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BAYES SIGNIFICATION TESTS IN LINEAR REGRESSION AND ECONOMIC APPLICATIONS

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Abstract: In this paper we will use the statistical inference, in particular the Bayes tests for the signification of the parameters of linear regression. We will consider the cases of known variance, and unknown variance respectively.
Because we replace the Student tests by the Z tests in practice if the involved number of degrees of freedom is at least 30, we can replace the case of unknown variance with that of known variance in our paper, if the above number of degrees of freedom is at least 30.
Because in economics we generally know the signs of the involved parameters, in practice the one-sided parametric tests are used. However, in this paper we will refer also to the two-sided Bayes test.

Keywords: Bayes test, signification test, two-sided test, left test, right test, wages, saving.

1. Introduction

Consider X a random variable whose distribution depends on the parameter $\theta\in\Theta$. If X is discrete, we denote by $p\left(x|\theta_0\right)=P\left(X=x|\theta=\theta_0\right)$, and, if X is continuous, by $f\left(x|\theta\right)$ the pdf. of X that is conditioned by the fact that $\theta=\theta_0$.

In the Bayesian inference the parameter θ is a random variable with the pdf $g\left(\theta\right)$, called priorly pdf. of θ. To determinate the posterior pdf. of θ we use the formula of Bayes, and we obtain [10, 8, 9]

$$g\left(\theta_0|x\right)=\frac{p\left(x|\theta_0\right)\cdot g\left(\theta_0\right)}{\int\limits_{\Theta}p\left(x|\theta\right)\cdot g\left(\theta\right)d\theta} \tag{1}$$

in the discrete case, respectively

$$g\left(\theta_0|x\right)=\frac{f\left(x|\theta_0\right)\cdot g\left(\theta_0\right)}{\int\limits_{\Theta}f\left(x|\theta\right)\cdot g\left(\theta\right)d\theta} \tag{1'}$$

in the continuous case.

Definition 1 ([10]). *A family of prior distribution $\mathcal{P}$ is called conjugated prior distribution for the pdfs family $\mathcal{F}=\left\{f\left(x|\theta\right)\middle|\theta\in\Theta\right\}$ if $(\forall g\in\mathcal{P})$ $(\forall f\in\mathcal{F})$ we have $g\left(\theta|x\right)\in\mathcal{P}$.*

A special case of conjugated family of prior distribution is that when P and F are identical to the normal family [10]: if $X\sim N\left(\theta,\sigma^2\right)$ with known, and $\theta\sim N\left(\mu,\tau^2\right)$ then

$$\theta|X\sim N\left(\frac{\tau^2\cdot X+\sigma^2\cdot\mu}{\sigma^2+\tau^2},\frac{\sigma^2\cdot\tau^2}{\sigma^2+\tau^2}\right) \tag{2}$$

For Bayes estimators we have to define first some loss functions, like $U\left(\theta_1,\theta_0\right)=\left(\theta_1-\theta_0\right)^2$, or $U\left(\theta_1,\theta_0\right)=\left|\theta_1-\theta_0\right|$ [10]. In these formulae $U\left(\theta_1,\theta_0\right)$ is the loss function if we decide that $\theta=\theta_1$ if in

fact $\theta=\theta_0$. Other loss function, as weighted quadratic one is used in [3]:

$$U(\theta_1,\theta_0)=w(\theta_0)(\theta_1-\theta_0)^2 \tag{3}$$

where $w(\theta)>0$ is a weight function.

Definition 2 ([10]). *Consider $g(\theta|x)$ the posterior pdf. of the parameter θ and $U(\theta_1,\theta_0)$ a loss function as above. The Bayes risk is $E(U(\theta,\theta_0))$ where θ_0 is the decided (chosen) value of θ, and the pdf. of the random variable θ is $g(\theta|x)$.*

Definition 3 ([10]). *The Bayes estimator of the parameter θ is the value θ_0 chosen such that the Bayes risk is minimum.*

If $U(\theta_1,\theta_0)=(\theta_1-\theta_0)^2$ the Bayes estimator is the expectation, and if $U(\theta_1,\theta_0)=|\theta_1-\theta_0|$ the Bayes estimator is the median.

In [10] a two-sided Bayes test is presented with the first degree error (the threshold) ε in the above case of normal distributions (the conditional distribution of $X|\theta$ and the prior distribution of θ). The conditioned pdf. $f(x|\theta)$, the prior pdf. of θ, $g(\theta)$, and the number $p_0\in(0,1)$ are considered. Because p_0 is interpreted as the prior probability to have $\theta=\theta_0$, we usually choose $p_0=0.5$ from the maximum entropy principle. The posterior probability to have $\theta=\theta_0$ is

$$P(\theta=\theta_0|X)=\frac{p_0\cdot f(x|\theta_0)}{p_0\cdot f(x|\theta_0)+(1-p_0)\int_\Theta f(x|\theta)\cdot g(\theta)\,d\theta} \tag{4}$$

We accept the null hypothesis $H_0:\theta=\theta_0$ against the alternative hypothesis $H_1:\theta\neq\theta_0$ with the threshold ε if and only if $P(\theta=\theta_0|X)>1-\varepsilon$.

Consider n points in $\mathbb{R}^{k+1}$, $X^{(1)},...,X^{(n)}$, where $X^{(i)}=(X_1^{(i)},X_2^{(i)},...,X_k^{(i)},Y_i)$. The regression hyper-plane is [7, 13]:

$$H:\ Y=A_0+\sum_{i=1}^{k}A_iX_i \text{ such that} \tag{5}$$

$$\sum_{i=1}^{n}u_i^2 \text{ is minimum,} \tag{5'}$$

where the residues u_i are given by the formula

$$u_i=Y_i-A_0-\sum_{j=1}^{k}A_jX_j^{(i)} \tag{5''}$$

For the computation of A_i from (5) we have to solve the system [11]

$$\sum_{j=0}^{k}\overline{X_i\cdot X_j}\cdot A_j=\overline{X_i\cdot Y},\ i=\overline{0,k} \tag{6}$$

where $\overline{X_0\cdot X_i}=\overline{X_i}$ and $\overline{X_0^2}=1$.

