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Production-inventory strategies for a linear reverse logistics system

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Abstract

The paper investigates an one-product reverse logistics system. The aim is to analyze the functioning of such a system with continuous time and linear cost structure. The costs consist of linear holding costs for both warehouse, and the linear remanufacturing, manufacturing, and disposal costs. The demand and return rates are known deterministic parameters. The problem is solved with the Pontyagin's maximum principle. Solving the control problem, it is corrected an earlier paper [1].

Mathematics Subject Classifications (2015). 90, 93

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1 Introduction

Reverse logistics is a term for manufacturing of materials and remanufacturing of from market returned reusable materials. The demand is to be satisfied with new manufactured (produced) and the remanufactured products, so there is no difference between manufactured and remanufactured items. In order the environment with waste not to burden, or the use of minerals to spare, it is a good opportunity to reuse the used items, if it is economical.

There are three types of deterministic reverse logistics models. EOQ-type reverse logistics models were studied extensively in the literature [2]-[9]. In this models the manufacturing and remanufacturing activities succeed each other, and the measure of effectiveness of such systems is the sum average setup and linear holding costs. The main question is the number of the manufacturing and remanufacturing batches. All these approaches assumed a predetermined control system and searched the optimal parameters for this control systems. There are no results regarding the structure of optimal policies [10].

A second research direction is the "dynamization" of the on classical EOQ based systems. On the basis of the Wagner-Whitin lot-size model, it can be

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generalized with two stores, and for the second store a "reverse" Wagner-Within model is to be introduced [11]. This research shows that either manufacturing or remanufacturing occur in a point of time.

The third possibility for generalization of the deterministic one-product reverse logistics system gives the optimal control theory. Optimal control characterizes the optimal trajectory in time. This direction supplies qualitative informations about the optimal path and decision rules in time. There are a linear [1] and a convex [12] contribution to this field.

The aim of the paper is to investigate the linear reverse logistics model and to correct of the results of [1]. The paper organizes as follows. The model is presented in Section 2. The next supplies the necessary and sufficient conditions for optimality, and gives the explicite solution of the model for manufacturing, remanufacturing and disposal and the inventory status for both stores. This section analyzes the dependence of the optimal strategy on the cost parameters. And last we summarize the results

2 The model

We will investigate a two-store reverse logistics model with continuous disposal. The model can be represented as an optimal control problem with two state variables (inventory status in the first and second store) and with three control variables (rate of manufacturing, remanufacturing and disposal). The objective is to minimize the sum of the holding costs in the stores and costs of the manufacturing, remanufacturing and disposal. The following parameters are in our model:

- T: length of the planning horizon,
- $S(t)$: the rate of demand, continuous differentiable,
- r: the return rate in the second store ($0 \leq r \leq 1$),
- τ: the delay of the return, nonnegative,
- $R(t)$: the rate of return, continuous differentiable,
- h_1: the inventory holding costs in the first store,
- h_2: the inventory holding costs in the second store,
- p_m: production cost for the remanufacturing,
- p_r: production cost for the remanufacturing,
- p_d: production cost for the disposal.

Decision variables:

- $I_1(t)$: inventory level in the first store, non-negative,

- $I_2(t)$: inventory level in the second store, non-negative,
- $P_r(t)$: rate of the remanufacturing, non-negative,
- $P_m(t)$: rate of the manufacturing (production), non-negative,
- $P_d(t)$: rate of the disposal, non-negative.

The functioning and the cost flow of the system is presented in Figure 1. The customers demand (outflow) is satisfied from the store 1. The manufactured and remanufactured items are transported to store 1 (inflow). The returned items are collected (inflow) in store 2, in order to either remanufacture or dispose of (outflow). The subsystem store 2 can be called as reverse production-inventory system. It is assumed that not all items return, and the return process occurs with a fix delay.

The model can be written in the following form:

$$\int_0^T \left[h_1 I_1(t) + h_2 I_2(t) + p_m P_m(t) + p_r P_r(t) + p_d P_d(t)\right] dt \to \min \tag{1}$$

such that

$$\begin{bmatrix} \dot{I}_1(t) \\ \dot{I}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} P_m(t) \\ P_r(t) \\ P_d(t) \end{bmatrix} + \begin{bmatrix} -S(t) \\ R(t) \end{bmatrix}, \begin{bmatrix} I_1(0) \\ I_2(0) \end{bmatrix} = \begin{bmatrix} I_{10} \\ 0 \end{bmatrix}, \tag{2}$$

$$\begin{bmatrix} I_1(t) \\ I_2(t) \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{3}$$

$$\begin{bmatrix} P_m(t) \\ P_r(t) \\ P_d(t) \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4}$$

It is assumed that the initial inventory level in store 2 is equal to zero. If not, then there exists no solution, as it is shown in Section 3.

3 Auxiliary problems

In this section we will investigate three types of model. First a model is analyzed without manufacturing and disposal, then we generalize this ground model in two directions: the ground model with manufacturing and disposal. These problems help us to understand the functioning of the problem, and they are very useful in constructing the optimal trajectory.

3.1 The linear remanufacturing model without manufacturing and disposal

The model can be described as follows

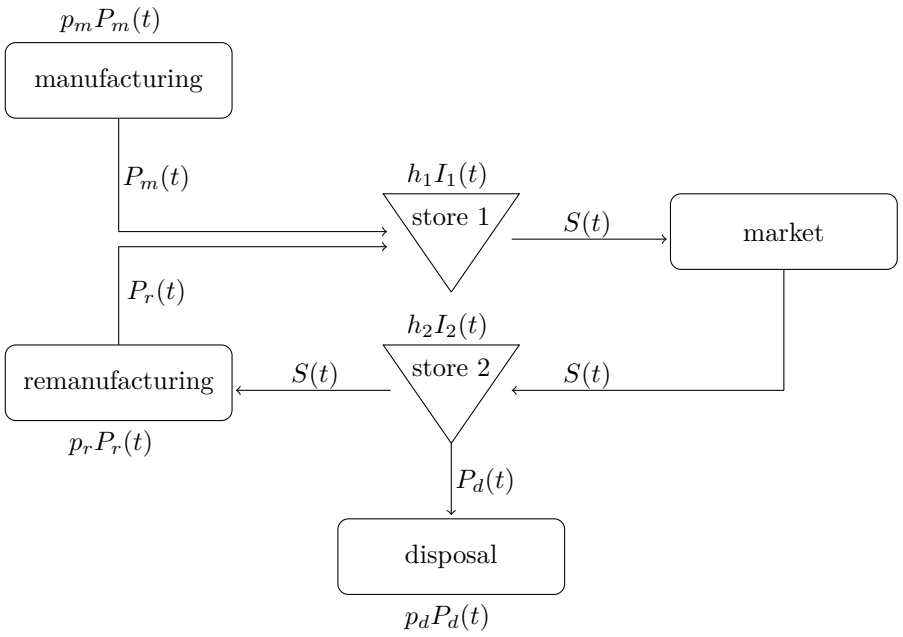


Figure 1: The material and cost flow of the model

$$\int_{t_1}^{t_2} [h_1 I_1(t) + h_2 I_2(t) + p_r P_r(t)]\, dt \to \min \tag{5}$$

such that

$$\dot{I}_1(t) = P_r(t) - S(t), \quad I_1(t_1) = 0 \ ,$$
$$\dot{I}_2(t) = -P_r(t) + R(t), \quad I_2(t_1) = 0 \ ,$$
$$I_1(t) \geq 0, \quad I_2(t) \geq 0 \ ,$$
$$P_r(t) \geq 0 \ .$$

It is intuitively clear that a solution to this problem exists if only

$$\int_{t_1}^{t} R(\tau) d\tau \geq \int_{t_1}^{t} S(\tau) d\tau, \quad \forall t \in [t_1, t_2]$$

which shows that the cumulated (returned) remanufacturing items must be greater than the cumulated demand.

