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An existence result for two-dimensional parabolic integro-differential equations involving CEV model

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Abstract. In this paper, we present an existence result of weak solutions for some parabolic equations involving the so-called CEV model with jumps.

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1. Introduction

In this paper, we deal with the two-dimensional Dirichlet problem associated to the following parabolic integro-differential equations (PIDE):

$$\begin{cases} \dfrac{\partial u}{\partial \tau} + Au + (\lambda_1 + \lambda_2)u = Iu & \tau \in]0,T], \quad (x,y) \in \mathbb{R}^2, \\ u(x,y,0) = u_0(x,y) & (x,y) \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

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where A and I are the operators defined by

$$\begin{aligned} Au &= -\frac{\sigma_1^2 e^{-2(\alpha_1-1)x}}{2}\frac{\partial^2 u}{\partial x^2} - \frac{\sigma_2^2 e^{-2(\alpha_2-1)y}}{2}\frac{\partial^2 u}{\partial y^2} - (1-\lambda_1\kappa_1 - \frac{1}{2}\sigma_1^2 e^{2(\alpha_1-1)x})\frac{\partial u}{\partial x} \\ &\quad -(1-\lambda_2\kappa_2 - \frac{1}{2}\sigma_2^2 e^{2(\alpha_2-1)y})\frac{\partial u}{\partial y} + \rho\sigma_1\sigma_2 e^{-(\alpha_1-1)x}e^{-(\alpha_2-1)y}\frac{\partial^2 u}{\partial x \partial y} + ru, \end{aligned}$$

and

$$Iu = \lambda_1 \int_{-\infty}^{+\infty} u(x+z,y,\tau)\Gamma_\delta(z)\,dz + \lambda_2 \int_{-\infty}^{+\infty} u(x,y+z,\tau)\Gamma_\delta(z)dz,$$

where

$$\Gamma_\delta(z) = \frac{1}{\sqrt{2\pi}\delta}\exp(-\frac{z^2}{2\delta^2}), \quad \forall \delta > 0.$$

The operator $G(u) = A(u) + (\lambda_1+\lambda_2)u - Iu$ appears when we studying the pricing option under the CEV model with jumps.

In the case of two dimensions, by introducing the correlation coefficient between the Brownian motions of two different underlying prices, the stochastic differential equations for the underlying prices S_1, S_2 can be written as:

$$\frac{dS_1}{S_1} = \mu_1 dt + \sigma_1 S_1^{\alpha_1-1} dW_1 + (\eta_1 - 1)dq_1, \tag{1.2}$$

$$\frac{dS_2}{S_2} = \mu_2 dt + \sigma_2 S_2^{\alpha_2-1} dW_2 + (\eta_2 - 1)dq_2, \tag{1.3}$$

$$dW_1 dW_2 = \rho dt. \tag{1.4}$$

where

$\mu_1, \mu_2 =$ drift rates,

$\sigma_1, \sigma_2 =$ underlying volatilities,

$dW_1, dW_2 =$ increments of standard Brownian motions,

$\eta_1 - 1, \eta_2 - 1 =$ impulse functions producing jumps from (S_1, S_2) to $(S_1\eta_1, S_2\eta_2)$,

$$dq_1 = \begin{cases} 0, & \text{with probability } 1-\lambda_1 dt, \\ 1, & \text{with probability } \lambda_1 dt, \end{cases}$$

$$dq_2 = \begin{cases} 0, & \text{with probability } 1-\lambda_2 dt, \\ 1, & \text{with probability } \lambda_2 dt, \end{cases}$$

$\lambda_1, \lambda_2 =$ mean arrival rates of Poisson jumps,

$\alpha_1, \alpha_2 =$ Parameters controls the relationship between volatility and underlying asset price and is characteristic to the CEV model with $0 < \alpha_1 < 1$ and $0 < \alpha_2 < 1$,

$\rho =$ correlation of two standard Brownian motions with $\rho \in [-1, 1]$.

The above model, (1.2), (1.3) and (1.4), is quite interesting because on the one hand it incorporates a local volatilities given by

$$\sigma_1(S_1, t) = \sigma_1 S_1^{\alpha_1-1} \quad \text{and} \quad \sigma_2(S_2, t) = \sigma_2 S_2^{\alpha_2-1}$$

which naturally captures implied volatilities skews. On the other hand, the integral term generally introduced to take into account certain rare events on the financial markets. Such models are more realistic than the Black-Scholes ones, see [2].
The main feature of the CEV part of the model is that it allows the volatility to change with the underlying price. For more details about such model, see for instance [5, 6]. The CEV model extends the Black-Scholes model by incorporating a deterministic relationship between the stock price and the volatility. When $\alpha_1 > 1$ and $\alpha_2 > 1$, the volatility increases as the stock price increases. If $\alpha_1 < 1$ and $\alpha_2 < 1$ then the volatility and the stock price are inversely related. Moreover, if $\alpha_1 = 1$ and $\alpha_2 = 1$, then the volatility is constant with respect to the stock price and the process degenerates to the Black-Scholes model with jumps see [1, 13, 11]. The same discussion can be renewed according to the value of λ_1, λ_2, (1.2), (1.3) and (1.4) becomes a CEV model.