The polynomial model is in fact the multilinear model (5) with the explanatory variables $X_1=X$, $X_2=X^2$ and so on [2].

For the obtained estimators of A_i using (6) and for the residues u_i we have the following hypotheses [7, 13]:

1. The estimators of A_i are linear.
2. The estimators of u_i have the expectation 0 and the same variance (homoscedasticity).
3. The estimators of u_i are normal.
4. The random variables u_i are independent.

From the above hypotheses and from Gauss-Markov theorem we obtain the following properties [7, 13]:

1. The estimators of A_i are consistent.
2. The estimators of A_i are unbiased.
3. The estimators of A_i have the minimum variance.
4. The estimators of A_i have the maximum likelihood.

If we denote by $\hat{A}$ the vector of the estimated parameters A_i, the variance-covariance matrix of $\hat{A}$ is [7]

$$Var\left(\hat{A}\right)=\sigma_e^2\left(XX'\right)^{-1} \tag{7}$$

If we denote by d_{ij} the value from the row i and column j in the matrix $\left(XX'\right)^{-1}$, we obtain the estimator of $\sigma_{A_i}^2$

$$S_{\hat{A}}^2=\sigma_u^2\cdot d_{ii} \tag{8}$$

where σ_u^2 is the estimated variance of the residues.

For testing the null hypothesis $H_0 : A_i=0$ against the alternative hypothesis $H_1 : A_i>0$ we use the Student test [7]. We compute

$$T_i=\frac{\hat{A}}{S_{\hat{A}}} \tag{9}$$

and we accept the null hypothesis (the parameter is not significant) if and only if $T_i<t_{n-k-1;\varepsilon}$, where n is the data size, k is the number of explanatory variables, and $t_{n-k-1;\varepsilon}$ is the quantile of the error ε of the Student distribution with $n-k-1$ degrees of freedom.

2. The Bayes Test for Normal Distribution

2.1. Case of Known Variance

If we expand the fraction of (4) by $\frac{g(\theta_0)}{\int_\Theta f(x|\theta)\cdot g(\theta)d\theta}$ we obtain the posterior probability to have $\theta=\theta_0$ in the case of two-sided Bayes test

$$P\left(\theta=\theta_0\middle|X\right)=\frac{p_0\cdot g\left(\theta_0\middle|X\right)}{p_0\cdot g\left(\theta_0\middle|X\right)+\left(1-p_0\right)\cdot g\left(\theta_0\right)} \tag{10}$$

In the following we consider the estimator $\hat{\theta}$ of θ which is unbiased and normal, with the known variance σ^2: $\hat{\theta}\big|\theta \sim N\left(\theta,\sigma^2\right)$. Consider also the prior distribution of θ being also normal, namely $N\left(\mu,\tau^2\right)$. Denoting by $\delta=\frac{1}{\sigma^2}$ we have [10]

$$\begin{cases} \hat{\theta}\big|\theta;\delta \sim N\left(\theta,\sigma^2=\frac{1}{\delta}\right) \\ \theta \sim N\left(\mu,\tau^2\right) \\ \theta\big|\hat{\theta};\delta \sim N\left(\frac{\delta\tau^2\hat{\theta}+\mu}{\delta\tau^2+1},\frac{\tau^2}{\delta\tau^2+1}\right) \end{cases} \tag{11}$$

If we replace in (10) the corresponding normal pdfs taking into account (11), denoted by $f\left(\hat{\theta}\big|\theta;\delta\right)$, $g(\theta)$, and $g\left(\theta\big|\hat{\theta};\delta\right)$ respectively, we obtain

$$P\left(\theta=\theta_0;\delta\big|\hat{\theta}\right)=\frac{p_0\cdot g\left(\theta_0\big|\hat{\theta};\delta\right)}{p_0\cdot g\left(\theta_0\big|\hat{\theta};\delta\right)+\left(1-p_0\right)\cdot g\left(\theta_0\right)} \tag{12}$$

Denote by $R_\delta\left(\hat{\theta},\mu,\tau^2\right)=\frac{g(\theta_0)}{g\left(\theta_0\big|\hat{\theta};\delta\right)}$. We obtain in this way

$$P\left(\theta=\theta_0\big|\hat{\theta};\delta\right)=\frac{p_0}{p_0+\left(1-p_0\right)R_\delta\left(\hat{\theta},\mu,\tau^2\right)} \tag{13}$$

The condition to accept the null hypothesis with the threshold ε is that $P\left(\theta=\theta_0;\delta\big|\hat{\theta}\right)>1-\varepsilon$. It is equivalent to

$$R_\delta\left(\hat{\theta},\mu,\tau^2\right)<\gamma_0\left(p_0,\varepsilon\right)=\frac{p_0}{1-p_0}\cdot\frac{\varepsilon}{1-\varepsilon} \tag{14}$$

Suppose $\gamma_0\left(p_0,\varepsilon\right)<1$, condition that can be obtained for small ε and $p_0=\frac{1}{2}$ (from the maximum entropy principle). From this point forward we consider $\theta_0=0$.

An acceptable condition is to accept the null hypothesis if $\hat{\theta}=0$, and to reject it if $\hat{\theta}=\mu$. But

$$R_\delta\left(\hat{\theta},\mu,\tau^2\right)=\frac{1}{\sqrt{\delta\tau^2+1}}\cdot\exp\left\{\frac{\left(\delta\tau^2\hat{\theta}+\mu\right)^2}{2\tau^2\left(\delta\tau^2+1\right)}-\frac{\mu^2}{2\tau^2}\right\} \tag{15}$$

and by substitution we obtain

$$\begin{cases} R_\delta\left(0,\mu,\tau^2\right)=\frac{1}{\sqrt{\delta\tau^2+1}}\cdot\exp\left\{-\frac{\delta\mu^2}{2\left(\delta\tau^2+1\right)}\right\} \\ R_\delta\left(\mu,\mu,\tau^2\right)=\frac{1}{\sqrt{\delta\tau^2+1}}\cdot\exp\left\{\frac{\delta\mu^2}{2}\right\} \end{cases} \tag{15'}$$

Consider now $\tau^2=\tau_0^2$ such that $\frac{1}{\sqrt{\delta\tau_0^2+1}}=\gamma_0\left(p_0,\varepsilon\right)$, i.e.

$$\tau_0^2=\frac{\frac{1}{\gamma_0^2\left(p_0,\varepsilon\right)}-1}{\delta} \tag{16}$$

We notice that we accept the null hypothesis if $\tau^2 > \tau_0^2$ and $\hat{\theta} = 0$ for all the values of μ, and we reject the null hypothesis if $\tau^2 < \tau_0^2$ and $\hat{\theta} = \mu$ for all the values of μ.

