

State Convertibility under Genuinely Incoherent Operations

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Abstract: State convertibility is fundamental in the study of resource theory of quantum coherence. It is aimed at identifying when it is possible to convert a given coherent state to another using only incoherent operations. In this paper, we give a complete characterization of state convertibility under genuinely incoherent operations. It is found that convexity of the robustness of coherence plays a central role. Based on this, the majorization condition of determining convertibility from pure states to mixed states under strictly incoherent operations is provided. Moreover, maximally coherent states in the set of all states with fixed diagonal elements are determined. It is somewhat surprising that convexity of the robustness of coherence can also decide conversion between off-diagonal parts of coherent states. This might be a big step to answer completely the question of state convertibility for mixed states under incoherent operations.

Keywords: quantum coherence; state convertibility; incoherent operations

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1. Introduction

The coherent superposition of states is one of the characteristic features that results in nonclassical phenomena [1,2]. Quantum coherence constitutes a powerful physical resource for implementing various tasks such as quantum algorithms [3–11], quantum metrology [12–20], quantum channel discrimination [21–27], witnessing quantum correlations [28–34], and quantum phase transitions and transport phenomena [35–40]. The resource theory of quantum coherence has been flourishing in recent years, it not only establishes a rigorous framework to quantify coherence but also provides a platform to understand quantum coherence from a different perspective [23,41].

Any quantum resource theory is described by two fundamental ingredients, namely, the free states and the free operations [42]. For the resource theory of coherence, the free states are quantum states that are diagonal in a prefixed reference basis. The free operations are not uniquely specified. Motivated by different physical considerations, several free operations are presented, such as incoherent operations (IOs) [41], maximally incoherent operations (MIOs) [43], strictly incoherent operations (SIOs) [44,45], dephasing-covariant incoherent operations (DIOs) [46–48], and genuinely incoherent operations (GIOs) [49].

Two fundamental problems in coherence resource theory are state convertibility and resource quantification [23,42]. The state convertibility problem is asking whether for two coherent states there exists a free operation converting one quantum state into the other. The goal of resource quantification is to quantify the amount of coherence in a quantum state. Recalling that coherent states cannot be created from incoherent states via free operations, it is intuitive to assume that

$$C(\rho) \geq C(\Phi(\rho)) \quad (1)$$

for any quantum state ρ and any free operation Φ . Quantifiers having this property are also called coherence monotones.

Both problems mentioned above—state convertibility and resource quantification—are in fact closely connected. A state ρ can be converted into σ via free operations if and only if

$$C(\rho) \geq C(\sigma) \quad (2)$$

holds true for all coherence monotones [50]. On the other hand, the fact that Eq. (2) holds for some coherence monotone C does not guarantee that the transformation $\rho \rightarrow \sigma$ is possible via free operations. The aim of state convertibility is to find a complete set of coherence monotones $\{C_i\}$ which can completely classify state transformation, that is,

$$\rho \rightarrow \sigma \Leftrightarrow C_i(\rho) \geq C_i(\sigma) \quad (3)$$

for all i .

The study of state convertibility is moving ahead since the question is proposed [41], it is completely answered in pure states or one-qubit case under IOs, SIOs, or MIOs [44,46,51–63]. The convertibility between mixed states seems to have remained unexplored territory. The difficulty lies in the complexity of pure state decomposition which results in an infinite number of measure conditions for characterizing convertibility of mixed states [57]. We investigate convertibility for coherent states under GIOs. In fact, GIOs are at the core of the resource theory of quantum coherence from both physical realization and dissecting the structure of SIOs and IOs [64]. Note that there is a hierarchical relationship between IOs, SIOs, MIOs, and GIOs [23],

$$\text{GIOs} \subseteq \text{SIOs} \subseteq \text{IOs} \subseteq \text{MIOs}. \quad (4)$$

For any $\mathcal{O} \in \{\text{IOs}, \text{SIOs}, \text{MIOs}\}$, we define $\rho \xrightarrow{\mathcal{O}} \sigma$ if there exists $\Phi \in \mathcal{O}$ such that $\Phi(\rho) = \sigma$. It is evident if $\rho \xrightarrow{\text{GIO}} \sigma$, then $\rho \xrightarrow{\mathcal{O}} \sigma$.

The complete set of coherence monotones for characterizing state convertibility under GIOs is found. In fact, convexity of the robustness of coherence is a good candidate. Moreover, it is also key to convert off-diagonal part of coherent states under more general free operations. Our results induce a useful tool for deciding maximally coherent states in the set of all states with fixed diagonal elements. This produces the so-called majorization condition of determining convertibility from pure states to mixed states under SIOs.

The paper is organized as follows. In Section 2, we briefly present the resource theory of quantum coherence. In Section 3, we will give our main results. Section 4 is a summary of our findings. The appendix is the proof of our results.

