t_3}{3} \right\}, \\ R_{3,ab}(x) = R_{3,a}(x) \cap R_{3,b}(x) = \emptyset.$$

In such case, we can evaluate M_3 as follows: For $x \in (0, \frac{1}{3})$,

$$M_3 = \frac{2}{4!} \int_R V(t_1, t_2, t_3) \tilde{\mathcal{I}}_3(x|t) dt_1 dt_2 dt_3 = \frac{2}{4!} \int_{R_3(x)} V(t_1, t_2, t_3) \tilde{\mathcal{I}}_2(x|t) dt_1 dt_2 dt_3 \\ = \frac{2}{4!} \int_{\Delta_3} V(t_1, t_2, t_3) \tilde{\mathcal{I}}_3(x|t) dt_1 dt_2 dt_3 \\ - \frac{2}{4!} \int_{R_{3,a}(x)} V(t_1, t_2, t_3) \tilde{\mathcal{I}}_3(x|t) dt_1 dt_2 dt_3 - \frac{2}{4!} \int_{R_{3,b}(x)} V(t_1, t_2, t_3) \tilde{\mathcal{I}}_3(x|t) dt_1 dt_2 dt_3 \\ = \frac{x(2340x^2 - 3341x + 1220)}{1513166054400} - \frac{x(135789030x^2 - 199046263x + 74163970)}{99166850541158400} - \frac{x^{14}}{691891200},$$

implying that

$$M_3 = -\frac{x^{14}}{691891200} + \frac{15013x^3}{84757991915520} - \frac{1531501x^2}{7628219272396800} + \frac{115799x}{1983337010823168}.$$

In summary,

$$M_1 + M_2 + M_3 = \frac{(1-x)^9 x^2 (33x^3 + 162x^2 + 72x + 8)}{40874803200}. \quad (112)$$

This indicates, together with Eq. (101), that

$$f(a) = \frac{\pi^5}{319334400} (1-a)^9 (33a^3 + 162a^2 + 72a + 8), \quad (113)$$

where $a \in (0, \frac{1}{3})$. $\square$

Remark 5. We see that the parameter x is restricted to the open interval $(0, \frac{1}{3})$ due to fact that the plane $|t_1 - \frac{t_3}{3}| = x$ intersects with R iff $x \in [0, \frac{1}{3}]$. Note that the integrations involved are performed by the mathematical software *MATHEMATICA*.

In the recent paper [7], Huong and Khoi published a proof of the exact separability probability $\frac{8}{33}$ for the two-qubit system, resolving a long-standing conjecture originally proposed by Slater. Their result demonstrates that within the Hilbert-Schmidt measure, the proportion of separable states among all two-qubit density matrices is precisely $\frac{8}{33}$. This breakthrough provides a definitive answer to a fundamental question in quantum information theory concerning the geometric prevalence of entanglement in simple quantum systems, achieved through a novel combination of advanced geometric probability techniques and symmetry arguments applied to the structure of the two-qubit state space.

Theorem 7. *The separability probability of all two-qubit states is given by*

$$P_{\text{sep}}^{(2 \times 2)} = \frac{\text{vol}_{\text{HS}}(\text{D}_{\text{sep}}(\mathbb{C}^2 \otimes \mathbb{C}^2))}{\text{vol}_{\text{HS}}(\text{D}(\mathbb{C}^2 \otimes \mathbb{C}^2))} = \frac{8}{33}. \quad (114)$$

Proof. From Theorem 6, we have seen that $P_{\text{sep}}^{(2 \times 2)}(a)$ is independent of $a \in [0, 1)$, which means that $P_{\text{sep}}^{(2 \times 2)}(a) = P_{\text{sep}}^{(2 \times 2)}(0)$ for all $a \in [0, 1)$. Then

$$\text{vol}_{\text{HS}}(\text{D}_{\text{sep}}^a(\mathbb{C}^2 \otimes \mathbb{C}^2)) = P_{\text{sep}}^{(2 \times 2)}(0) \text{vol}_{\text{HS}}(\text{D}^a(\mathbb{C}^2 \otimes \mathbb{C}^2)). \quad (115)$$