In the case $\tau^2 > \tau_0^2$ we take into account that $R_\delta\left(\mu, \mu, \tau^2\right)$ is increasing on μ^2, and it has the value $\frac{1}{\sqrt{\delta\tau^2+1}}$ for $\mu^2 = 0$ (accepting the null hypothesis for $\mu \to 0$), and tends to ∞ for $\mu^2 \to \infty$. Hence there exists $\mu^2 = \mu_{\min}^2$ such that $R_\delta\left(\mu, \mu, \tau^2\right) = \gamma_0\left(p_0, \varepsilon\right)$. In fact we have

$$\mu_{\min}^2 = \frac{\ln\left(\delta\tau^2+1\right) + 2\ln\gamma_0\left(p_0, \varepsilon\right)}{\delta} \tag{17}$$

and we take $\mu_{\min}$ having the same sign of $\hat{\theta}$. Therefore, if $\hat{\theta} > 0$ we have $\mu_{\min} = \sqrt{\mu_{\min}^2}$ and if $\hat{\theta} < 0$ $\mu_{\min}$ becomes $\mu_{\max} = -\sqrt{\mu_{\min}^2}$.

Analogously, if $\tau^2 < \tau_0^2$ we obtain

$$\mu_{\min}^2 = \frac{-\left(\delta\tau^2+1\right)\ln\left(\delta\tau^2+1\right) - 2\left(\delta\tau^2+1\right)\ln\gamma_0\left(p_0, \varepsilon\right)}{\delta} \tag{17'}$$

and we take in the same way $\mu_{\min}$, respectively $\mu_{\max}$ depending on the sign of $\hat{\theta}$.

If $\tau^2 = \tau_0^2$ we treat initially both cases: the first one for $\tau^2 = 0.95 \cdot \tau_0^2$ and the second one for $\tau^2 = 1.05 \cdot \tau_0^2$, and we obtain $\mu_{\min}$, and $\mu_{\max}$ respectively as the average of the two cases.

Now, with the above τ^2 and μ fixed, we notice that we accept the null hypothesis if $\hat{\theta} = 0$, and we reject it if $\hat{\theta} = \mu$. The function R_δ is obviously increasing on $[0, \mu]$ with respect to $\hat{\theta}$. It results that there exists $\hat{\theta}_1 < 0 < \hat{\theta}_2$ such that we accept the null hypothesis if and only if $\hat{\theta}_1 < \hat{\theta} < \hat{\theta}_2$. In fact, $\hat{\theta}_1$ and $\hat{\theta}_2$ are obtained from the second degree equation resulting from $R_\delta\left(\hat{\theta}, \mu, \tau^2\right) = \gamma_0\left(p_0, \varepsilon\right)$, and next we consider only the root with the same sign of μ. The other limit of the interval is the maintained root with the reverse sign. This removing of one of the solutions of the second degree equation (with one solution negative and one positive) is done due to the fact that if $\hat{\theta} < 0$ we change the sign of μ. Therefore $\hat{\theta}_1 = -\hat{\theta}_2$.

The power of the two-sided Bayes test is

$$\pi_\delta\left(\tilde{\theta}\right) = 1 + \Phi\left(\sqrt{\delta}\left(\hat{\theta}_1 - \tilde{\theta}\right)\right) - \Phi\left(\sqrt{\delta}\left(\hat{\theta}_2 - \tilde{\theta}\right)\right) \tag{18}$$

i.e. the probability that $\hat{\theta} \in \left(\hat{\theta}_1, \hat{\theta}_2\right)$ according to the pdf $f\left(\hat{\theta}\middle|\tilde{\theta}\right)$ ($\tilde{\theta}$ is the real value of θ). Of course, we denote by Φ the cdf of the standard normal distribution.

The standard normal cdf Φ computed for the above two values are computed by the Monte Carlo method [12] as follows. We generate first 10000 standard normal variables, using either the central limit method, the Box-Muler method, or the Butcher 1 method. We use in our $C++$ program the Box-Muler method, because is the most rapid. The value $\Phi(\alpha)$ is equal to the ratio of the generated values less than or equal to α. We use the same methodology for the left sided and for the right sided Bayes test that follows.

Definition 4. *The left-sided Bayes test checks with the threshold* ε *the null hypothesis* $H_0 : \theta = \theta_0$ *against the alternative hypothesis* $H_1 : \theta < \theta_0$.

In this case we consider the prior distribution of θ having the support $(-\infty,\theta_0)$, and we replace this prior pdf and the posterior pdf of θ (which has the same support) in (12). The differences to the two-sided test are that in the case of (11) both distributions of θ (prior and posterior) have the support, $\mathbb{R}$.