2. Definition and Basic Properties

Throughout the paper, we consider the d dimensional Hilbert space $\mathcal{H}$ and adopt the computational basis $\{|i\rangle\}_{i=1}^d$ as the incoherent basis [41]. Thus all diagonal density operators in this basis constitute the set of all incoherent states denoted as $\mathcal{I}$. IOs are specified by a set of Kraus operators $\{K_j\}$ such that $K_j \rho K_j^\dagger / \text{Tr}(K_j \rho K_j^\dagger) \in \mathcal{I}$ for all $\rho \in \mathcal{I}$, $\Phi(\rho) = \sum_j K_j \rho K_j^\dagger$. Such operation elements $\{K_j\}$ are called incoherent. An incoherent operation is strictly incoherent if both K_j and $K_j^\dagger$ are incoherent. The MIOs are known as incoherent states preserving operations. GIOs are operations that fix all incoherent states, that is,

$$\Phi(\rho) = (\rho) \quad (5)$$

for any incoherent state $\rho \in \mathcal{I}$. Since GIOs do not allow for transformations between different incoherent states, notably, for example, between the energy eigenstates (when coherence is measured with respect to the eigenbasis of the Hamiltonian of the system), they capture the framework of coherence in the presence of additional constraints, such as energy conservation. For other important type of incoherent operations, we refer the reader to the review article [23].

In order to characterize conversion of coherent states under GIOs, we need a key measure originated from the task of maximizing the mean value of an observable [65]. Let $|\psi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle$, it is well-known that $|\psi^+\rangle\langle\psi^+|$ is a maximally coherent state under IOs, that is, a state from which all other states can be created via IOs [41]. It is easy to see that $U|\psi^+\rangle\langle\psi^+|U^\dagger$ is maximally coherent under IOs for any diagonal unitary matrix U . Let Ω be the set of convex hull of $U|\psi^+\rangle\langle\psi^+|U^\dagger$. For every $M \in \Omega$, define

$$C_M^{\text{GIOs}}(\rho) = \max_{\Phi \in \text{GIOs}} \text{tr}(\Phi(\rho)M) - \frac{1}{d}. \quad (6)$$

In the following, we list some elementary properties of $C_M^{\text{GIOs}}(\cdot)$ and discuss its relationship with other coherence measures (see appendix for the proof).

- (i) $C_M^{\text{GIOs}}(\rho) \geq 0$ for every quantum state ρ and $C_M^{\text{GIOs}}(\rho) = 0$ if $\rho \in \mathcal{I}$;
(ii) Monotonicity under all GIOs Φ :

$$C_M^{\text{GIOs}}(\Phi(\rho)) \leq C_M^{\text{GIOs}}(\rho); \quad (7)$$

- (iii) Monotonicity for average coherence:

$$\sum_j p_j C_M^{\text{GIOs}}(\rho_j) \leq C_M^{\text{GIOs}}(\rho) \quad (8)$$

for all $\{K_j\}$ specifying every GIO, where $\rho_j = \frac{K_j \rho K_j^\dagger}{p_j}$ and $p_j = \text{Tr}(K_j \rho K_j^\dagger)$;

- (iv) Non-increasing under mixing of quantum states:

$$C_M^{\text{GIOs}}(\sum_j p_j \rho_j) \leq \sum_j p_j C_M^{\text{GIOs}}(\rho_j) \quad (9)$$

for any set of states $\{\rho_j\}$ and any $p_j \geq 0$ with $\sum_j p_j = 1$;

- (v) $C_M^{\text{GIOs}}(\rho)$ is related to the l_1 -norm of coherence by the inequality

$$\begin{aligned} \frac{C_{l_1}(\rho)}{d-1} \min_{1 \leq i \neq j \leq d} \{|M_{ij}|\} &\leq C_M^{\text{GIOs}}(\rho) \\ &\leq C_{l_1}(\rho) \max_{1 \leq i \neq j \leq d} \{|M_{ij}|\}, \end{aligned} \quad (10)$$

here $M = (M_{ij})$ and $C_{l_1}(\rho) = \sum_{i \neq j} |\rho_{ij}|$ is the l_1 -norm of coherence;

- (vi) $C_M^{\text{GIOs}}(\rho)$ is also related to the robustness of coherence by the inequality

$$0 \leq C_M^{\text{GIOs}}(\rho) \leq \frac{C_{\text{ROC}}(\rho)}{d}, \quad (11)$$

here

$$C_{\text{ROC}} = \min_{\tau \in \mathcal{S}} \{s : \frac{\rho + s\tau}{1+s} \in \mathcal{I}\} = \min_{\delta \in \mathcal{I}} \{s : \rho \leq (1+s)\delta\}$$

is the robustness of coherence [66].

Specially, if $M = |\psi^+\rangle\langle\psi^+|$, then combining Theorem 1 with Theorem 2 of [67], we have

$$C_{|\psi^+\rangle\langle\psi^+|}^{\text{GIOs}}(\rho) = \frac{C_{\text{ROC}}(\rho)}{d} \quad (12)$$

For general $M \in \Omega$, there exist a probability distribution $\{p_i\}$ and diagonal unitary matrices $\{U_i\}$ such that $M = \sum_i p_i U_i |\psi^+\rangle\langle\psi^+| U_i^\dagger$. That is, M is a convexity of maximally coherent states. In this sense, we say C_M^{GIOs} is a convexity of the robustness of coherence. It is found that such measures play a key role in studying state convertibility under GIOs.

3. Main Results

Now, we are in a position to give our main result.