This problem can be simplified in two ways in dependence of the linear inventory holding tags, and with the nonnegative excess function

$$\widetilde{I}(t) = \int_{t_1}^{t} (R(\tau) - S(\tau)) d\tau, \quad t \in [t_1, t_2] \ .$$

Case 1.: $h_1 \geq h_2$

$$\dot{I}_1(t) = P_r(t) - S(t), \quad I_1(t_1) = 0 \ ,$$
$$0 \leq I_1(t) \leq \widetilde{I}(t) \ ,$$
$$P_r(t) \geq 0$$

.

The problem is a classical "direct" linear production-inventory model with warehousing constraint. In linear case the warehousing constraint is always inactive, so the optimal solution for this case is

$$\left.\begin{array}{l} I_1^0(t) = 0 \\ I_2^0(t) = \widetilde{I}(t) \\ P_r^0(t) = S(t) \end{array}\right\} \quad t \in [t_1, t_2]$$

Case 2.: $h_1 < h_2$

The problem described below.

$$\int_{t_1}^{t_2} \left[(h_2 - h_1)\, I_2(t) + p_r P_r(t)\right] dt + \int_{t_1}^{t_2} h_1 \widetilde{I}(t) dt \to \min$$

such that

$$\dot{I}_2(t) = -P_r(t) + R(t), \quad I_2(t_1) = 0 \ ,$$
$$0 \leq I_2(t) \leq \widetilde{I}(t) \ ,$$
$$P_r(t) \geq 0 \ .$$

This problem can be called, as the "reverse" production-inventory problem with warehousing constraint. In this model the inflow (return rate) is given, and the outflow (remanufacturing) is decision variable. The solution to this problem is in Appendix 1.

As we see, the optimal inventory level is equal to zero and the warehousing constraint is inactive. The optimal remanufacturing rate is positive, there is no interval, where the remanufacturing rate is equal to zero. It shows us, that there exists no optimal solution, if the initial inventory level is strict greater than zero, so $I_{20} = I(t_1) = 0$.

3.2 The linear remanufacturing model with manufacturing and without disposal

The model can be described as follows

$$\int_{t_1}^{t_2} \left[h_1 I_1(t) + h_2 I_2(t) + p_m P_m(t) + p_r P_r(t)\right] dt \to \min$$

such that

$$\dot{I}_1(t) = P_m(t) + P_r(t) - S(t), \quad I_1(t_1) = 0$$

,

$$\dot{I}_2(t) = -P_r(t) + R(t), \quad I_2(t_1) = 0 \ ,$$
$$I_1(t) \geq 0, \quad I_2(t) \geq 0 \ ,$$
$$P_m(t) \geq 0, \quad P_r(t) \geq 0 \ .$$

This problem can be simplified in two ways in dependence of the linear inventory holding tags, and with the sum of the inventory levels

$$\widetilde{I}(t) = I_1(t) + I_2(t), \quad t \in [t_1, t_2]$$

Two new problems are presented below. To solve the problem, we use the results of Appendix 2. The auxiliary problem of Appendix 2. plays an important role in solution of the following problem.

Case 1.: $h_1 \geq h_2$

$$\int_{t_1}^{t_2} \left[(h_1 - h_2) I_1(t) + h_2 \widetilde{I}(t) + p_m P_m(t) + p_r P_r(t) \right] dt \to \min$$

such that

$$\dot{I}_1(t) = P_m(t) + P_r(t) - S(t), \quad I_1(t_1) = 0 \ ,$$
$$\dot{\tilde{I}}(t) = P_m(t) - (S(t) - R(t)), \quad \tilde{I}(t_1) = 0 \ ,$$
$$0 \leq I_1(t) \leq \tilde{I}(t) \ ,$$
$$P_m(t) \geq 0, \quad P_r(t) \geq 0 \ .$$

This case consists of two subproblems: subproblem as in Appendix 2. and the following subproblem

$$\int_{t_1}^{t_2} \left[(h_1 - h_2) I_1(t) + p_r P_r(t) \right] dt \to \min$$

such that

$$\dot{I}_1(t) = P_r(t) - \left(S(t) - P_m^0(t) \right), \quad I_1(t_1) = 0 \ ,$$
$$0 \leq I_1(t) \leq \tilde{I}(t) \ ,$$
$$P_r(t) \geq 0 \ .$$

This subproblem is easy to solve, because the optimal manufacturing rate is not higher than the demand rate $S(t)$, so it is a classical linear problem. The optimal solution for this case is

$$I_1^0(t) = 0$$
$$I_2^0(t) = \begin{cases} 0 & t \in [\tau_{i-1} + \varepsilon_{i-1}, \tau_i) \\ \int_{\tau_i}^{t} (R(\tau) - S(\tau)) d\tau & t \in [\tau_i, \tau_i + \varepsilon_i) \end{cases}$$
$$P_m^0(t) = \begin{cases} S(t) - R(t) & t \in [\tau_{i-1} + \varepsilon_{i-1}, \tau_i) \\ 0 & t \in [\tau_i, \tau_i + \varepsilon_i) \end{cases}$$
$$P_r^0(t) = S(t) - P_m^0(t)$$

So we have solved this case.

Case 2.: $h_1 < h_2$

$$\int_{t_1}^{t_2} \left[h_1 \widetilde{I}(t) + (h_2 - h_1)\, I_2(t) + p_m P_m(t) + p_r P_r(t) \right] dt \to \min$$

such that

$$\dot{\tilde{I}}(t) = P_m(t) - (S(t) - R(t)), \quad \tilde{I}\,(t_1) = 0\ ,$$
$$\dot{I}_2(t) = -P_r(t) + R(t), \quad I_2\,(t_1) = 0\ ,$$
$$0 \le I_2(t) \le \tilde{I}(t)\ ,$$
$$P_m(t) \ge 0, \quad P_r(t) \ge 0\ .$$

The solution of this case is very easy, because it consists of two independent solved subproblem. The warehousing constraints are inactive, so the solution is

$$I_1^0(t) = \begin{cases} 0 & t \in [\tau_{i-1} + \varepsilon_{i-1}, \tau_i) \\ \int_{\tau_i}^{t} (R(\tau) - S(\tau)) d\tau & t \in [\tau_i, \tau_i + \varepsilon_i) \end{cases}$$
$$I_2^0(t) = 0$$
$$P_m^0(t) = \begin{cases} S(t) - R(t) & t \in [\tau_{i-1} + \varepsilon_{i-1}, \tau_i) \\ 0 & t \in [\tau_i, \tau_i + \varepsilon_i) \end{cases}$$
$$P_r^0(t) = R(t)$$

As we see, in the optimal solution the initial inventory level is equal to zero, otherwise there exists no solution.

3.3 The linear remanufacturing model without manufacturing and with disposal

The model can be described as follows

$$\int_{t_1}^{t_2} \left[h_1 I_1(t) + h_2 I_2(t) + p_r P_r(t) + p_d P_d(t) \right] dt \to \min$$

such that

$$\dot{I}_1(t) = P_r(t) - S(t), \quad I_1\,(t_1) = 0\ ,$$
$$\dot{I}_2(t) = -P_r(t) - P_d(t) + R(t), \quad I_2\,(t_1) = 0\ ,$$
$$I_1(t) \ge 0, \quad I_2(t) \ge 0\ ,$$
$$P_r(t) \ge 0, \quad P_d(t) \ge 0\ .$$

A solution to this problem exists if only

$$\int_{t_1}^{t} R(\tau) d\tau \ge \int_{t_1}^{t} S(\tau) d\tau, \quad \forall t \in [t_1, t_2]$$

which shows that the cumulated (returned) remanufacturing items must be greater than the cumulated demand.

This problem can be simplified as for the model without manufacturing and disposal

$$\tilde{I}(t) = I_1(t) + I_2(t), \quad t \in [t_1, t_2] \ .$$

Two new problems are presented below. To solve the problem, we discuss an auxiliary problem in Appendix 3.