If we denote by φ the pdf of the standard normal distribution and we consider the prior distribution of θ as $N(\mu,\tau^2)$ left-truncated in θ_0, we can prove as in the case of two-sided Bayes test that the posterior distribution of θ is $N\left(\frac{\delta\tau^2\bar{\theta}+\mu}{\delta\tau^2+1},\frac{\tau^2}{\delta\tau^2+1}\right)$, also left-truncated in θ_0. Because $\theta_0=0$ we obtain

$$\begin{cases} f\left(\hat{\theta}\middle|\theta;\delta\right)=\sqrt{\delta}\varphi\left(\sqrt{\delta}\left(\hat{\theta}-\theta\right)\right) \\ g\left(\theta\right)=\dfrac{\varphi\left(\frac{\theta-\mu}{\tau}\right)}{\tau\cdot\Phi\left(\frac{-\mu}{\tau}\right)} \\ g\left(\theta\middle|\hat{\theta};\delta\right)=\dfrac{\sqrt{\delta\tau^2+1}}{\tau}\cdot\dfrac{\varphi\left(\frac{\sqrt{\delta\tau^2+1}}{\tau}\left(\theta-\frac{\delta\tau^2\hat{\theta}+\mu}{\delta\tau^2+1}\right)\right)}{\Phi\left(-\frac{\delta\tau^2\hat{\theta}+\mu}{\tau\sqrt{\delta\tau^2+1}}\right)} \end{cases} \tag{19}$$

In this case it results that

$$\begin{cases} R_\delta\left(\hat{\theta},\mu,\tau^2\right)=\dfrac{1}{\sqrt{\delta\tau^2+1}}\cdot\dfrac{\varphi\left(-\frac{\mu}{\tau}\right)}{\varphi\left(-\frac{\delta\tau^2\hat{\theta}+\mu}{\tau\sqrt{\delta\tau^2+1}}\right)}\cdot\dfrac{\Phi\left(-\frac{\delta\tau^2\hat{\theta}+\mu}{\tau\sqrt{\delta\tau^2+1}}\right)}{\Phi\left(-\frac{\mu}{\tau}\right)} \\ R_\delta\left(0,\mu,\tau^2\right)=\dfrac{1}{\sqrt{\delta\tau^2+1}}\cdot\dfrac{\varphi\left(-\frac{\mu}{\tau}\right)}{\varphi\left(-\frac{\mu}{\tau\sqrt{\delta\tau^2+1}}\right)}\cdot\dfrac{\Phi\left(-\frac{\mu}{\tau\sqrt{\delta\tau^2+1}}\right)}{\Phi\left(-\frac{\mu}{\tau}\right)} \\ R_\delta\left(\mu,\mu,\tau^2\right)=\dfrac{1}{\sqrt{\delta\tau^2+1}}\cdot\dfrac{\varphi\left(-\frac{\mu}{\tau}\right)}{\varphi\left(-\frac{\mu\sqrt{\delta\tau^2+1}}{\tau}\right)}\cdot\dfrac{\Phi\left(-\frac{\mu\sqrt{\delta\tau^2+1}}{\tau}\right)}{\Phi\left(-\frac{\mu}{\tau}\right)} \end{cases} \tag{20}$$

where φ is the standard normal pdf.

Now, due to the fact that the function $\frac{\varphi}{\Phi}$ is decreasing on $\mathbb{R}$, having the values 1 for $x=0$ and 0 for $x\to\infty$, we compute first the value τ_0^2 as in the case of the two-sided test. We can now compute for any τ^2 the value $\mu_{\max}<0$ such that if $\mu<\mu_{\max}$ we accept the null hypothesis if $\hat{\theta}=0$, and we reject it if $\hat{\theta}=\mu$. The function R_δ is decreasing on $[\mu,0]$ with respect to $\hat{\theta}$. We obtain then by the bisection method $\hat{\theta}_1<0$ such that we accept the null hypothesis if and only if $\hat{\theta}_1<\hat{\theta}\le 0$. The power of the test becomes

$$\pi\left(\tilde{\theta}\right)=\Phi\left(\sqrt{\delta}\left(\hat{\theta}-\tilde{\theta}\right)\right) \tag{21}$$

In the case of the left sided Bayes test we can take also a prior distribution such that $-\theta$ is an exponential random variable. It results that $\theta|\hat{\theta};\delta$ has the distribution $N\left(\hat{\theta}+\frac{\lambda}{\delta},\frac{1}{\delta}\right)$ left-truncated in 0.

Therefore the involved pdfs are

$$\begin{cases} f\left(\hat{\theta}\middle|\theta;\delta\right)=\sqrt{\delta}\varphi\left(\sqrt{\delta}\left(\hat{\theta}-\theta\right)\right) \\ g\left(\theta\right)=\lambda\cdot e^{\lambda\theta} \\ g\left(\theta\middle|\hat{\theta};\delta\right)=\sqrt{\delta}\cdot\dfrac{\varphi\left(\sqrt{\delta}\left(\theta-\hat{\theta}-\frac{\lambda}{\delta}\right)\right)}{\Phi\left(\sqrt{\delta}\left(-\hat{\theta}-\frac{\lambda}{\delta}\right)\right)} \end{cases} \tag{19'}$$

It results that

$$\begin{cases} R_{\delta}\left(\hat{\theta},\lambda\right)=\dfrac{\lambda}{\sqrt{\delta}}\cdot\dfrac{\Phi\left(-\sqrt{\delta}\left(\hat{\theta}+\frac{\lambda}{\delta}\right)\right)}{\varphi\left(-\sqrt{\delta}\left(\hat{\theta}+\frac{\lambda}{\delta}\right)\right)} \\ R_{\delta}\left(0,\lambda\right)=\dfrac{\lambda}{\sqrt{\delta}}\cdot\dfrac{\Phi\left(-\frac{\lambda}{\sqrt{\delta}}\right)}{\varphi\left(-\frac{\lambda}{\sqrt{\delta}}\right)} \end{cases} \tag{22}$$

From the above monotony of $\frac{\varphi(x)}{\Phi(x)}$ on $\mathbb{R}$ it results that $-R_{\delta}\left(0,\lambda\right)$ is increasing on $-\frac{\lambda}{\sqrt{\delta}}$. Therefore $R_{\delta}\left(0,\lambda\right)$ is increasing on λ. Because $R_{\delta}\left(0,0\right)=0$ and $\lim_{\lambda\to\infty}R_{\delta}\left(0,\lambda\right)=\infty$, it results that there exists $\lambda_{\max}>0$ such that if $\lambda<\lambda_{\max}$ we accept the null hypothesis if $\hat{\theta}=0$.