The interest in these processes (Black-Scholes model with jumps) increased dramatically after the 1987 stock market's crash, the event that has underscored the inability of pure diffusion models. In the last years these processes started appearing more and more frequently in energy applications. For more details about these type of problem, one can see [8, 12, 7].
Under the assumption of Merton's paper [13], the option values can be computed as the expectations of the option payoffs in the risk-neutral world, in which price evolution is governed by the process (1.2) and (1.3) (with $\alpha_1 = \alpha_2 = 1$) when the drift parameter was replaced by

$$\mu_1 = r - \lambda_1\kappa_1 \text{ and } \mu_2 = r - \lambda_2\kappa_2.$$

In this paper, we are interested in the studding of the following model combining the CEV and the jump models under the Merton's assumptions cited above

$$\frac{dS_1}{S_1} = (r - \lambda_1\kappa_1)dt + \sigma_1 S_1^{\alpha_1 - 1} dW_1 + (\eta_1 - 1)dq_1, \tag{1.5}$$

$$\frac{dS_2}{S_2} = (r - \lambda_2\kappa_2)dt + \sigma_2 S_2^{\alpha_2 - 1} dW_2 + (\eta_2 - 1)dq_2, \tag{1.6}$$

$$dW_1 dW_2 = \rho dt, \tag{1.7}$$

where

$$\begin{aligned}
r &= \text{ risk free rate,}\\
\kappa_1 &= E\left[\eta_1 - 1\right], \text{ where } E\left[\eta_1\right] = \int_0^\infty \eta_1 \tilde{\Gamma}_\delta\left(\eta_1\right) d\eta_1,\\
\kappa_2 &= E\left[\eta_2 - 1\right], \text{ where } E\left[\eta_2\right] = \int_0^\infty \eta_2 \tilde{\Gamma}_\delta\left(\eta_2\right) d\eta_2,\\
\tilde{\Gamma}_\delta(\eta) &= \text{ probability density function of a bivariate normal distribution, given by}\\
\tilde{\Gamma}_\delta(\eta) &= \frac{\exp(-\frac{1}{2}(\frac{\log\eta}{\delta})^2)}{\sqrt{2\pi}\delta\eta}.
\end{aligned}$$

Our aim is to study the existence of weak solutions for the model (1.1), which is the corresponding PDE to the stochastic model (1.5, 1.6 and 1.7), in the sense of Theorem 2.1.
Problem (1.1) is already covered in dimension one see [9].

2. Main Results

Let v be the value of an European contract which depends both on S_1, S_2 and time t. It's easy to prove using Itô's formula, that $v(S_1, S_2, \tau)$ satisfies the following Partial Integral Differential Equation:

$$(\mathcal{P})\begin{cases} \dfrac{\partial v}{\partial \tau} - \dfrac{\sigma_1^2 S_1^{2\alpha_1}}{2}\dfrac{\partial^2 v}{\partial S_1^2} - \dfrac{\sigma_2^2 S_2^{2\alpha_2}}{2}\dfrac{\partial^2 v}{\partial S_2^2} - (r-\lambda_1\kappa_1)S_1\dfrac{\partial v}{\partial S_1} \\ \qquad -(r-\lambda_2\kappa_2)S_2\dfrac{\partial v}{\partial S_2} - \rho\sigma_1\sigma_2 S_1^{\alpha_1} S_2^{\alpha_2}\dfrac{\partial^2 v}{\partial S_2 \partial S_1} + (r+\lambda_1+\lambda_2)v \\ = \lambda_1 \displaystyle\int_0^{+\infty} v(S_1\eta_1, S_2, \tau)\tilde{\Gamma}_\delta(\eta_1)\, d\eta_1 + \lambda_2 \int_0^{+\infty} v(S_1, S_2\eta_2, \tau)\tilde{\Gamma}_\delta(\eta_2)\, d\eta_2, \\ \qquad\qquad \tau \in [0,T], S_1, S_2 > 0 \\ v(S_1, S_2, 0) = v_0(S_1, S_2), \qquad S_1, S_2 > 0, \end{cases} \tag{2.1}$$