3.3.1 Case 1.: $h_1 \geq h_2$

$$\int_{t_1}^{t_2} \left[(h_1 - h_2)\, I_1(t) + h_2 \widetilde{I}(t) + p_r P_r(t) + p_d P_d(t) \right] dt \to \min$$

such that

$$\dot{I}_1(t) = P_r(t) - S(t), \quad I_1(t_1) = 0 \ ,$$

$$\dot{\tilde{I}}(t) = -P_d(t) + (R(t) - S(t)), \quad \tilde{I}(t_1) = 0 \ ,$$

$$0 \leq I_1(t) \leq \tilde{I}(t) \ ,$$

$$P_r(t) \geq 0, \quad P_d(t) \geq 0 \ .$$

This case consists of two subproblems: subproblem of Appendix 3. and the following subproblem

$$\int_{t_1}^{t_2} \left[(h_1 - h_2)\, I_1(t) + p_r P_r(t) \right] dt \to \min$$

such that

$$\dot{I}_1(t) = P_r(t) - S(t), \quad I_1(t_1) = 0 \ ,$$

$$0 \leq I_1(t) \leq \tilde{I}(t) \ ,$$

$$P_r(t) \geq 0 \ .$$

This subproblem is easy to solve, because the optimal manufacturing rate is not higher than the demand rate $S(t)$, so it is a classical linear problem. The optimal solution for this case is

$$I_1^0(t) = 0$$

$$I_2^0(t) = \begin{cases} 0 & t \in [\tau_{i-1}, \tau_i - \varepsilon_i) \\ \int_{\tau_i - \varepsilon_j}^{t} (S(\tau) - R(\tau)) d\tau & t \in [\tau_i - \varepsilon_i, \tau_i) \end{cases}$$

$$P_r^0(t) = S(t)$$

$$P_d^0(t) = \begin{cases} R(t) - S(t) & t \in [\tau_{i-1}, \tau_i - \varepsilon_i) \\ 0 & t \in [\tau_i - \varepsilon_i, \tau_i) \end{cases}$$

So we have solved this case.

3.3.2 Case 2.: $h_1 < h_2$

$$\int_{t_1}^{t_2} \left[h_1 \widetilde{I}(t) + (h_2 - h_1)\, I_2(t) + p_r P_r(t) + p_d P_d(t) \right] dt \to \min$$

such that

$$\dot{\tilde{I}}(t) = -P_d(t) + (R(t) - S(t)), \quad \tilde{I}\,(t_1) = 0\ ,$$

$$\dot{I}_2(t) = -P_r(t) - P_d(t) + R(t), \quad I_2\,(t_1) = 0\ ,$$

$$0 \le I_2(t) \le \tilde{I}(t)\ ,$$

$$P_r(t) \ge 0, \quad P_d(t) \ge 0\ .$$

The solution of this case is very easy, because it consists of two independent solved subproblem. The warehousing constraints are inactive, so the solution is

$$I_1^0(t) = \begin{cases} 0 & t \in [\tau_{i-1}, \tau_i - \varepsilon_i) \\ \int_{\tau_i - \varepsilon_i}^{t} (R(\tau) - S(\tau)) d\tau & t \in [\tau_i - \varepsilon_i, \tau_i) \end{cases}$$

$$I_2^0(t) = 0$$

$$I_1^0(t) = \begin{cases} 0 & t \in [\tau_{i-1}, \tau_i - \varepsilon_i) \\ \int_{\tau_i - \varepsilon_i}^{t} (R(\tau) - S(\tau)) d\tau & t \in [\tau_i - \varepsilon_i, \tau_i) \end{cases}$$

$$I_2^0(t) = 0$$

$$P_r^0(t) = S(t) t \in [\tau_{i-1}, \tau_i - \varepsilon_i)\, R(t), \qquad t \in [\tau_i - \varepsilon_i, \tau_i)$$

$$P_d^0(t) = \begin{cases} R(t) - S(t) & t \in [\tau_{i-1}, \tau_i - \varepsilon_i) \\ 0 & t \in [\tau_i - \varepsilon_i, \tau_i) \end{cases}$$

As we see, in the optimal solution the initial inventory level is equal to zero, otherwise there exists no solution.

4 Some characteristics of the model

Let us first analyze the model. We investigate the model in dependence of the cost parameters. The cost function can be reformulated in the following way:

$$\int_0^T [h_1 I_1(t) + h_2 I_2(t) + p_m(P_m(t) + P_r(t)) + p_d(P_d(t) + P_r(t)) + \\ + (p_r - p_m - p_d) P_r(t)] dt$$

With this reformulation we can characterize the optimal solution.

LEMMA 1 *If $p_r > p_m + p_d$, then it is optimal not to remanufacture.*

Proof. The proof is obvious. We can see that it is optimal not to remanufacture, because the cost cofficient of the remanufacturing rate is positive. In this case two problems are independent for the two stores, and optimal solution is not to store in the first and second warehouse. The independent problems are for store 1

$$\int_0^T [h_1 I_1(t) + p_m P_m(t)]\, dt \to \min$$

such that

$$\dot{I}_1(t) = P_m(t) - S(t), \quad I_1(0) = I_{10}$$
$$I_1(t) \geq 0$$
$$P_m(t) \geq 0$$

and for the store 2

$$\int_0^T [h_2 I_2(t) + p_d P_d(t)]\, dt \to \min$$

so that

$$\dot{I}_2(t) = -P_d(t) + R(t), \quad I_2(0) = I_{20}$$
$$I_2(t) \geq 0$$
$$P_d(t) \geq 0$$

□

This second problem can be called, as the "reverse" production-inventory problem. (See Appendix 1.) In this model the inflow (return rate) is given, and the outflow (disposal or remanufacturing) is decision variable. The solution of the first problem is known from the production-inventory literature.

The optimal inventory levels are equal to zero. The optimal disposal (remanufacturing) is positive, there is no an interval, where the disposal rate is equal to zero. It shows us, that there exists no optimal solution, if the initial inventory level is strict greater than zero. In the following we assume that the initial inventory level in the second store is zero, otherwise there is no optimal solution.

Now we can define the optimal solution for this cost coefficient structure. Let us define the following point of time:

$$t_1 = \left\{ t \mid I_{10} = \int_0^t S(t)dt \right\}$$

The optimal solution is for the manufacturing, remanufacturing and disposal rate

$$P_m^0(t) = \begin{cases} 0 & 0 \leq t < t_1 \\ S(t) & t_1 \leq t \leq T \end{cases}$$

$$\left.\begin{array}{l} P_r^0(t) = 0 \\ P_d^0(t) = R(t) \end{array}\right\} \quad 0 \leq t \leq T$$

and the inventory status

$$I_1^0(t) = \begin{cases} I_{10} - \int_0^t S(\tau)d\tau & 0 \le t < t_1 \\ 0 & t_1 \le t \le T \end{cases}$$

$$I_2^0(t) = 0 \quad 0 \le t \le T$$

This optimal policy can be called, as no-inventory policy.

Another case is

LEMMA 2 *If $p_r = p_m + p_d$, then the optimal solution is*

$$P_r^0(t) = \begin{cases} 0 & 0 \le t < t_1 \\ \widetilde{P}(t) & 0 \le \widetilde{P}(t) \le \min\{S(t), R(t)\}, \quad t_1 \le t \le T \end{cases}$$

$$P_m^0(t) = \begin{cases} 0 & 0 \le t < t_1 \\ S(t) - P_r^0(t) & t_1 \le t \le T \end{cases}$$

$$P_d(t) = R(t) - P_r^0(t) \quad 0 \le t \le T$$

and the inventory status

$$I_1^0(t) = \begin{cases} I_{10} - \int_0^t S(\tau)d\tau & 0 \le t < t_1 \\ 0 & t_1 \le t \le T \end{cases}$$

$$I_2^0(t) = 0 \quad 0 \le t \le T$$

Proof. The proof is obvious. Studying the cost function, we have obtained a similar problem, as investigated in Lemma 1. The difference is that the remanufacturing rate varies between an upper and a lower bound.