Now we fix $\lambda<\lambda_{\max}$. Because the second fraction in (22) is increasing on the function argument, it results that we can find the interval $\left(\hat{\theta}_1,0\right)$ such that $R_{\delta}\left(\hat{\theta}_1,\lambda\right)=\gamma_0\left(p_0,\varepsilon\right)$.

Definition 5. *The right sided Bayes test checks with the threshold* ε *the null hypothesis* $H_0:\theta=\theta_0$ *against the alternative hypothesis* $H_1:\theta>\theta_0$.

By symmetry reasons, in this case we consider the prior distribution of θ having the support $\left(\theta_0,\infty\right)$, and we replace this prior pdf and the posterior pdf of θ (which has the same support) in (12).

The considered prior distributions are also symmetric to those of the case of the left sided Bayes test, namely the right-truncated in θ_0 prior distribution, respectively the prior distribution such that $\theta-\theta_0\sim\exp\left(\lambda\right)$. The relations (19) and (19') become

$$\begin{cases} f\left(\hat{\theta}\middle|\theta;\delta\right)=\sqrt{\delta}\varphi\left(\sqrt{\delta}\left(\hat{\theta}-\theta\right)\right) \\ g\left(\theta\right)=\dfrac{\varphi\left(\frac{\theta-\mu}{\tau}\right)}{\tau\left(1-\Phi\left(\frac{-\mu}{\tau}\right)\right)}=\dfrac{\varphi\left(\frac{\theta-\mu}{\tau}\right)}{\tau\cdot\Phi\left(\frac{\mu}{\tau}\right)} \\ g\left(\theta\middle|\hat{\theta};\delta\right)=\dfrac{\sqrt{\delta\tau^2+1}}{\tau}\cdot\dfrac{\varphi\left(\frac{\sqrt{\delta\tau^2+1}}{\tau}\left(\theta-\frac{\delta\tau^2\hat{\theta}+\mu}{\delta\tau^2+1}\right)\right)}{1-\Phi\left(-\frac{\delta\tau^2\hat{\theta}+\mu}{\tau\sqrt{\delta\tau^2+1}}\right)}=\dfrac{\sqrt{\delta\tau^2+1}}{\tau}\cdot\dfrac{\varphi\left(\frac{\sqrt{\delta\tau^2+1}}{\tau}\left(\theta-\frac{\delta\tau^2\hat{\theta}+\mu}{\delta\tau^2+1}\right)\right)}{\Phi\left(\frac{\delta\tau^2\hat{\theta}+\mu}{\tau\sqrt{\delta\tau^2+1}}\right)} \end{cases} \text{, and} \tag{23}$$

$$\begin{cases} f\left(\hat{\theta}\middle|\theta;\delta\right)=\sqrt{\delta}\varphi\left(\sqrt{\delta}\left(\hat{\theta}-\theta\right)\right) \\ g\left(\theta\right)=\lambda\cdot e^{-\lambda\theta} \\ g\left(\theta\middle|\hat{\theta};\delta\right)=\sqrt{\delta}\cdot\dfrac{\varphi\left(\sqrt{\delta}\left(\hat{\theta}-\frac{\lambda}{\delta}-\theta\right)\right)}{\Phi\left(\sqrt{\delta}\left(\hat{\theta}-\frac{\lambda}{\delta}\right)\right)} \end{cases} \tag{23'}$$

Due to the same symmetry reasons we compute τ_0^2 and $\mu_{\min}^2$ and $\lambda_{\max}$ as in the case of the left-sided Bayes test. The interval becomes $\left(0,\hat{\theta}_2\right)$, and the power of the test is

$$\pi_\delta\left(\tilde{\theta}\right)=1-\Phi\left(\sqrt{\delta}\left(\hat{\theta}_2-\tilde{\theta}\right)\right) \tag{24}$$

2.2. Case of Unknown Variance

Consider $\hat{\theta}$ having the same distribution from (11), but the variance σ^2 is unknown. Suppose that there exists S^2 an estimator of σ^2 with m degrees of freedom, i.e. $\frac{\hat{\theta}-\theta}{S}\sim t_m$ (the Student distribution with m degrees of freedom). It results that $\frac{m\cdot S^2}{\sigma^2}\sim\chi_m^2$, and if we take the prior distribution of δ as $\Gamma\left(a,b\right)$ we obtain

$$\begin{cases} S^2\middle|\,\delta\sim\Gamma\left(\frac{m}{2},\frac{2}{m\cdot\delta}\right) \\ \delta\sim\Gamma\left(a,b\right) \\ \delta\middle|\,S^2\sim\Gamma\left(\frac{m}{2}+a,\frac{2b}{mbS^2+2}\right) \end{cases} \tag{25}$$

We apply now the total probability formula for (13) and we obtain

$$P\left(\theta=\theta_0\middle|\hat{\theta}\right)=E_\delta\left\{P\left(\theta=\theta_0\middle|\hat{\theta};\delta\right)\right\} \tag{26}$$

i.e. the expectation of $P\left(\theta=\theta_0\middle|\hat{\theta};\delta\right)$ using the conditional pdf of δ from (25).

Due to the fact that $\frac{1}{\sqrt{\delta\tau^2+1}}$ is decreasing on τ^2, it results that $\frac{p_0}{p_0+\frac{1-p_0}{\sqrt{\delta\tau^2+1}}}$ is increasing on τ^2, and from here we derive that $E_\delta\left\{\frac{p_0}{p_0+\frac{1-p_0}{\sqrt{\delta\tau^2+1}}}\right\}$ is increasing on τ^2. Because the above expectation is equal to p_0 for $\tau^2=0$, and it tends to 1 for $\tau^2\to\infty$, it results that there exists τ_0^2 such that

$$E_\delta\left\{\frac{p_0}{p_0+\dfrac{1-p_0}{\sqrt{\delta\tau_0^2+1}}}\right\}=1-\varepsilon \tag{27}$$

In the case of the two-sided Bayes test we notice first that R_δ from (15') and P have inverse monotony: for fixed δ, when R_δ is increasing on a parameter, P from (13) is decreasing, and vice versa.