Theorem 1. *There exists some GIO Φ such that*

$$\Phi(\rho) = \sigma \Leftrightarrow C_M^{\text{GIOs}}(\rho) \geq C_M^{\text{GIOs}}(\sigma) \quad (13)$$

for any $M \in \Omega$, $\rho_{ii} = \sigma_{ii}$ ($i = 1, 2, \dots, d$).

Theorem 1 tells convertibility between pure states is impossible except for diagonal-unitary equivalent states. A parallel result in multipartite entanglement is almost all n -qubit pure states with $n \geq 3$ can neither be reached nor be converted into any other LU-inequivalent state via deterministic LOCC [68]. On the other hand, deterministic convertibility between incoherent-unitary inequivalent pure states is possible under IOs, DIOs, SIOs, and MIOs [46,51]. Thus, compared with other free operations in the coherence resource theory, GIOs are more matching to LOCC in multipartite entanglement theory from the point of state convertibility.

For one-parameter maximally mixed states [52,67]

$$\rho_p = p|\psi^+\rangle\langle\psi^+| + \frac{1-p}{d}I, \quad (14)$$

Theorem 1 shows that

$$\rho_p \xrightarrow{\text{GIO}} \rho_q \Leftrightarrow q \leq p. \quad (15)$$

Based on Theorem 1, we can provide a nice majorization condition that determines the convertibility from pure states to mixed states under SIOs, IOs, and MIOs.

Theorem 2. For $|\psi\rangle = \sum_{i=1}^d \psi_i|i\rangle$, $\sigma = (\sigma_{ij})$,

$$(|\psi_1|^2, \dots, |\psi_d|^2)^t \prec (\sigma_{11}, \dots, \sigma_{dd})^t \Rightarrow |\psi\rangle\langle\psi| \xrightarrow{\text{SIO}} \rho \quad (16)$$

here $\prec$ denotes the majorization relation between probability vectors.

By the hierarchical relationship $\text{SIOs} \subseteq \text{IOs} \subseteq \text{MIOs}$, there exists some IO or MIO Φ with $\Phi(|\psi\rangle\langle\psi|) = \sigma$ if $(|\psi_1|^2, \dots, |\psi_d|^2)^t \prec (\sigma_{11}, \dots, \sigma_{dd})^t$.

For $|\psi^+\rangle = \sum_{i=1}^d \frac{1}{\sqrt{d}}|i\rangle$, it is evident that

$$\left(\frac{1}{d}, \dots, \frac{1}{d}\right)^t \prec (\sigma_{11}, \dots, \sigma_{dd})^t \quad (17)$$

for any quantum state σ . A direct consequence of Theorem 2 is that $|\psi^+\rangle\langle\psi^+|$ is maximally coherent under IOs which is an important conclusion of [41].

It is well-known that convertibility between pure states is completely characterized by majorization relation [51]. Theorem 2 can be regarded as an extension when the output state is mixed. Although a structural characterization of coherence conversion for the output mixed state is provided in terms of a finite number of measure conditions [57], such conditions are somewhat hard to verify because pure state decomposition is involved. In comparison, Theorem 2 is more handy because we need only to check a majorization relation.

The core for the proof of Theorem 2 is to find maximally coherent states (MCS) in the set of all states S with fixed diagonal elements, here an MCS means a state from which all other states of S can be created via GIOs.

We remark that the existence of MCS in a particular set of states S has independent meaning, because one may not be able to prepare all states of choice in many situations. Suppose we are bound to a particular set of states S , can we find a notion of maximally coherent state in S . By Theorem 1, a natural choice of S is the set of all states with fixed diagonal elements, that is, $S = \{(\rho_{ij}) : \rho_{ii} = p_i, i = 1, 2, \dots, d\}$, here $\{p_i\}$ is a fixed probability distribution. In fact, there exists an MCS in S . Our result reads as follows.

Theorem 3. Let $|\psi\rangle = \sum_{i=1}^d \sqrt{p_i}|i\rangle$, $S = \{(\rho_{ij}) : \rho_{ii} = p_i, i = 1, 2, \dots, d\}$. Then for any $\rho \in S$, there exists a GIO Φ such that $\Phi(|\psi\rangle\langle\psi|) = \rho$.

By the hierarchical relationship

$$\text{GIOs} \subseteq \text{SIOs} \subseteq \text{IOs} \subseteq \text{MIOs},$$

$$\text{GIOs} \subseteq \text{SIOs} \subseteq \text{DIOs} \subseteq \text{MIOs},$$

we can obtain $|\psi\rangle = \sum_{i=1}^d \sqrt{p_i}|i\rangle$ is also maximally coherent in S under SIOs, DIOs, IOs, and MIOs.

Theorem 1 and Theorem 3 show that coherent mixed states cannot be converted into pure states in general. This is a parallel result of no-go theorem of purification for coherent mixed states of discrete-variable and Gaussian systems [60,69]. It shows a strong limit on the efficiency of perfect coherent purification under GIOs.

We also remark that parallel discussion of Theorem 3 in quantum entanglement is the existence of a maximally entangled state within a given set of states with fixed spectrum. This is the Problem 5 in the Open Quantum Problems List maintained by the Institute for Quantum Optics and Quantum Information (IQOQI) in Vienna [70,71]. It is newly shown that maximally entangled mixed states for a fixed spectrum do not always exist [72].