By multiplying $\frac{\pi}{2} a^2$ on both sides above, and taking the integration over $[0, 1)$, we see that

$$\begin{aligned} \frac{\pi}{2} \int_0^1 a^2 \text{vol}_{\text{HS}}(\text{D}_{\text{sep}}^a(\mathbb{C}^2 \otimes \mathbb{C}^2)) da &= P_{\text{sep}}^{(2 \times 2)}(0) \frac{\pi}{2} \int_0^1 a^2 \text{vol}_{\text{HS}}(\text{D}^a(\mathbb{C}^2 \otimes \mathbb{C}^2)) da \\ &= P_{\text{sep}}^{(2 \times 2)}(0) \text{vol}_{\text{HS}}(\text{D}(\mathbb{C}^2 \otimes \mathbb{C}^2)), \end{aligned}$$

where we used the result in Proposition 13. Analogously,

$$\frac{\pi}{2} \int_0^1 a^2 \text{vol}_{\text{HS}}(\text{D}_{\text{sep}}^a(\mathbb{C}^2 \otimes \mathbb{C}^2)) da = \text{vol}_{\text{HS}}(\text{D}_{\text{sep}}(\mathbb{C}^2 \otimes \mathbb{C}^2)).$$

These formulas indicate that $P_{\text{sep}}^{(2 \times 2)} = P_{\text{sep}}^{(2 \times 2)}(0)$. With Eq. (115), it suffices to calculate

$$P_{\text{sep}}^{(2 \times 2)}(0) = \frac{\text{vol}_{\text{HS}}(\text{D}_{\text{sep}}^0(\mathbb{C}^2 \otimes \mathbb{C}^2))}{\text{vol}_{\text{HS}}(\text{D}^0(\mathbb{C}^2 \otimes \mathbb{C}^2))}. \quad (116)$$

To this end, it suffices to calculate $\text{vol}_{\text{HS}}(\text{D}_{\text{sep}}^0(\mathbb{C}^2 \otimes \mathbb{C}^2))$ since we have already know the denominator $\text{vol}_{\text{HS}}(\text{D}^0(\mathbb{C}^2 \otimes \mathbb{C}^2)) = \frac{\pi^5}{9676800}$ from Eq. (77). Clearly, from Proposition 16,

$$f(0) = \text{vol}_{\text{HS}}(\text{D}_{\text{sep}}^0(\mathbb{C}^2 \otimes \mathbb{C}^2)) = \frac{\pi^5}{39916800},$$

implying that

$$P_{\text{sep}}^{(2 \times 2)}(0) = \frac{\text{vol}_{\text{HS}}(\text{D}_{\text{sep}}^0(\mathbb{C}^2 \otimes \mathbb{C}^2))}{\text{vol}_{\text{HS}}(\text{D}^0(\mathbb{C}^2 \otimes \mathbb{C}^2))} = \frac{\frac{\pi^5}{39916800}}{\frac{\pi^5}{9676800}} = \frac{8}{33}. \quad (117)$$

This completes the proof. $\square$

7. Concluding Remarks

This study establishes a comprehensive geometric derivation of the exact separability probability $8/33$ for two-qubit states under the Hilbert-Schmidt measure — a rigorously confirmed result previously inaccessible. By leveraging the intrinsic connection between Hilbert-Schmidt geometry and symplectic geometry formalized through the Duistermaat-Heckman (DH) measure, and explicitly computing the Hilbert-Schmidt volume of the full state space, volumes of critical substructures (flag manifolds and regular adjoint orbits), and the symplectic volumes of corresponding regular co-adjoint orbits, we integrate these perspectives to isolate the volume of separable states and rigorously confirm the $8/33$ ratio.

This work achieves pedagogical clarity through a self-contained derivation of a fundamental quantum information constant and conceptual synthesis by unifying Hilbert-Schmidt geometry, symplectic mechanics, and representation-theoretic tools within a coherent probabilistic framework. Beyond providing an alternative pathway to this constant, our results elucidate the geometric structure governing the transition from classical correlation to quantum entanglement. Future work may extend this framework to higher-dimensional systems, alternative metrics (e.g., Bures), and generalized entanglement witnesses, underscoring its potential as a universal paradigm for probing the classical-quantum correlation boundary.

Author Contributions

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Conflicts of Interest Statement

The authors declare no conflict of interest.

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