This is a no-inventory policy, as well, but the possible rates of manufacturing, remanufacturing and disposal are a set in dependence of the remanufacturing rate. □

In the next investigations we assume that $p_r < p_m + p_d$, with which we neglect the cases with special cost structure. This case can be called, as general case. The following lemmas explain, when manufacturing or disposal is not necessary to satisfy the demand.

LEMMA 3 *If $I_{10} + \int_0^t (R(\tau) - S(\tau))d\tau \ge 0$, $\forall t \in [0, T]$, then the manufacturing rate is equal to zero in the planning horizon.*

Proof. To prove the lemma, we use the set of possible manufacturing, remanufacturing and disposal rates:

$$I_{10} + \int_0^t P_m(\tau)d\tau + \int_0^t P_r(\tau)d\tau - \int_0^t S(\tau)d\tau \ge 0$$

and

$$-\int_0^t P_d(\tau)d\tau - \int_0^t P_r(\tau)d\tau + \int_0^t R(\tau)d\tau \ge 0.$$

These two inequalities come the integral form of the differential equations. From these it follows

$$\int_0^t S(\tau)d\tau - I_{10} - \int_0^t P_m(\tau)d\tau \leq \int_0^t P_r(\tau)d\tau \leq -\int_0^t P_d(\tau)d\tau + \int_0^t R(\tau)d\tau.$$

If we sum up the inequalities

$$\int_0^t S(\tau)d\tau - I_{10} \leq \int_0^t P_r(\tau)d\tau \leq \int_0^t R(\tau)d\tau$$

we have achieved the condition of the lemma. □

The result of this lemma can be interpreted, as manufacturing is not needed, if the sum of initial inventory level for the end product and the returned items is not lower than the cumulated demand in the planning horizon, i.e.

$$I_{10} + \int_0^t R(\tau)d\tau \geq \int_0^t S(\tau)d\tau \ .$$

Another meaning of the lemma is that in such a cases there is disposal activity, as it is considered in Section 3.2. The next lemma gives condition for no disposal activity.

LEMMA 4 *If* $I_{10} + \int_0^T (R(\tau) - S(\tau))d\tau \leq 0$*, then the disposal rate is equal to zero in the planning horizon.*

Proof. The proof of the lemma is obvious. If we all returned items remanufacture in the planning horizon, and the demand cannot be satisfied with them, then the disposal activity increases the costs. The disposed items must be manufactured moreover the residue of the not satisfied demand. The solution for this case is given in Section 3.3. □

In the following section we give the general solution to this problem, without any special cost and system structure, i.e. it is assumed that the results of the lemmas are tested before.

5 Solution of the model

To solve the problem, we apply the Pontryagin's maximum principle. In order to optimize, we must introduce two auxiliary function: Hamilton function and Lagrange function. The Lagrange function is an extension of the Hamilton function with the inventory constraints.

Hamilton function

$$H\left(I_1(t), I_2(t), P_m(t), P_r(t), P_r(t), \psi_1(t), \psi_2(t), t\right) =$$

$$-\left[\begin{bmatrix} h_1 & h_2 \end{bmatrix}\begin{bmatrix} I_1(t) \\ I_2(t) \end{bmatrix} + \begin{bmatrix} p_m & p_r & p_d \end{bmatrix}\begin{bmatrix} P_m(t) \\ P_r(t) \\ P_d(t) \end{bmatrix}\right]$$

$$+\begin{bmatrix} \psi_1(t) & \psi_2(t) \end{bmatrix}\left\{\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} P_m(t) \\ P_r(t) \\ P_d(t) \end{bmatrix} + \begin{bmatrix} -S(t) \\ R(t) \end{bmatrix}\right\}$$

The next following Lagrange function is a useful tool for construction of adjoint variables.

Lagrange function

$$L\left(I_1(t), I_2(t), P_m(t), P_r(t), P_d(t), \psi_1(t), \psi_2(t), t\right) =$$

$$-\left[\begin{bmatrix} h_1 & h_2 \end{bmatrix}\begin{bmatrix} I_1(t) \\ I_2(t) \end{bmatrix} + \begin{bmatrix} p_m & p_r & p_d \end{bmatrix}\begin{bmatrix} P_m(t) \\ P_r(t) \\ P_d(t) \end{bmatrix}\right]$$

$$+\begin{bmatrix} \psi_1(t) & \psi_2(t) \end{bmatrix}\left\{\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} P_m(t) \\ P_r(t) \\ P_d(t) \end{bmatrix} + \begin{bmatrix} -S(t) \\ R(t) \end{bmatrix}\right\}$$

$$+\begin{bmatrix} \lambda_1(t) & \lambda_2(t) \end{bmatrix}\begin{bmatrix} I_1(t) \\ I_2(t) \end{bmatrix}$$

Our problem is a convex (linear) optimal control problem. The advantage of a convex problem that necessary conditions are sufficient, as well. So it is true [13-15]:

THEOREM 1 *In order for* $\left\{I_1^0(t), I_2^0(t), P_m^0(t), P_r^0(t), P_d^0(t)\right\}$ *to be optimal the problem (1)-(4) it is necessary and sufficient that there exist functions* $\psi_1(t)$ *and* $\psi_2(t)$*, where for all* $0 \le t \le T$ *we have* $\begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(a) stationarity for the inventory levels:

$$\begin{bmatrix} \dot{\psi}_1(t) \\ \dot{\psi}_2(t) \end{bmatrix} = -\frac{\partial\left(I_1^0(t), I_2^0(t), P_m^0(t), P_r^0(t), P_d^0(t), \psi_1(t), \psi_2(t), \lambda_1(t), \lambda_2(t), t\right)}{\partial\begin{bmatrix} I_1(t) \\ I_2(t) \end{bmatrix}}$$

$$\begin{bmatrix} \dot{\psi}_1(t) \\ \dot{\psi}_2(t) \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} - \begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix},$$

(b) maximum principle for the manufacturing, remanufacturing and disposal rates:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \le \begin{bmatrix} P_{P_\ell}(t) \\ P_P(t) \\ P_d(t) \end{bmatrix} \max \left\{H\left(I_1^0(t), I_2^0(t), P_m(t), P_r(t), , P_d(t)\psi_1(t), \psi_2(t), t\right)\right\} =$$

$$H\left(I_1^0(t), I_2^0(t), P_m^0(t), P_r^0(t), P_d^0(t), \psi_1(t), \psi_2(t), t\right)$$

$$\max_{0\le P_m(t)} \left\{\left(\psi_1(t) - p_m\right) P_m(t)\right\} = \left(\psi_1(t) - p_m\right) P_m^0(t) = 0$$

$$\max_{0\le P_r(t)} \left\{\left(\psi_1(t) - \psi_2(t) - p_r\right) P_r(t)\right\} = \left(\psi_1(t) - \psi_2(t) - p_r\right) P_r^0(t) = 0$$

$$\max_{0\le P_d(t)} \left\{-\left(\psi_2(t) + p_d\right) P_d(t)\right\} = -\left(\psi_2(t) + p_d\right) P_d^0(t) = 0\ ,$$

(c) discontinuity of the adjoint variables in some points:

$$\begin{bmatrix} \psi_1\left(t_j^-\right) \\ \psi_2\left(t_j^-\right) \end{bmatrix} \ge \begin{bmatrix} \psi_1\left(t_j^+\right) \\ \psi_2\left(t_j^+\right) \end{bmatrix}, \quad t_j \in (0,T), \quad j \in J, \begin{bmatrix} I_1^0\left(t_j\right) \\ I_2^0\left(t_j\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\ ,$$

(d) constraint qualification for the inventory levels:

$$\begin{bmatrix} \lambda_1(t) & \lambda_2(t) \end{bmatrix} \begin{bmatrix} I_1^0(t) \\ I_2^0(t) \end{bmatrix} = 0, \begin{bmatrix} \lambda_1(t) & \lambda_2(t) \end{bmatrix} \ge \begin{bmatrix} 0 & 0 \end{bmatrix}\ ,$$

(e) terminal conditions:

$$\begin{bmatrix} \psi_1(T) & \psi_2(T) \end{bmatrix} \begin{bmatrix} I_1^0(T) \\ I_2^0(T) \end{bmatrix} = 0, \quad [\psi_1(T) \quad \psi_2(T)] \ge \begin{bmatrix} 0 & 0 \end{bmatrix}\ .$$

Proof. Condition (a) is the moving rule of the adjoint variables. Conditions (d) are constraint qualification for the inventory status, and conditions (e) are the terminal conditions for the state variables. The adjoint variables can be interpreted, as the shadow prices in stores 1 and 2 From conditions (b) the optimal manufacturing and remanufacturing rates can be written in simpler way:

$$P_m^0(t) = \begin{cases} 0 & \psi_1(t) \le p_m \\ \text{undefined} & \psi_1(t) = p_m \end{cases}$$

$$P_r^0(t) = \begin{cases} 0 & \psi_1(t) - \psi_2(t) \le p_r \\ \text{undefined} & \psi_1(t) - \psi_2(t) = p_r \end{cases}$$

and

$$P_d^0(t) = \begin{cases} 0 & \psi_2(t) \ge -p_d \\ \text{undefined} & \psi_2(t) = -p_d \end{cases}$$

The conditions are depicted in Figure 2.