By Theorem 1, if diagonal elements of ρ and σ are not completely equal in the same position, then both $\rho \nrightarrow \sigma$ and $\sigma \nrightarrow \rho$ under GIOs hold true. However, exact conditions for realizing conversion between off-diagonal parts of coherent states can also be found.

For any $\mathcal{O} \in \{\text{GIOs, DIOs, MIOs}\}$, we define

$$C_M^{\mathcal{O}}(\rho) = \max_{\Phi \in \mathcal{O}} \text{tr}(\Phi(\rho)M) - \frac{1}{d} \quad (19)$$

for $M \in \Omega$. By the hierarchical relationship between GIOs, DIOs, and MIOs [23], we know that each $C_M^{\mathcal{O}}(\cdot)$ is a coherence measure. Based on this, we actually have the following result.

Theorem 4. *There exists some $\Phi \in \mathcal{O}$ such that*

$$\Phi(\rho) - \Delta(\Phi(\rho)) = \sigma - \Delta(\sigma) \Leftrightarrow C_M^{\mathcal{O}}(\rho) \geq C_M^{\mathcal{O}}(\sigma), \quad (20)$$

for any $M \in \Omega$, here Δ is the dephasing operation defined by $\Delta(\rho) = \sum_{i=1}^d |i\rangle\langle i|\rho|i\rangle\langle i|$.

Imaginaryity as resource is a hot topic and recently receives much attention (see [73] and the references therein). For any coherence measure C ,

$$C(\rho) = C(\rho^*) \quad (21)$$

is an axiomatic assumption proposed in [73] for studying coherence and imaginaryity of quantum states, here ρ^* is the complex conjugate of ρ . The intuition tells us (21) is right. Actually, the author has checked that all existing important coherence measures such as the l_1 -norm of coherence, the relative entropy of coherence [41], the Tsallis relative entropy of coherence [74], the robustness of coherence, the geometric coherence [75], the coherence weight [76], and coherence measures from the convex roof construction [77] satisfying $C(\rho) = C(\rho^*)$. From the point of state convertibility, we need only to prove

$$\rho \xrightarrow{\Phi_1} \rho^*, \rho^* \xrightarrow{\Phi_2} \rho, \quad (22)$$

$\Phi_1, \Phi_2 \in \{\text{GIOs, DIOs, MIOs}\}$. By Theorem 4, we need to check $C_M^{\mathcal{O}}(\rho) = C_M^{\mathcal{O}}(\rho^*)$. However, we can find that $C_M^{\text{GIOs}}(\cdot)$ has a distinguished property

$$C_M^{\text{GIOs}}(\rho) \neq C_M^{\text{GIOs}}(\rho^*) \quad (23)$$

for some ρ and $M \in \Omega$ (see the appendix for an example). This shows the peculiarity of $C_M^{\text{GIOs}}(\cdot)$ and the necessity of assumption $C(\rho) = C(\rho^*)$.

4. Summary and Discussion

Among the most fundamental questions in quantum coherence theory is state convertibility, it is aimed to study whether incoherent operations can introduce an order on the set of coherent states, that is, whether, given two coherent states ρ and σ , either ρ can be transformed into σ or vice versa. Since the question of state convertibility in coherence resource theory is proposed [41], understanding the exact conditions for existence of incoherent transformations between coherent states has attracted a lot of work [23]. In this work, we have determined the exact conditions for coherence conversion under GIOs. Our conditions show that coherence measures from convexity of the robustness of coherence are central. Based on these conditions, maximally incoherent states in a particular set are classified. This induces the majorization condition of determining the convertibility from pure states to mixed states under SIOs. Furthermore, conditions of conversion between off-diagonal parts of coherent states are also characterized. The study of state convertibility for general resource theory has also been discussed recently [50,78].

There still exist some interesting open questions. First, note that the existence proof of our Theorem 1 is not constructive, given two states satisfying coherence order, a problem is how to construct desired GIOs realizing convertibility. Second, can we offer an efficient algorithm to compute $C_M^{\text{GIOs}}(\cdot)$? Note that $C_M^{\text{GIOs}}(\cdot)$ is a generalization of quantum coherence fraction which quantifies the closeness between a given state and the set of maximally coherent states [67]. Therefore an efficient algorithm of $C_M^{\text{GIOs}}(\cdot)$ is also efficient for quantum coherence fraction which is key in the framework of coherence theory.

Proofs of all results in this paper are given in Appendix A.

Before giving the proof of our main results, we firstly recall some fundamental properties of GIOs. In fact, the notion of GIOs is equivalent to the Schur channels [79–81]. Suppose Φ is trace-preserving completely positive maps on density operators, the following statements are equivalent:

- (1) Φ is a GIO, that is, a Schur channel;
- (2) Φ preserves incoherent basis states, that is, $\Phi(|i\rangle\langle i|) = |i\rangle\langle i|$ for all i ;
- (3) For every Kraus representation of $\Phi(\rho) = \sum_j K_j \rho K_j^\dagger$, all Kraus operators $\{K_j\}$ are diagonal;
- (4) Φ can be written as a Schur product form: $\Phi(\rho) = \tau \circ \rho$, where the matrix τ is positive semidefinite such that its diagonals are all equal to 1, and the Schur product is denoted by $\tau \circ \rho = (\tau_{ij} \rho_{ij})$.