In this way we obtain the existence of $\mu_{\min}^2$, and from it the value $\mu_{\min}$ for $\hat{\theta}>0$, and $\mu_{\max}$ respectively for $\hat{\theta}>0$ (the sample value) such that we accept the null hypothesis if $\hat{\theta}\to 0$ (the imposed value), and we reject the null hypothesis if $\hat{\theta}=\mu$. Note that $\mu_{\min}$, and $\mu_{\max}$ respectively

does not depend on δ, due to the expectation in (26). In the same way we obtain $\hat{\theta}_1 < 0 < \hat{\theta}_2$ such that we accept the null hypothesis if and only if $\hat{\theta}_1 < \hat{\theta} < \hat{\theta}_2$. For the non-dependence of $\mu_{\min}$, and $\mu_{\max}$ respectively as well as of $\hat{\theta}_j$ on δ we have to solve the equations

$$P\left(\theta = \theta_0 \middle| \hat{\theta} = 0\right) = 1 - \varepsilon \tag{28}$$

if $\tau^2 < \tau_0^2$, respectively

$$P\left(\theta = \theta_0 \middle| \hat{\theta} = \mu\right) = 1 - \varepsilon \tag{28'}$$

if $\tau^2 > \tau_0^2$. If $\tau^2 = \tau_0^2$ we solve none of the above equations: we solve initially both equations, first one for $\tau^2 = 0.95 \cdot \tau_0^2$ and the second one for $\tau^2 = 1.05 \cdot \tau_0^2$, and we obtain $\mu_{\min}$, and $\mu_{\max}$ respectively as the average of the two cases. Next, for obtaining $\hat{\theta}_1$ or $\hat{\theta}_2$ (depending on the sample value of $\hat{\theta}$) we have to solve the equation

$$P\left(\theta = \theta_0 \middle| \hat{\theta}\right) = 1 - \varepsilon \tag{28''}$$

Due to the fact that the interval $\left(\hat{\theta}_1, \hat{\theta}_2\right)$ does not depend on δ, we use π_δ from (18), and we obtain

$$\pi\left(\tilde{\theta}\right) = E_\delta\left\{\pi_\delta\left(\tilde{\theta}\right)\right\} \tag{29}$$

In the case of the left-sided Bayes test we obtain analogously the existence of τ_0^2, and of $\mu_{\max} < 0$ such that for μ and τ^2 fixed we accept the null hypothesis if $\hat{\theta} = 0$, and we reject the null hypothesis if $\hat{\theta} = \mu$. We obtain also $\hat{\theta}_1 < 0$ such that we accept the null hypothesis if and only if $\hat{\theta}_1 < \hat{\theta} < 0$.

If we use the prior and posterior pdfs from (19') we obtain first $\lambda_{\max}$ such that for $0 < \lambda < \lambda_{\max}$ we accept the null hypothesis if $\hat{\theta} = 0$. Next we obtain the value $\hat{\theta}_1 < 0$ such that we accept the null hypothesis if and only if $\hat{\theta}_1 < \hat{\theta} < 0$.

For the right-sided Bayes test we obtain, using also the symmetry used in the case of known δ, it results the existence of τ_0^2 and of $\mu_{\min} > 0$ in the case of right-truncated normal prior (and posterior) distribution, and of $\lambda_{\max}$ in the case of exponential prior distribution such that if μ and τ^2 are (respectively λ is) fixed we accept the null hypothesis if $\hat{\theta} = 0$. Finally, we obtain $\hat{\theta}_2 > 0$ such that we accept the null hypothesis if and only if $0 < \hat{\theta} < \hat{\theta}_2$.

In the cases of the left-sided and right-sided Bayes test $\hat{\theta}_1$, and $\hat{\theta}_2$ respectively do not depend on δ, due to equation (28''). Therefore we apply (29) for (21) and (24) for obtaining the power of the test in each case.

3. Applications

Example 1. *Consider [7] the linear regression between the household savings (EP) and net nominal average earning per month (SNN). The considered period is between January 1997 and August 2003. Denoting by* $D(EP)$ *and* $D(SNN)$ *the differences between the current value and those of the previous month, we obtain* 79 *data points.*
Considering the explanatory variable $X = D(SNN)$ *and the resulting variable* $Y = D(EP)$ *we obtain the linear regression [7]*

$$Y = 0.90349 + 3.23 \cdot X,$$

with the estimated standard deviations for the above coefficients 0.13296, *respectively* 0.904.

The Student statistics having 77 degrees of freedom obtained in [7] are 6.795, *respectively* 3.573.

With our $C++$ program we obtain the linear regression

$$Y = 0.90342 + 3.23042 \cdot X,$$

with the estimated standard deviations for the above coefficients 0.13297, respectively 0.90414, and finally the Student statistics having 77 degrees of freedom 6.79415, respectively 3.57293.

In the case of the intercept the standard normal cdf is equal to the Student with 77 degrees of freedom, and the common value is $F(6.79415) = 0.99999$. In the case of the coefficient of X we obtain $\Phi(3.57293) = 0.99982$ for standard normal cdf, and $F(3.57293) = 0.99969$ for the Student cdf.

The power of the test in Table 1 and Table 2 are computed if the real value of θ is zero, if the real value of θ is equal to the right limit of the acceptance interval (the interval such that we accept the null hypothesis $H_0 : \theta = 0$ if and only if θ is in the interval), and if the real value of θ is equal to the estimated value respectively. For instance, in the case of known variance of the intercept (Table 1), the first accepting interval is $(-0.04076, 0.04076)$. The power of the test is 0.769 if the real expectation is $\theta = 0$, 0.7691 if the real expectation is $\theta = 0.04076$, respectively $> 1 - 5 \cdot 10^{-6}$ (the $C++$ program gives the power equal to one, with five significant decimals) if the real expectation is $\theta = 0.90342$.

In Table 1 that follows we present the Bayes test in the case of the intercept if we know the variance (hence δ is known).