Before constructing the optimal path for manufacturing, remanufacturing and disposal, let us investigate the possible path of adjoint variables $\psi_1(t)$ and $\psi_2(t)$. The adjoint variables are in linear connection, so we can distinguish three cases:

Case 1.:

$$\psi_1(t) = p_r - p_d, \psi_2(t) = -p_d, t \in [t_1, t_2]$$

$$P_m^0(t) = 0, P_r^0(t) > 0, P_d^0(t) > 0, t \in [t_1, t_2]$$

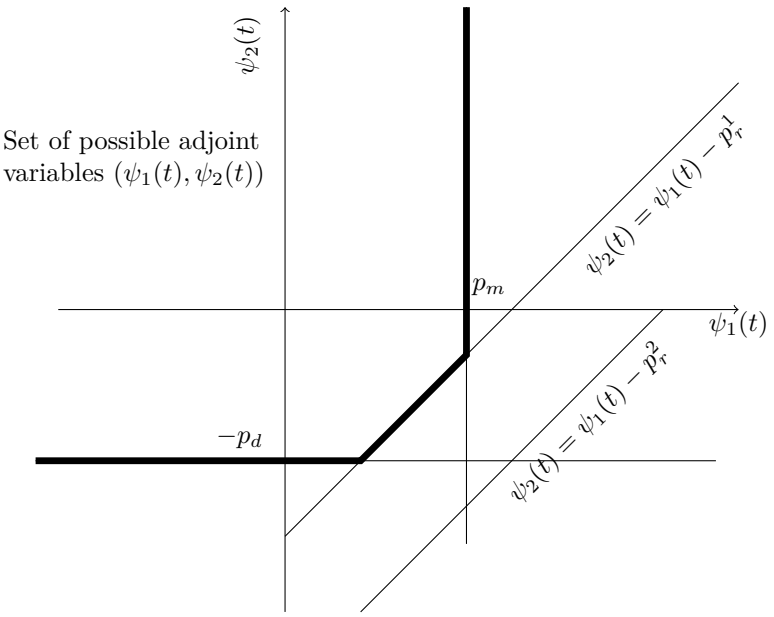


Figure 2: Graphical representation of the conditions of optimality

This point is a solution for the adjoint variables. (See Figure 2.) After differentiation we have the following identities

$$0 = \dot{\psi}_1(t) = h_1 - \lambda_1(t), \quad 0 = \dot{\psi}_2(t) = h_2 - \lambda_2(t)$$

which means that

$$h_1 = \lambda_1(t) > 0, \; h_2 = \lambda_2(t) > 0 \; .$$

For this case we can use the constraint qualification, and in such an interval the inventory levels are equal to zero, and the optimal manufacturing and remanufacturing rates are

$$\left.\begin{array}{l} I_1^0(t) = 0 \\ I_2^0(t) = 0 \\ P_m^0(t) = 0 \\ P_r^0(t) = S(t), \\ P_d^0(t) = R(t) - S(t) \geq 0 \end{array}\right\} \quad t \in [t_1, t_2]$$

Case 2.:

$$\psi_1(t) - \psi_2(t) = p_r, \quad \psi_1(t) < p_m, \quad \psi_2(t) > p_d \quad t \in [t_1, t_2]$$

$$P_m^0(t) = 0, \quad P_r^0(t) > 0, \quad P_d^0(t) = 0, \quad t \in [t_1, t_2] \; .$$

This solution for the adjoint variables is a set of points between the arrows. (See Figure 2.) After differentiation we have the following identities

$$0 = \dot{\psi}_1(t) - \dot{\psi}_2(t) = h_1 - h_2 - \lambda_1(t) + \lambda_2(t)$$

or

$$0 = h_1 - h_2 - \lambda_1(t) + \lambda_2(t) = h_1 - h_2 + \lambda_2(t) = \lambda_1(t) > 0 \ .$$

If we assume that $h_1 > h_2$, then the optimal solution

$$\left.\begin{array}{l} I_1^0(t) = 0, \\ I_2^0(t) \geq 0, \\ P_m^0(t) = 0, \\ P_r^0(t) = S(t), \\ P_d^0(t) = 0, \end{array}\right\} \quad t \in [t_1, t_2]$$

or if $h_1 \leq h_2$

$$\left.\begin{array}{l} I_1^0(t) \geq 0, \\ I_2^0(t) = 0, \\ P_m^0(t) = 0, \\ P_r^0(t) = R(t), \\ P_d^0(t) = 0, \end{array}\right\} \quad t \in [t_1, t_2]$$

The adjoint variables for the first case are

$$\left.\begin{array}{l} \psi_1(t) = \psi_{10} + h_2\,(t - t_1) \\ \psi_2(t) = \psi_1(t) - p_r \end{array}\right\} \quad t \in [t_1, t_2]$$

and for the second case

$$\left.\begin{array}{l} \psi_1(t) = \psi_{10} + h_1\,(t - t_1) \\ \psi_2(t) = \psi_1(t) - p_r \end{array}\right\} \quad t \in [t_1, t_2]$$

Case 3.:

$$\psi_1(t) = p_m, \quad \psi_2(t) = p_m - p_r, \quad t \in [t_1, t_2]$$

$$P_m^0(t) > 0, \quad P_r^0(t) > 0, \quad P_d^0(t) = 0, \quad t \in [t_1, t_2] \ .$$

This point is a solution for the adjoint variables. (See Figure 2.) After differentiation we have the following identities

$$\begin{array}{l} 0 = \dot{\psi}_1(t) = h_1 - \lambda_1(t), \\ 0 = \dot{\psi}_2(t) = h_2 - \lambda_2(t) \end{array}$$

which means that

$$\begin{array}{l} h_1 = \lambda_1(t) > 0 \\ h_2 = \lambda_2(t) > 0 \ . \end{array}$$

For this case we can use the constraint qualification, and in such an interval the inventory levels are equal to zero, and the optimal remanufacturing and disposal rates are

$$\left.\begin{array}{l} I_1^0(t) = 0, \\ I_2^0(t) = 0, \\ P_m^0(t) = S(t) - R(t) \geq 0, \\ P_r^0(t) = R(t), \\ P_d^0(t) = 0, \end{array}\right\} \quad t \in [t_1, t_2] \ .$$