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Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data Availability Statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix A. Proof of Our Results

Appendix A.1. Proof of Properties of $C_M^{\text{GIOs}}(\rho)$

$$\begin{aligned} (i) \quad C_M^{\text{GIOs}}(\rho) &= \max_{\Phi \in \text{GIOs}} \text{tr}(\Phi(\rho)M) - \frac{1}{d} \\ &= \max_{\tau \geq 0, \{\tau_{ii}=1\}_{i=1}^d} \text{tr}((\rho_{ij}\tau_{ij})M) - \frac{1}{d} \\ &= \max_{\tau \geq 0, \{\tau_{ii}=1\}_{i=1}^d} \sum_{i \neq j} \rho_{ij}\tau_{ij}M_{ji} \end{aligned}$$

It is evident if $\rho \in \mathbf{I}$, then

$$\rho_{ij} = 0 \quad (1 \leq i \neq j \leq d)$$

and so $C_M^{\text{GIOs}}(\rho) = 0$. Otherwise, choosing $\tau_{ij}(i \neq j)$ such that $\rho_{ij}\tau_{ij}M_{ji} = |\rho_{ij}\tau_{ij}M_{ji}|$ and $\tau_{ii} = 1$, then $C_M^{\text{GIOs}}(\rho) \geq 0$.

(ii) Note that the composition of two GIOs is also a GIO, monotonicity of $C_M^{\text{GIOs}}(\cdot)$ under all GIOs is evident.

(iii) Combining Theorem 1 [65] and $\text{GIOs} \subseteq \text{IOs}$, we get the desired.

(iv) It is easy to check that $C_M^{\text{GIOs}}(\cdot)$ is non-increasing under mixing of quantum states.

(v) By (i),

$$C_M^{\text{GIOs}}(\rho) = \max_{\tau \geq 0, \{\tau_{ii}=1\}_{i=1}^d} \sum_{i \neq j} \rho_{ij}\tau_{ij}M_{ji}.$$

Combining $\tau \geq 0$ with $\tau_{ii} = 1 (1 \leq i \leq d)$, we have $|\tau_{ij}| \leq 1$. Therefore

$$\sum_{i \neq j} \rho_{ij}\tau_{ij}M_{ji} \leq \sum_{i \neq j} |\rho_{ij}| |M_{ji}|.$$

It is evident that

$$C_M^{\text{GIOs}}(\rho) \leq C_{l_1}(\rho) \max_{1 \leq i \neq j \leq d} \{|M_{ij}|\}.$$

Choosing τ as the Lemma 1 [82], a direct computation shows

$$\frac{C_{l_1}(\rho)}{d-1} \min_{1 \leq i \neq j \leq d} \{|M_{ij}|\} \leq C_M^{\text{GIOs}}(\rho).$$

(vi) By the form of M and Theorem 2 [67], (11) and (12) can be obtained.

Proof of Theorem 1. " $\Rightarrow$ " Assume there exists a GIO Φ with $\Phi(\rho) = \sigma$, by the monotonicity of $C_M^{\text{GIOs}}(\rho)$ under GIOs, we have $C_M^{\text{GIOs}}(\sigma) \leq C_M^{\text{GIOs}}(\rho)$. Note that every GIO is a Schur channel, that is, $\Phi(\rho) = \tau \circ \rho$, thus $\rho_{ii} = \sigma_{ii}$ ($i = 1, 2, \dots, d$).

" $\Leftarrow$ " By the definition of $C_M^{\text{GIOs}}(\rho)$, it is easy to see that $C_M^{\text{GIOs}}(\rho) = C_{UMU^\dagger}^{\text{GIOs}}(\rho)$ for any diagonal unitary matrix. Therefore

$$\max_{\Phi \in \text{GIOs}} \text{tr}(\Phi(\rho)UMU^\dagger) \geq \text{tr}(\sigma UMU^\dagger).$$

This implies

$$\min_{M \in \Omega} \max_{\Phi \in \text{GIOs}} \text{tr}((\Phi(\rho) - \sigma)UMU^\dagger) \geq 0,$$

here the optimization is overall convex combination of maximally coherent states. Note that GIOs are compact and convex, and Ω is convex, by the fundamental von Neumann's minimax theorem [83],

$$\max_{\Phi \in \text{GIO}} \min_{M \in \Omega} \text{tr}((\Phi(\rho) - \sigma)UMU^\dagger) \geq 0.$$