where T is the maturity date and $\tau = T - t$ where t is the current time. In the case of a put option, the boundary conditions are:

$$\begin{aligned} v(S_1, S_2, \tau) &= K\exp(-r\tau) \quad \text{as} \quad (S_1, S_2) \to (0,0), \\ v(S_1, S_2, \tau) &\approx 0 \quad \text{as} \quad S_1, S_2 \to +\infty, \\ v_0(S_1, S_2) &= \min(S_1 + S_2 - K, 0). \end{aligned}$$

where K is the strike price.
In [3, 4] the authors studied problem $(\mathcal{P})$, in the case where $\alpha_1 = \alpha_2 = 1$, which corresponds to the standard Black-Scholes equation with jumps in the case of two dimeonsion. Furthermore, they have treated also the problem $(\mathcal{P})$ using the viscosity solutions, (see also the recent work of [12] dealing with some kind of parabolic equations involving jumps).

Express the option price in term of the logarithmic value of the stock price: we therefore use the logarithmic change of variable $\big(x = log(S_1), y = log(S_2)\big)$ and the unknown function

$$u(x, y, \tau) = v(e^x, e^y, \tau) = v(S_1, S_2, \tau),$$

it follows that

$$\frac{\partial v}{\partial \tau}(S_1, S_2, \tau) = \frac{\partial u}{\partial \tau}(x, y, \tau),$$

and

$$\frac{\sigma_1^2 S_1^{2\alpha_1}}{2}\frac{\partial^2 v}{\partial S_1^2}(S_1, S_2, \tau) = \frac{\sigma_1^2 e^{2\alpha_1 x}}{2}\Big(\frac{\partial^2 u}{\partial x^2}(x, y, \tau)e^{-2x} - e^{-2x}\frac{\partial u}{\partial x}(x, y, t)\Big),$$

and

$$\frac{\sigma_2^2 S_2^{2\alpha_2}}{2}\frac{\partial^2 v}{\partial S_2^2}(S_1, S_2, \tau) = \frac{\sigma_2^2 e^{2\alpha_2 y}}{2}\Big(\frac{\partial^2 u}{\partial y^2}(x, y, \tau)e^{-2y} - e^{-2y}\frac{\partial u}{\partial y}(x, y, \tau)\Big),$$

also we have

$$(r-\lambda_1\kappa_1)S_1\frac{\partial v}{\partial S_1}(S_1, S_2, \tau) = (r-\lambda_1\kappa_1)\frac{\partial u}{\partial x}(x, y, \tau)$$

and

$$(r-\lambda_2\kappa_2)S_2\frac{\partial v}{\partial S_2}(S_1,S_2,\tau)=(r-\lambda_2\kappa_2)\frac{\partial u}{\partial y}(x,y,\tau)$$

and

$$\rho\sigma_1\sigma_2S_1^{\alpha_1}S_2^{\alpha_2}\frac{\partial^2 v}{\partial S_2\partial S_1}(S_1,S_2,\tau)=\rho\sigma_1\sigma_2e^{(\alpha_1-1)x}e^{(\alpha_2-1)y}\frac{\partial^2 u}{\partial x\partial y}(x,y,\tau)$$

and

$$-(r+\lambda_1+\lambda_2)v(S_1,S_2,\tau)=-(r+\lambda_1+\lambda_2)u(x,y,\tau)\cdot$$

Let $\eta_1=e^z$, then $d\eta_1=e^z dz$, therefore

$$\lambda_1\int_0^{+\infty}v(S_1\eta_1,S_2,\tau)\tilde{\Gamma}_\delta(\eta_1)\,d\eta_1=\lambda_1\int_{-\infty}^{+\infty}u(x+z,y,\tau)\tilde{\Gamma}_\delta(e^z)e^z\,dz.$$

Similarly, we have

$$\lambda_2\int_0^{+\infty}v(S_1,\eta_2S_2,\tau)\tilde{\Gamma}_\delta(\eta_2)\,d\eta_2=\lambda_2\int_{-\infty}^{+\infty}u(x,y+z,\tau)\tilde{\Gamma}_\delta(e^z)e^z\,dz.$$