□

From the three cases it follows immediately

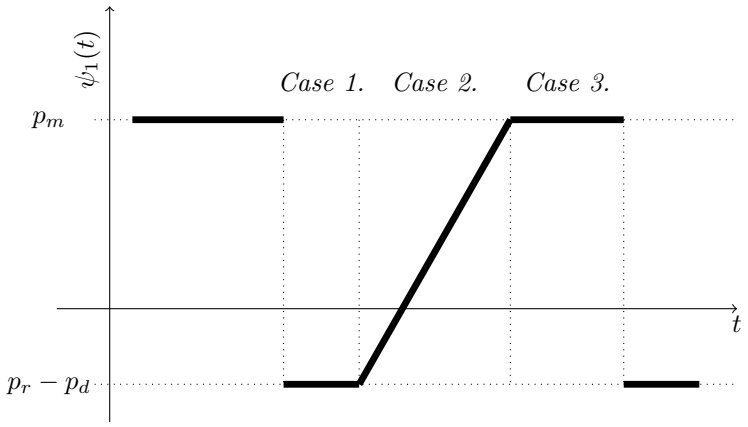


Figure 3: A possible path of adjoint variable $\psi_1(t)$

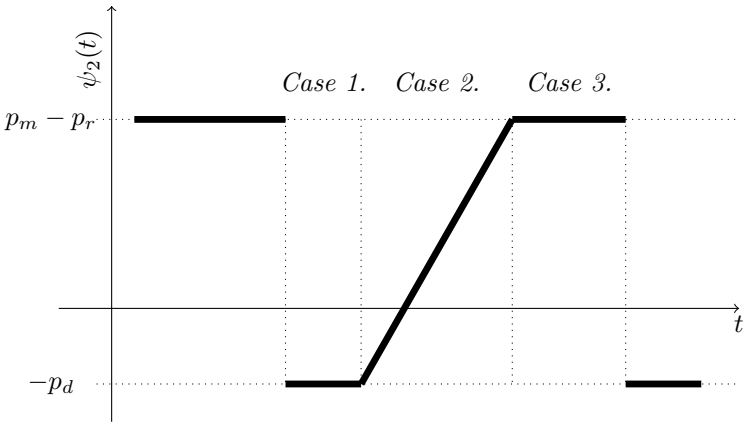


Figure 4: A possible path of adjoint variable $\psi_2(t)$

LEMMA 5 *The optimal controls satisfy* $P_m^0(t)P_d^0(t) = 0$ *for every* $t \in [0, T]$.

The meaning of the lemma is that it is either manufactured or disposed in a point of time, but both together never. The next question is how long can be remanufactured without manufacturing and disposal.

LEMMA 6 *The maximal length of time* τ *for only remanufacturing is*

$$\tau = \begin{cases} \frac{p_m + p_d - p_r}{h_2} & h_1 > h_2 \\ \frac{p_m + p_d - p_r}{h_1} & h_1 \leq h_2 \end{cases}$$

Proof. For the proof of this lemma, let us construct a possible path of the adjoint variable. When we schedule the three cases, we obtain a possible path. (See Figure 3. and 4.)

In case 2. the adjoint variables are linear function with tangent h_2, if $h_1 > h_2$, and this fact proves the lemma. As we see, the adjoint variables can have discontinuities, if there is a switching point between either cases 3. and 1. or cases 3. and 2. or cases 2. and 1. □

Next we investigate the switching points

LEMMA 7 *There can be switching points between cases 3. and 1., if function*

$$\widetilde{I}(t) = I_{10} + \int_0^t (R(\tau) - S(\tau))d\tau$$

has a local maximum.

Proof. In case 3.$P_m^0(t) = S(t) - R(t) \geq 0$, and in case 1. $P_d^0(t) = R(t) - S(t) \geq 0$. If point t_1 is this switching point, then in point t_1 it is true $R(t_1) - S(t_1) = 0$. This fact proves the proposition. □

In the following we give some examples to construct the optimal solution.

6 Numerical examples

In this section we give three example to solve the problem. Before constructing the optimal solution, we must check the results of Lemmas 3. and 4. Let us assume that the conditions of the lemmas do not fulfill, and $h_1 > h_2$. The other case can be handled similarly.

6.1 Example 1: there is manufacturing, remanufacturing and disposal

The data for model specification is shown in Table 1. The Figure 5. shows function which plays an important role in constructing the optimal manufacturing, remanufacturing and disposal rates.

Demand	$S(t) = 1 + 0.5sin(t)$
Rate of return	$R(t) = 0.7(1 + 0.5sin(t - \pi)), t > \pi$, $R(t) = 0, t \leq \pi$
Inventory holding cost in store 1	$h_1 = 2$
Inventory holding cost in store 2	$h_2 = 1$
Cost of manufacturing	$p_m = 5$
Cost of remanufacturing	$p_r = 2$
Cost of disposal	$p_d = 3$
Initial inventory level in store 1	$I_0 = 1.2$
Planning horizon	$T = 12$

Table 1: Parameter specification for example 1.

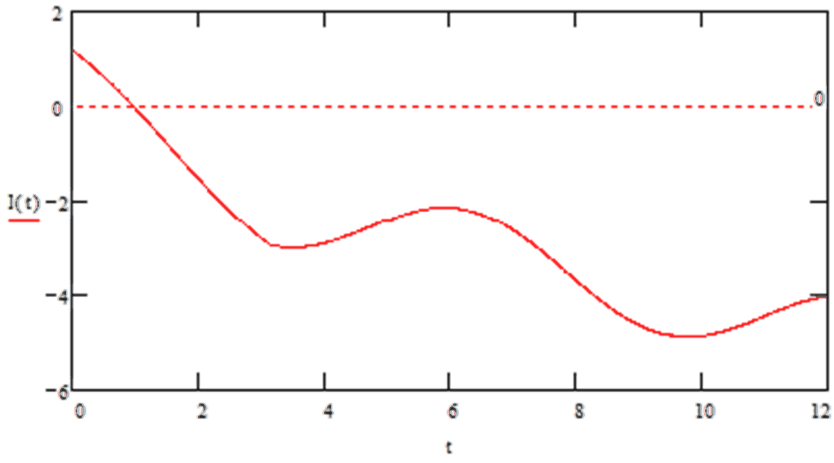


Figure 5: The function $\widetilde{I}(t)$ to this example

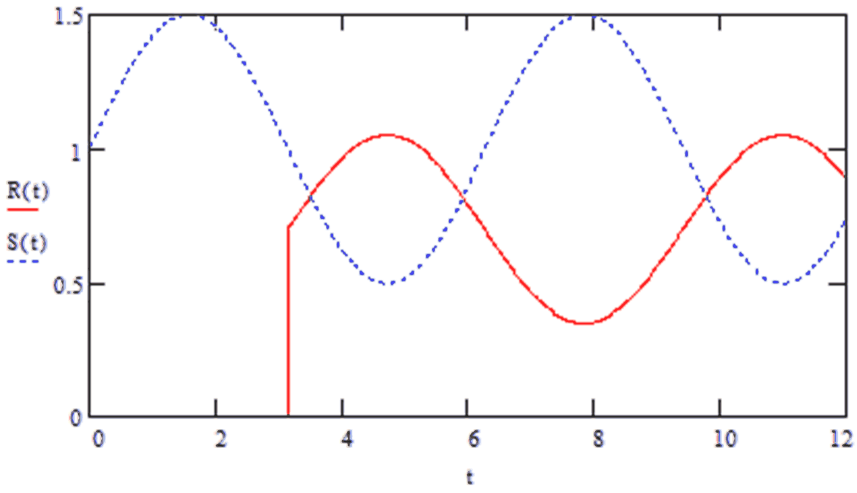


Figure 6: Demand and return rate to example 1

Function $\widetilde{I}(t)$ has two minimums in points of time 3.502 and 9.785. First we give the optimal manufacturing rate:

$$P_m^0(t) = \begin{cases} 0 & 0 \le t < 0.979 \\ S(t) - R(t) & 0.979 \le t < 3.502 \\ 0 & 3502 \le t < 7.398 \\ S(t) - R(t) & 7.398 \le t < 9.785 \\ 0 & 9.785 \le t \le 12 \ , \end{cases}$$

The optimal disposal rate is

$$P_d^0(t) = \begin{cases} 0 & 0 \le t < 9.785 \\ R(t) - S(t) & 9.785 \le t \le 12 \ , \end{cases}$$

and the remanufacturing rate is

$$P_r^0(t) = \begin{cases} 0 & 0 \le t < 0.979 \\ R(t) & 0.979 \le t < 3502 \\ S(t) & 3.502 \le t < 7.398 \\ R(t) & 7.398 \le t < 9.785 \\ S(t) & 9.785 \le t \le 12 \ . \end{cases}$$

6.2 Example 2: there are manufacturing and disposal

The data for model specification is shown in Table 2. The Figure 5. shows function which plays an important role in constructing the optimal manufacturing, remanufacturing and disposal rates.