Thus there exists a GIO Φ_0 such that

$$\text{tr}((\Phi_0(\rho) - \sigma)UMU^\dagger) \geq 0$$

for all $M \in \Omega$. In particular, we have

$$\text{tr}((\Phi_0(\rho) - \sigma)U|\psi^+\rangle\langle\psi^+|U^\dagger) \geq 0$$

for all diagonal unitary matrices U . In the following, we will show $\Phi_0(\rho) = \sigma$ by the mathematical induction. For the induction step, it is firstly shown that $\Phi_0(\rho) = \sigma$ for a three-level system. Secondly, we deduce an l dimensional system satisfies the assertion by assuming an $l-1$ dimensional system does. Let

$$\Phi_0(\rho) - \sigma = \begin{pmatrix} 0 & a_{12} + ib_{12} & a_{13} + ib_{13} \\ a_{12} - ib_{12} & 0 & a_{23} + ib_{23} \\ a_{13} - ib_{13} & a_{23} - ib_{23} & 0 \end{pmatrix},$$

a_{ij}, b_{ij} are all real numbers, and $U = \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix}$, a direct computation shows

$$\begin{aligned} & \text{Re}(a_{12} + ib_{12})e^{i(\theta_2 - \theta_1)} + \text{Re}(a_{13} + ib_{13})e^{i(\theta_3 - \theta_1)} + \\ & \text{Re}(a_{23} + ib_{23})e^{i(\theta_3 - \theta_2)} \geq 0. \end{aligned}$$

That is

$$\begin{aligned} & a_{12} \cos(\theta_2 - \theta_1) - b_{12} \sin(\theta_2 - \theta_1) + a_{13} \cos(\theta_3 - \theta_1) - \\ & b_{13} \sin(\theta_3 - \theta_1) + a_{23} \cos(\theta_3 - \theta_2) - b_{23} \sin(\theta_3 - \theta_2) \geq 0. \end{aligned}$$

Choosing $\theta_1 = \theta_2 = \theta_3 = 0$, we have

$$a_{12} + a_{13} + a_{23} \geq 0. \quad (\text{A1})$$

Picking $(\theta_1, \theta_2, \theta_3) = (0, \pi, \pi)$, we can obtain

$$a_{23} - a_{12} - a_{13} \geq 0. \quad (\text{A2})$$

Selecting $(\theta_1, \theta_2, \theta_3) = (0, 0, \pi)$, we get

$$a_{12} - a_{13} - a_{23} \geq 0. \quad (\text{A3})$$

Let $(\theta_1, \theta_2, \theta_3) = (0, \pi, 0)$, we have

$$-a_{12} + a_{13} - a_{23} \geq 0. \quad (\text{A4})$$

It is evident that

$$(A1) + (A2) \Rightarrow a_{23} \geq 0,$$

$$(A1) + (A3) \Rightarrow a_{12} \geq 0,$$

$$(A1) + (A4) \Rightarrow a_{13} \geq 0.$$

A direct computation shows that

$$(A2) + (A3) \Rightarrow a_{13} \leq 0,$$

$$(A2) + (A4) \Rightarrow a_{12} \leq 0,$$

$$(A3) + (A4) \Rightarrow a_{23} \leq 0.$$

Therefore $a_{12} = a_{13} = a_{23} = 0$. Analogously, we can also obtain

$$b_{12} = b_{13} = b_{23} = 0,$$

and so $\Phi_0(\rho) = \sigma$.

Let $U = \sum_{i=1}^l e^{i\theta_i} |i\rangle\langle i|$, $\Phi_0(\rho) - \sigma = (a_{ij} + ib_{ij})$, here both a_{ij} and b_{ij} ($1 \leq i \neq j \leq l$) are real numbers. We assume

$$\text{tr}(U^\dagger(\Phi_0(\rho) - \sigma)U|\psi^+\rangle\langle\psi^+|) \geq 0. \quad (A5)$$

A direct computation shows that condition (A5) is equivalent to

$$\sum_{1 \leq i < j \leq l} (a_{ij} \cos(\theta_j - \theta_i) - b_{ij} \sin(\theta_j - \theta_i)) \geq 0. \quad (A6)$$

It is easy to see

$$\begin{aligned} & \sum_{1 \leq i < j \leq l} (a_{ij} \cos(\theta_j - \theta_i) - b_{ij} \sin(\theta_j - \theta_i)) \\ &= \sum_{1 \leq i < j \leq l-1} (a_{ij} \cos(\theta_j - \theta_i) - b_{ij} \sin(\theta_j - \theta_i)) + \\ & \sum_{1 \leq i < l} (a_{il} \cos(\theta_l - \theta_i) - b_{il} \sin(\theta_l - \theta_i)) \geq 0. \end{aligned}$$

By the arbitrariness of θ_l , substituting θ_l for $\pi + \theta_l$, we have

$$\begin{aligned} & \sum_{1 \leq i < j \leq l-1} (a_{ij} \cos(\theta_j - \theta_i) - b_{ij} \sin(\theta_j - \theta_i)) - \\ & \sum_{1 \leq i < l} (a_{il} \cos(\theta_l - \theta_i) - b_{il} \sin(\theta_l - \theta_i)) \geq 0. \end{aligned}$$

Therefore

$$\sum_{1 \leq i < j \leq l-1} (a_{ij} \cos(\theta_j - \theta_i) - b_{ij} \sin(\theta_j - \theta_i)) \geq 0.$$

By our induction, we have $a_{ij} = b_{ij} = 0$ ($1 \leq i \neq j \leq l-1$). This implies

$$\sum_{1 \leq i < l} (a_{il} \cos(\theta_l - \theta_i) - b_{il} \sin(\theta_l - \theta_i)) = 0. \quad (A7)$$