Table 1

The Bayes test for the intercept in the case of known variance

τ_0^2 / $\lambda_{\max}$	τ^2 / λ	Type of the test	$\mu > \mu_{\min}$	Probability of accepting H_0	Interval for accepting H_0	Power of the test
$\tau_0^2 =$ 6.36523	$\tau^2 = 6$	Two-sided	$1 > 0.59545$	$< 5 * 10^{-6}$	$(-0.04076, 0.04076)$	0.759
						0.7619
						$\geq 1 - 5 * 10^{-6}$
		Right-sided	$0.1 > 0.09473$	$< 5 * 10^{-6}$	$(0, 0.00027)$	0.5027
						0.4941
						$\geq 1 - 5 * 10^{-6}$
	$\tau^2 =$ 6.36523	Two-sided	$0.5 > 0.29314$	$< 5 * 10^{-6}$	$(-0.025, 0.025)$	0.845
						0.8392
						$\geq 1 - 5 * 10^{-6}$
		Right-sided	$0.05 > 0.04344$	$< 5 * 10^{-6}$	$(0, 0.0025)$	0.4864
						0.4934
						$\geq 1 - 5 * 10^{-6}$
	$\tau^2 = 7$	Two-sided	$0.05 > 0.04095$	$< 5 * 10^{-6}$	$(-0.04095, 0.04095)$	0.7594
						0.7584
						$\geq 1 - 5 * 10^{-6}$
		Right-sided	$0.01 > 0.00819$	$< 5 * 10^{-6}$	$(0, 0.00827)$	0.4772
						0.4936
						$\geq 1 - 5 * 10^{-6}$
$\lambda_{\max} =$ 0.32685	$\lambda = 0.25$	Right-sided		$< 5 * 10^{-6}$	$(0, 0.04095)$	0.3898
						0.5039
						$\geq 1 - 5 * 10^{-6}$

Analogous results are obtained in Table 2 in the case of the coefficient of X (A_1).

Table 2

The Bayes test for A_1 in the case of known variance

τ_0^2 / $\lambda_{\max}$	τ^2 / λ	Type of the test	$\mu > \mu_{\min}$	Probability of accepting H_0	Interval for accepting H_0	Power of the test
$\tau_0^2 =$ 294.28825	$\tau^2 = 290$	Two-sided	$2.25 > 2.06324$	0.03094	$(-0.04172, 0.04172)$	0.9532
						0.9541
						$> 1 - 5 * 10^{-6}$
		Right-sided	$0.25 > 0.16457$	0.01611	$(0, 0.00433)$	0.501
						0.4986
						0.9998
	$\tau^2 =$ 294.28825	Two-sided	$2.25 > 1.99322$	0.03116	$(-0.1125, 0.1125)$	0.9041
						0.8995
						0.9995
		Right-sided	$0.5 > 0.29537$	0.01637	$(0, 0.025)$	0.4904
						0.4945
						0.9999
	$\tau^2 = 300$	Two-sided	$0.25 > 0.12519$	0.03186	$(-0.12536, 0.12536)$	0.8883
						0.8939
						0.9996
		Right-sided	$0.05 > 0.01141$	0.01626	$(0, 0.01331)$	0.4843
						0.4999
						0.9998
$\lambda_{\max} =$ 0.04807	$\lambda = 0.025$	Right-sided		0.03133	$(0, 0.61912)$	0.25
						0.5013
						0.9976

As we can see in the above tables, we consider seven prior distributions for each coefficient. The first three are for the two-sided Bayes test (one for $\tau^2 < \tau_0^2$, one for $\tau^2 > \tau_0^2$ and one for $\tau^2 = \tau_0^2$), and next three are analogous, but for the right-sided Bayes test. The last prior distribution considered is exponential, in the case of the right-sided Bayes test.

For the intercept, $A_0 = 0.90342$ with the estimated variance 0.01768, we consider the prior distributions for the two-sided Bayes test $N(\mu, \tau^2)$ being $N(1,6)$, $N(0.05,7)$, and $N(0.5,6.35232)$ respectively. We obtain the posterior distributions $N(0.90371, 0.01763)$, $N(0.90127, 0.01764)$, and $N(0.9023, 0.01763)$ respectively. For the right-sided Bayes test with left-truncated prior distribution we consider $N(0.1,6)$ left-truncated in zero, $N(0.01,7)$ left-truncated in zero, and $N(0.05,6.36523)$ left-truncated in zero, obtaining the posterior distributions $N(0.90106, 0.01763)$ left-truncated in zero, $N(0.90117, 0.01764)$ left-truncated in zero, and $N(0.90106, 0.01763)$ left-truncated in zero respectively. For the exponential case we consider prior distribution $\exp(0.25)$, and we obtain the posterior distribution $N(0.899, 0.01768)$ left-truncated in zero.

For the coefficient of X, $A_1 = 3.23042$ with the estimated variance 0.81747, we obtain analogous results. For the two-sided Bayes test we consider prior distributions $N(2.25,290)$, $N(0.25,300)$, and $N(2.25,294.28825)$, obtaining the posterior distributions $N(3.22767,0.81517)$, $N(3.22232,0.81525)$, and $N(3.22771,0.8152)$ respectively. For the right-sided Bayes test with left-truncated prior distribution we consider $N(0.25,290)$ left-truncated in zero, $N(0.05,300)$ left-truncated in zero, and $N(0.5,294.28825)$ left-truncated in zero, obtaining the posterior distributions $N(3.22204,0.81517)$ left-truncated in zero, $N(3.22178,0.81525)$ left-truncated in zero, and $N(3.22286,0.81521)$ left-truncated in zero respectively. For the exponential case we consider prior distribution $\exp(0.025)$, and we obtain the posterior distribution $N(3.20999,0.81747)$ left-truncated in zero.

In the case of unknown variance, we consider prior distribution of δ being $\Gamma(0.5,1)$ for both coefficients, and we obtain the posterior distribution $\Gamma(38.5,0.59498)$ in the case of intercept, and $\Gamma(38.5,0.0308)$ in the case of the coefficient of X respectively. The results are listed in Tables 3 and 4.