Demand	$S(t) = t^2(t^2 - 144)$
Rate of return	$R(t) =$ $0.9(t-1)^2((t-1)^2 - 144)$, if $t > 1$, $R(t) = 0, t \le 1$
Inventory holding cost in store 1	$h_1 = 2$
Inventory holding cost in store 2	$h_2 = 1$
Cost of manufacturing	$p_m = 5$
Cost of remanufacturing	$p_r = 2$
Cost of disposal	$p_d = 3$
Initial inventory level in store 1	$I_0 = 6478$
Planning horizon	$T = 12$

Table 2: Parameter specification for example 2.

Function $\widetilde{I}(t)$ has a minimum in point of time 9.757. First we give the optimal manufacturing rate:

$$P_m^0(t) = \begin{cases} 0 & 0 \le t < 8.281 \\ S(t) - R(t) & 8.281 \le t < 9.757 \\ 0 & 9.757 \le t \le 12 \ , \end{cases}$$

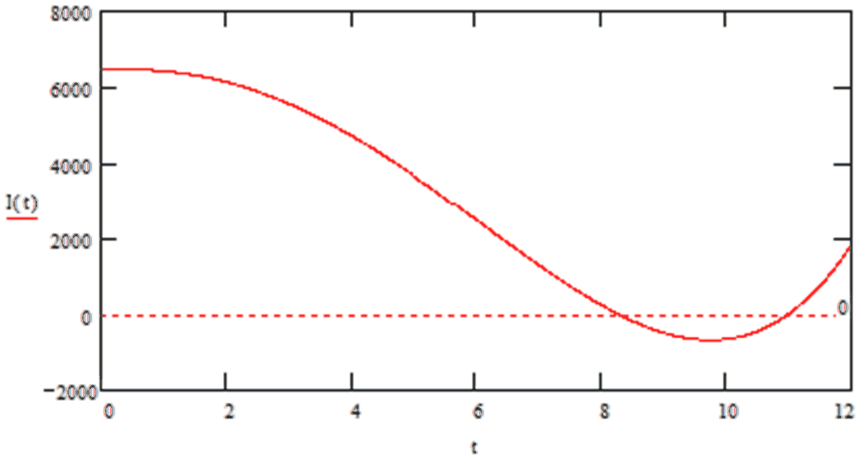


Figure 7: The function $\widetilde{I}(t)$ to this example 2

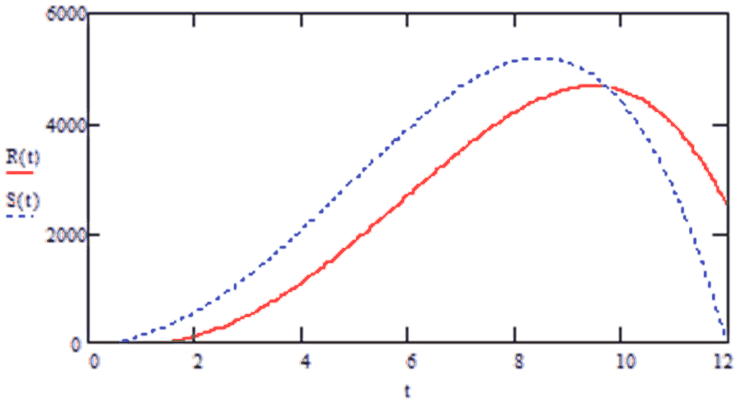


Figure 8: Demand and return rate to example 2

The optimal disposal rate is

$$P_d^0(t) = \begin{cases} 0 & 0 \le t < 9.757 \\ R(t) - S(t) & 9.757 \le t \le 12 \ , \end{cases}$$

and the remanufacturing rate is

$$P_r^0(t) = \begin{cases} S(t) & 0 \le t < 8.281 \\ R(t) & 8.281 \le t < 9.757 \\ S(t) & 9.757 \le t \le 12 \ . \end{cases}$$

6.3 Example 3: there is no manufacturing

The data for model specification is shown in Table 3. The Figure 5. shows function which plays an important role in constructing the optimal manufacturing, remanufacturing and disposal rates.

Demand	$S(t) = (1 + 0.05t)(1 + \sin(t))$
Rate of return	$R(t) =$ $(1 + 0.05(t-1))(1 + \sin(t-1))$, if $t > 1$, $R(t) = 0$ otherwise.
Inventory holding cost in store 1	$h_1 = 2$
Inventory holding cost in store 2	$h_2 = 1$
Cost of manufacturing	$p_m = 5$
Cost of remanufacturing	$p_r = 2$
Cost of disposal	$p_d = 3$
Initial inventory level in store 1	$I_0 = 3.7$
Planning horizon	$T = 12$

Table 3: Parameter specification for example 1.

Function $\widetilde{I}(t)$ strictly greater than zero, so it is not manufactured: $p_m^0(t) = 0$. The optimal disposal rate is

$$P_d^0(t) = \begin{cases} 0 & 0 \le t < 8.628 \\ R(t) - S(t) & 8.628 \le 10.943 \\ 0 & 10.943 \le t \le 10 \ , \end{cases}$$

and the remanufacturing rate is

$$P_r^0(t) = \begin{cases} S(t) & 0 \le t < 8.628 \\ R(t) - S(t) & 8.628 \le t < 10.943 \\ S(t) & 10.943 \le t \le 12 \ . \end{cases}$$

7 Conclusions

In this paper we have solved a reverse logistics problem. Because of the convexity (linearity) of manufacturing, remanufacturing and disposal costs, the necessary

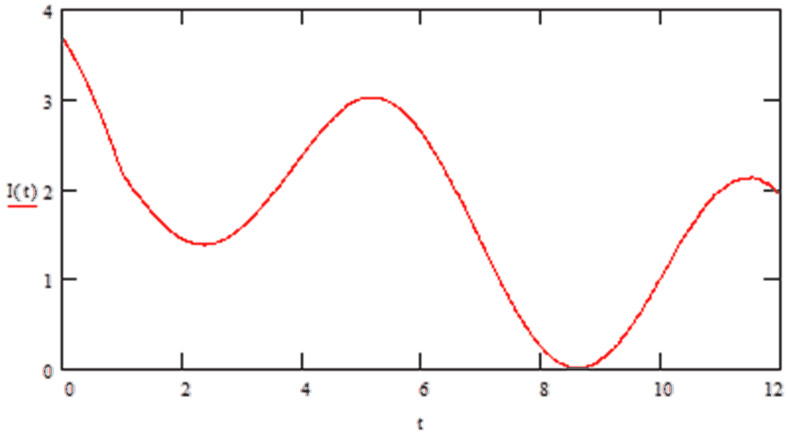


Figure 9: The function $\widetilde{I}(t)$ to this example 3

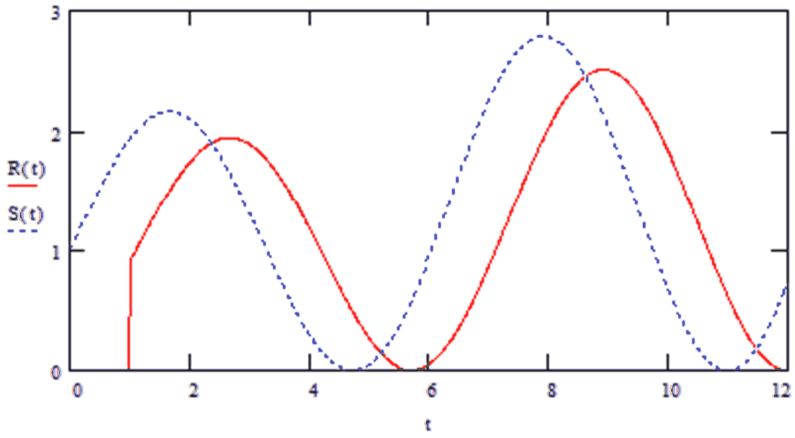


Figure 10: Demand and return rate to example 3

conditions of optimality are sufficient, as well. After characterizing the model, it was given some example to construct optimal trajectories. A result is that the behavior of reverse logistics problems is affected by inventory holding costs for new and returned products. Remanufacturing controls the optimal path of the system, and the role of manufacturing is to broaden the production facilities, if the demand is not to satisfy from the remanufactured returned items.