Choosing $\theta_1 = \theta_2 = \dots = \theta_l = 0$ in (A7), we can obtain

$$\sum_{1 \leq i < l} a_{il} = 0. \quad (A8)$$

Picking $\theta_1 = \pi, \theta_2 = \theta_3 = \dots = \theta_{l-1} = 0, \theta_l = \pi$ in (A7), one has

$$a_{1l} - \sum_{2 \leq i < l} a_{il} = 0. \quad (A9)$$

It is evident that

$$(A8) + (A9) \Rightarrow a_{1l} = 0.$$

Similarly, $a_{2l} = a_{3l} = \dots = a_{l-1l} = 0$, and so

$$\sum_{1 \leq i < l} b_{il} \sin(\theta_l - \theta_i) = 0 \quad (\text{A10})$$

from (A7). Using analogous treatments, we also have $b_{1l} = b_{2l} = \dots = b_{l-1l} = 0$. This completes the proof. $\square$

The proof of Theorem 2 depends on Theorem 3, so we firstly give the proof of Theorem 3.

Proof of Theorem 3. Assume $\rho = (\rho_{ij})$, by Theorem 1, we need only to prove

$$C_M^{\text{GIOs}}(\rho) \leq C_M^{\text{GIOs}}(|\psi\rangle\langle\psi|), \quad |\psi\rangle = \sum_{i=1}^d \sqrt{\rho_{ii}} |i\rangle.$$

From the proof of property (i) of $C_M^{\text{GIOs}}(\rho)$, we have

$$C_M^{\text{GIOs}}(\rho) = \max_{\tau \geq 0, \{\tau_{ii}=1\}_{i=1}^d} \sum_{1 \leq i \neq j \leq d} \tau_{ij} \rho_{ij} M_{ji}.$$

Similarly

$$C_M^{\text{GIOs}}(|\psi\rangle\langle\psi|) = \max_{\tau \geq 0, \{\tau_{ii}=1\}_{i=1}^d} \sum_{1 \leq i \neq j \leq d} \tau_{ij} \sqrt{\rho_{ii} \rho_{jj}} M_{ji}.$$

We divide the proof into two cases.

Case 1. All $\rho_{ii} \neq 0$ ($i = 1, 2, \dots, d$).

Write

$$C_M^{\text{GIOs}}(\rho) = \max_{\tau \geq 0, \{\tau_{ii}=1\}_{i=1}^d} \sum_{1 \leq i \neq j \leq d} \tau_{ij} \frac{\rho_{ij}}{\sqrt{\rho_{ii} \rho_{jj}}} \sqrt{\rho_{ii} \rho_{jj}} M_{ji},$$

then the (i, j) position of $\tau \circ \tau_0$ is $\tau_{ij} \frac{\rho_{ij}}{\sqrt{\rho_{ii} \rho_{jj}}}$, here

$$\tau_0 = \begin{pmatrix} 1 & \frac{\rho_{12}}{\sqrt{\rho_{11} \rho_{22}}} & \dots & \frac{\rho_{1d}}{\sqrt{\rho_{11} \rho_{dd}}} \\ \frac{\rho_{21}}{\sqrt{\rho_{11} \rho_{22}}} & 1 & \dots & \frac{\rho_{2d}}{\sqrt{\rho_{22} \rho_{dd}}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\rho_{d1}}{\sqrt{\rho_{11} \rho_{dd}}} & \frac{\rho_{d2}}{\sqrt{\rho_{22} \rho_{dd}}} & \dots & 1 \end{pmatrix},$$

and $\circ$ denotes the Schur product. Note that

$$\tau_0 = \begin{pmatrix} \frac{1}{\rho_{11}} & \frac{1}{\sqrt{\rho_{11} \rho_{22}}} & \dots & \frac{1}{\sqrt{\rho_{11} \rho_{dd}}} \\ \frac{1}{\sqrt{\rho_{11} \rho_{22}}} & \frac{1}{\rho_{22}} & \dots & \frac{1}{\sqrt{\rho_{22} \rho_{dd}}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{\rho_{11} \rho_{dd}}} & \frac{1}{\sqrt{\rho_{22} \rho_{dd}}} & \dots & \frac{1}{\rho_{dd}} \end{pmatrix} \circ \rho \geq 0,$$

this is due to the fact the Schur product of two positive semidefinite is also positive semidefinite [84]. Hence

$$\tau \circ \tau_0 \geq 0.$$

This implies

$$C_M^{\text{GIOs}}(\rho) \leq C_M^{\text{GIOs}}(|\psi\rangle\langle\psi|).$$

Case 2. $\rho_{ii} = 0$ for some i .

For clarity, we firstly treat the qutrit case with $\rho_{22} = 0$. It is easy to see $\rho_{12} = \rho_{23} = 0$. Choosing

$$\tau_0 = \begin{pmatrix} 1 & 0 & \frac{\rho_{13}}{\sqrt{\rho_{11} \rho_{33}}} \\ 0 & 1 & 0 \\ \frac{\rho_{31}}{\sqrt{\rho_{11} \rho_{33}}} & 0 & 1 \end{pmatrix},$$

the (1, 3) and (3, 1) positions of $\tau \circ \tau_0$ have desired property as the case 1. Hence

$$C_M^{\text{GIOs}}(\rho) \leq C_M^{\text{GIOs}}(|\psi\rangle\langle\psi|).$$

For the general case, we can choose τ_0 as follows:

- (1) Non-diagonal elements of the i th row and the i th column are all 0;
- (2) All diagonal elements are 1;
- (3) Other entries are defined as the case 1.