Table 3

The Bayes test for the intercept in the case of unknown variance

τ_0^2 / $\lambda_{\max}$	τ^2 / λ	Type of the test	$\mu > \mu_{\min}$	Probability of accepting H_0	Interval for accepting H_0	Power of the test
$\tau_0^2 =$ 15.97486	$\tau^2 = 15$	Two-sided	$1.25 > 0.97509$	0.00387	$(-0.03768, 0.03768)$	0.85665
						0.85947
						0.99997
		Right-sided	$0.25 > 0.17384$	0.00225	$(0, 0.00413)$	0.49986
						0.50808
						0.99997
	$\tau^2 =$ 15.97486	Two-sided	$0.5 > 0.47295$	0.00413	$(-0.02258, 0.02258)$	0.9142
						0.91398
						0.99998
		Right-sided	$0.1 > 0.08131$	0.00217	$(0, 0.00516)$	0.49994
						0.50988
						0.99997
	$\tau^2 = 20$	Two-sided	$0.1 > 0.09962$	0.00467	$(-0.09808, 0.09808)$	0.63957
						0.67443
						0.99988
		Right-sided	$0.05 > 0.02901$	0.00231	$(0, 0.03066)$	0.4999
						0.55762
						0.99996
$\lambda_{\max} =$ 0.20704	$\lambda = 0.1$	Right-sided		0.00189	$(0, 0.00001)$	0.49989
						0.49987
						0.99997

Table 4

The Bayes test for A_1 in the case of unknown variance

τ_0^2 / $\lambda_{\max}$	τ^2 / λ	Type of the test	$\mu > \mu_{\min}$	Probability of accepting H_0	Interval for accepting H_0	Power of the test
$\tau_0^2 =$ 309.75413	$\tau^2 = 305$	Two-sided	$2 > 1.62499$	0.05227	$(-0.0315, 0.0315)$	0.97273
						0.97283
						0.99991
		Right-sided	$0.25 > 0.22071$	0.02801	$(0, 0.00819)$	0.4997
						0.50355
						0.99961
	$\tau^2 =$ 309.75413	Two-sided	$2.25 > 1.96249$	0.05158	$(-0.13184, 0.13184)$	0.88642
						0.88705
						0.99967
		Right-sided	$0.5 > 0.31447$	0.02738	$(0, 0.02575)$	0.49992
						0.51119
						0.99961
	$\tau^2 = 315$	Two-sided	$0.25 > 0.10448$	0.05341	$(-0.12312, 0.12312)$	0.89318
						0.89448
						0.9997
		Right-sided	$0.01 > 0.00991$	0.02989	$(0, 0.00858)$	0.50009
						0.50374
						0.99962
$\lambda_{\max} =$ 0.04685	$\lambda = 0.025$	Right-sided		0.05049	$(0, 0.61179)$	0.25256
						0.50007
						0.99723

4. Conclusions

The condition that must be fulfilled to have good properties of the test (small errors of the first/second type if we accept/reject the null hypothesis) is that the prior expectation is different enough of θ_0. This is the explanation of the choosing for $\mu > \mu_{\min}$ and $0 < \lambda < \lambda_{\max}$: the prior expectation is μ in the first case, and $\frac{1}{\lambda}$ in the second case respectively. The fact that there exists a value $\left(\tau_0^2\right)$ for each coefficient such that $\mu_{\min} = 0$ in the cases of two-sided and right-sided Bayes tests can be explained by the denominator equal to $2 \cdot \tau^2$ in the prior pdf formula (and analogous for the posterior pdf).

The formula (2) from [10] was obtained also in [6], but in the case of $\overline{X}$ instead of X, hence the known variance becomes $\frac{\sigma^2}{n}$. But, due to the estimation of the variance of each regression coefficient, according (7) and (8) we prefer to consider σ^2 the known variance of the current coefficient (the case of known variance). In the case of unknown variance, we consider the unbiased estimator of the variance.

Of course, in both cases of known/unknown variance the power of the test depends on the real value of θ, $\tilde{\theta}$. In our $C++$ program we will compute this power if $\tilde{\theta} = 0$, if $\tilde{\theta}$ is a limit of the acceptance interval different to zero (in the case of right-sided Bayes test), and if $\tilde{\theta} = \hat{\theta}$ respectively, to illustrate the performances of the test. We notice that the power is high in the last case, which is a good thing, taking into account that we reject the null hypothesis for both coefficients. Unfortunately the power is too high if $\tilde{\theta} = 0$ (Tables $1 \to 4$), and an open problem is how to choose the prior distribution to decrease the power in this case. The same thing we can say about the two-sided Bayes test in the case of the limit of the interval. For the right-sided

Bayes test the good properties of the test remains in the third case, and in the second case (the limit of the acceptance interval) the power becomes arround 0.5 (which is good, taking into account that we accept the null hypothesis $\theta = 0$ if and only if $\hat{\theta}$ is in the interior of the interval). In the first case $\left(\tilde{\theta} = 0\right)$ with prior distribution left-truncated normal, the power is arround 0.5, which is also too high. In this case the best results are obtained for exponential prior distribution: 0.3898 in Table 1, 0.25 in Table 2, and 0.25356 in Table 4 respectively.

In the case of the Student test we know that the unkown variance case can be reduced to the known variance case if the involved number of degrees of freedom is at least 30. In our paper case we notice that the results of Tables 2 and 4 are closer than that of Tables 1 and 3. An explanation can be found in the low of large numbers and formula (25): the posterior expectation of δ tends to $\frac{1}{S^2}$ while $\frac{m}{2} + a$ and $\frac{mbS^2+2}{2b}$ tend to ∞. We can say the same thing about the convergence to zero of the variance. In our case only $\frac{m}{2} + a = 38.5$ is large enough in both cases, but the second term is $\frac{1}{0.59498} = 1.68073$ for the intercept, respectively $\frac{1}{0.0308} = 32.76532$ for the coefficient of X (the slope). This is because the value S^2 with which we multiply $\frac{m}{2}$ is too small $\left(0.01768\right)$ for the intercept, and it compensates the large size of $\frac{m}{2}$. It is not the case of the slope, when $S^2 = 0.81747$, which is around one. An open problem is if we can obtain in the case of the intercept (and even in the case of the slope) better results if we use non-informative prior distribution [5].

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