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Appendix 1. The "reverse" linear production-inventory problem

The problem described as follows.

$$\int_{t_1}^{t_2} \left[(h_2 - h_1)\, I_2(t) + p_r P_r(t) \right] dt \to \min$$

such that

$$\dot{I}_2(t) = -P_r(t) + R(t), \quad I_2(t_1) = 0 \ ,$$
$$0 \geq I_2(t) \ ,$$
$$P_r(t) \geq 0 \ .$$

This problem can be called, as the "reverse" production-inventory problem. In this model the inflow (return rate) is given, and the outflow (remananufacturing) is decision variable. Let us now solve the reverse problem.

The Hamilton- and Lagrange-functions are presented as

Hamilton-function:

$$H\left(I_2(t), P_r(t), \psi_2(t), t\right) = -\left[(h_2 - h_1)\, I_2(t) + p_r P_r(t) \right] + \psi_2(t) \left[-P_r(t) + R(t) \right] \ ,$$

Lagrange function:

$$L\left(I_2(t), P_r(t), \psi_2(t), \lambda_1(t), t\right) =$$
$$-\left[(h_2 - h_1)\, I_2(t) + p_r P_r(t) \right] + \psi_2(t) \left[-P_r(t) + R(t) \right] + \lambda_1(t) I_2(t) \ .$$

The necesssary and sufficient conditions for optimality are

1. for the adjoint variable: $\dot{\psi}_2(t) = (h_2 - h_1) - \lambda_1(t)$,
2. the maximum principle:
   $\max_{P_r(t) \geq 0} \left\{ -\left(p_r + \psi_2(t)\right) P_r(t) \right\} = -\left(p_r + \psi_2(t)\right) P_r^0(t) = 0$,

3. constraints qualifications: $\lambda_1(t)I_2^0(t) = 0, \lambda_1(t) \geq 0,$

4. terminal condition: $\psi_2(t_2)\, I_2^0(t_2) = 0, \psi_2(t_2) \geq 0.$

The optimal solution is

$$\left.\begin{array}{l} I_2^0(t) = 0 \\ P_r^0(t) = R(t) \\ \psi_2(t) = -p_d \end{array}\right\} \quad \forall t \in [t_1, t_2]\,.$$

As we see, the optimal inventory level is equal to zero and the warehousing cosntraint is inactive. The optimal remanufacturing is positive, there is no an interval, where the remanufacturing rate is equal to zero. It show us, that there exists no optimal solution, if the initial inventory level is strict greater than zero, so $I_{20} = I(t_1) = 0$.

Appendix 2. The "direct" linear production-inventory model with generalized demand

The following problem is a generalization of the classical linear problem, where the modified demand rate can be negative, if the return rate is higher than the demand rate. The solution is a little bit other than those for non-negative demand rate. For this problem we do not use the maximum principle, but the following solution can be achieved with it.

$$\int_{t_1}^{t_2} \left[h\widetilde{I}(t) + p_m P_m(t)\right] dt \to \min$$

such that

$$\dot{\widetilde{I}}(t) = P_m(t) - (S(t) - R(t)), \quad \widetilde{I}\,(t_1) = \widetilde{I}_1\ ,$$

$$0 \leq \widetilde{I}(t), \quad P_m(t) \geq 0\ .$$

The main idea of the algorithm is that the manufacturing rate must be zero, if the modified demand rate is negative. In this case the change of the inventory level is positive, so the inventory level is increasing. If the manufacturing rate would be positive, then the increasing of the inventory level would be higher, and with the choice of zero manufacturing the objective function is lower, and we do not produce until the inventory level is zero again.

Let us first find the initial points of such an interval where the modified demand rate is negative. It is easy to see that there is a maximum of the cumulated modified demand, and the demand rate is equal to zero. Let these points be $\{\tau_i\}_{i=1}^n$, and points $\{\tau_i + \varepsilon_i\}_{i=1}^n$ where the inventory level achieves first the zero or the end of the planning period, if the zero inventory level is not

attained. So the optimal solution is

$$\widetilde{I}^0(t) =$$
$$\begin{cases} I_{10}+I_{20}+\int_{t_1}^{t}(R(\tau)-S(\tau))d\tau, \text{ if } I_{10}+I_{20}+\int_{t_1}^{t}(R(\tau)-S(\tau))d\tau \geq 0 \\ 0, \text{ if } t \in [\tau_{i-1}+\varepsilon_{i-1}, \tau_i) \\ \int_{\tau_i}^{t}(R(\tau)-S(\tau))d\tau, \text{ if } t \in [\tau_i, \tau_i+\varepsilon_i) \end{cases}$$

$$P_m^0(t) = \begin{cases} 0, \text{ if } tI_{10}+I_{20}+\int_{t_1}^{t}(R(\tau)-S(\tau))d\tau \\ S(t)-R(t), \text{ if } tt \in [\tau_{i-1}+\varepsilon_{i-1}, \tau_i) \\ 0, \text{ if } tt \in [\tau_i, \tau_i+\varepsilon_i) \end{cases} .$$

Appendix 3. The "reverse" linear production-inventory model with generalized demand

This problem is a generalization of the reverse linear problem, where the modified return rate can be negative, if the the demand rate is higher than return rate. The solution is a little bit other than those for non-negative return rate. For this problem we do not use the maximum principle, but the following solution can be achieved with it. Before solving the problem, we give a necessary condition for existence a solution:

$$I_{10}+I_{20}+\int_{t_1}^{t}(R(\tau)-S(\tau))d\tau \geq 0, \quad \forall t \in [t_1, t_2] .$$

If this condition does not hold, there don not exist any solution of the problem.

$$\int_{t_1}^{t_2}\left[h\widetilde{I}(t)+p_dP_d(t)\right]dt \to \min$$

such that

$$\dot{\widetilde{I}}(t) = -P_d(t)+(R(t)-S(t)), \quad \widetilde{I}(t_1) = \widetilde{I}_1,$$
$$0 \leq \widetilde{I}(t), \quad P_d(t) \geq 0 .$$

The main idea of the algorithm is that the disposal rate must be zero, if the modified return rate is negative. In this case the change of the inventory level is positive, so the inventory level is increasing. If the disposal rate would be positive, then the increasing of the inventory level would be higher, and with the choice of zero disposal the objective function is lower, and we do not dispose until the inventory level is zero again.

We repeat the algorithm showed above, but "reverse" (backward). Let us first find the final points of such an interval where the modified disposal rate is positive. It is easy to see that there is a minimum of the cumulated modified return, and the modified return rate is equal to zero.

Let these points be $\{\tau_i\}_{i=1}^n$, and points $\{\tau_i-\varepsilon_i\}_{i=1}^n$ where the inventory level achieves first the zero or the beginning of the planning period, if the zero

inventory level is not attained. So the optimal solution is

$$\widetilde{I}^0(t) = \begin{cases} 0, \text{ if } t \in [\tau_{i-1}, \tau_i - \varepsilon_i) \\ \int_{\tau_i - \varepsilon_i}^{t} (R(\tau) - S(\tau)) d\tau, \text{ if } t \in [\tau_i - \varepsilon_i, \tau_i) \end{cases}$$

$$P_d^0(t) = \begin{cases} R(t) - S(t), \text{ if } t \in [\tau_{i-1}, \tau_i - \varepsilon_i) \\ 0, \text{ if } t \in [\tau_i - \varepsilon_i, \tau_i) \end{cases}$$