It is easy to check that such τ_0 has the property as the case 1. Therefore

$$C_M^{\text{GIOs}}(\rho) \leq C_M^{\text{GIOs}}(|\psi\rangle\langle\psi|).$$

Based on Theorem 3, we can prove Theorem 2. $\square$

Proof of Theorem 2. Assume

$$(|\psi_1|^2, \dots, |\psi_d|^2)^t \prec (\sigma_{11}, \dots, \sigma_{dd})^t,$$

then there exists a SIO Φ_1 such that [46]

$$\Phi_1(|\psi\rangle\langle\psi|) = |\eta\rangle\langle\eta|, |\eta\rangle = \sum_{i=1}^d \sqrt{\sigma_{ii}} |i\rangle$$

By Theorem 3, there exists some GIO Φ_2 such that $\Phi_2(|\eta\rangle\langle\eta|) = \sigma$. Let Φ be the composition of Φ_1 and Φ_2 , it is easy to see that Φ is an SIO and $\Phi(|\psi\rangle\langle\psi|) = \sigma$. $\square$

Proof of Theorem 4. By the compactness of DIO and MIO, the sufficiency can be followed from the proof of Theorem 3.1. For the necessity, we claim that

$$\Phi^\dagger(M)_{ii} = \frac{1}{d}$$

for $\Phi \in \text{MIO}$. Indeed, for arbitrary state τ , we have

$$\text{tr}(\Phi^\dagger(M)\Delta(\tau)) = \text{tr}(M\Phi(\Delta(\tau))) = \frac{1}{d}.$$

Thus $\Phi^\dagger(M)_{ii} = \frac{1}{d}$. Now, assume that

$$\Phi_0(\rho) - \Delta(\Phi_0(\rho)) = \sigma - \Delta(\sigma)$$

for some $\Phi_0 \in \mathcal{O}$. Then

$$\begin{aligned} C_M^{\mathcal{O}}(\rho) &\geq C_M^{\mathcal{O}}(\Phi_0(\rho)) \\ &= \max_{\Phi \in \mathcal{O}} \text{tr}(\Phi(\Phi_0(\rho))M) - \frac{1}{d} \\ &= \max_{\Phi \in \mathcal{O}} \text{tr}(\Phi_0(\rho)\Phi^\dagger(M)) - \frac{1}{d} \\ &= \max_{\Phi \in \mathcal{O}} \text{tr}(\sigma\Phi^\dagger(M)) - \frac{1}{d} = C_M^{\mathcal{O}}(\sigma). \end{aligned}$$

In the following, we give an example to show

$$C_M^{\text{GIOs}}(\rho) \neq C_M^{\text{GIOs}}(\rho^*).$$

$\square$

Example A1. Taking

$$M = \frac{1}{3}U_1|\psi^+\rangle\langle\psi^+|U_1^\dagger + \frac{2}{3}U_2|\psi^+\rangle\langle\psi^+|U_2^\dagger,$$

here $U_1 = \text{diag}(1, \frac{3+4i}{5}, 1)$ and $U_2 = \text{diag}(1, 1, \frac{3+4i}{5})$. That is $M = (M_{ij})$ with

$$\begin{aligned} M_{12} &= \frac{13}{45} - \frac{4}{45}i, \\ M_{13} &= \frac{11}{45} - \frac{8}{45}i, \\ M_{23} &= \frac{1}{5} - \frac{4}{45}i, \\ M_{11} &= M_{22} = M_{33} = 1. \end{aligned}$$

One can check that

$$\begin{aligned} C_M^{GLOs}(M) &= \max_{\Phi \in GLOs} \text{tr}(\Phi(M)M) - \frac{1}{3} \\ &= \max_{\tau \geq 0, \tau_{ii}=1} \sum_{i \neq j} \tau_{ij} |M_{ji}|^2 = \sum_{i \neq j} |M_{ji}|^2. \\ C_M^{GLOs}(M^*) &= \max_{\Phi \in GLOs} \text{tr}(\Phi(M^*)M) - \frac{1}{3} \\ &= \max_{\tau \geq 0, \tau_{ii}=1} \sum_{i \neq j} \tau_{ij} (M_{ji})^2 \\ &< \sum_{i \neq j} |M_{ji}|^2. \end{aligned}$$

The last strict inequality holds true because $\max_{\tau \geq 0, \tau_{ii}=1} \sum_{i \neq j} \tau_{ij} (M_{ji})^2 \leq \sum_{i \neq j} |M_{ji}|^2$ and the equation holds true iff $(M_{ji}^2) = U(|M_{ji}|^2)U^\dagger$ for some diagonal unitary U . To show this, one only need a direct computation saying $(M_{12}M_{23}\overline{M_{13}})^2$ is not a real number.

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