We deduce that

$$\begin{aligned}&\frac{\partial u}{\partial\tau}(x,y,\tau)-\frac{\sigma_1^2e^{2(\alpha_1-1)x}}{2}\frac{\partial^2u}{\partial x^2}(x,y,\tau)-\frac{\sigma_2^2e^{2(\alpha_2-1)y}}{2}\frac{\partial^2u}{\partial y^2}(x,y,\tau)-\rho\sigma_1\sigma_2e^{(\alpha_1-1)x}e^{(\alpha_2-1)y}\frac{\partial^2u}{\partial x\partial y}(x,y,\tau)\\&-(r-\lambda_1\kappa_1-\frac{\sigma_1^2e^{2(\alpha_1-1)x}}{2})\frac{\partial u}{\partial x}(x,y,\tau)-(r-\lambda_2\kappa_2-\frac{\sigma_2^2e^{2(\alpha_2-1)y}}{2})\frac{\partial u}{\partial y}(x,y,\tau)\\&+(r+\lambda_1+\lambda_2)u(x,y,t)=\lambda_1\int_{-\infty}^{+\infty}u(x+z,y,\tau)\Gamma_\delta(z)\,dz+\lambda_2\int_{-\infty}^{+\infty}u(x,y+z,\tau)\Gamma_\delta(z)\,dz.\end{aligned}$$

with $\tilde{\Gamma}_\delta(e^z)e^z=\Gamma_\delta(z)$.
We conclude that the problem $(\mathcal{P})$ becomes according to u:

$$(\mathcal{E})\begin{cases}\dfrac{\partial u}{\partial\tau}+Au+(\lambda_1+\lambda_2)u=Iu, & \tau\in[0,T],(x,y)\in\mathbb{R}^2,\\ u(x,y,0)=u_0(x,y), & (x,y)\in\mathbb{R}^2.\end{cases}\tag{2.2}$$

Before giving the principal result of this paper, let's remark that the introduction of the jump places us in the incomplete markets. Thus, the pricing by a portfolio replication is impossible and the uniqueness of an optimal strategy is not possible.

Theorem 2.1. *Let $Q_{\mathbb{R}}$ the infinite cylinder $\mathbb{R}^2\times(0,T)$.*
Assume that $u_0(x,y)e^{-(\alpha_1-1)x}e^{-(\alpha_2-1)y}\in L^2(\mathbb{R}^2)$, then problem (2.2) has a solution in the following sense:

$$\begin{cases}u\in L^2(0,T;H^1_{loc}(\mathbb{R}^2)),\\ -\displaystyle\int_{Q_{\mathbb{R}}}u\frac{\partial v}{\partial\tau}\,dx\,dy\,dt+\langle Au,v\rangle+(\lambda_1+\lambda_2)\int_{Q_{\mathbb{R}}}uv\,dx\,dy\,dt=\langle Iu,v\rangle,\end{cases}$$

For all $v\in\mathcal{D}(Q_{\mathbb{R}})$.

Remark 1. *If we take $\alpha_1<1$ and $\alpha_2<1$ with $u_0(x,y)=\min(\exp(x)+\exp(y)-K,0)$, one can see easily that $u_0(x,y)e^{-(\alpha_1-1)x}e^{-(\alpha_2-1)y}\in L^2(\mathbb{R}^2)$.*

Proof of Theorem 2.1.

Step 1: Approximate problem. For any $m \in \mathbb{N}^*$, let's denote $Q_m = B_m \times (0,T)$, $H = L^2(B_m)$ and $V = H^1(B_m)$ with $B_m = (-m,m)^2$. We need the following lemma.

Lemma 2.1. *We consider the following approximate parabolic problem*

$$(P_m)\begin{cases} \dfrac{\partial u_m}{\partial \tau} + Au_m + (\lambda_1+\lambda_2)u_m = I_m u_m & \text{in } Q_m, \\ u_m(x,y,\tau) = 0, & \text{on } \partial Q_m, \\ u_m(x,y,0) = u_0(x,y), & \text{in } B_m, \end{cases} \tag{2.3}$$

with

$$I_m(u_m) = \lambda_1\Big(\int_{-m}^{m} u_m(z,y,t)\Gamma_\delta(x-z)dz\Big) + \lambda_2\Big(\int_{-m}^{m} u_m(x,z,t)\Gamma_\delta(y-z)dz\Big).$$

By applying classical Lions Theorem (see [10]*), the problem (P_m) admits at least one weak solution $u_m \in L^2(0,T;V) \cap C^0([0,T];H)$ such that $\dfrac{\partial u_m}{\partial \tau} \in L^2(0,T;V')$.*

Step 2: A priori estimation. Let $Q_m^\omega = B_m \times (0,\omega)$ with $0 < \omega < T$. By choosing $v(x,y,t) = u_m(x,y,t)\theta^2(x,y)\chi_{(0,\omega)}(t)$ as a test function for the approximate problem (2.3), where $\theta(x,y) = e^{-(\alpha_1-1)x}e^{-(\alpha_2-1)y}$, we obtain

$$\begin{aligned} &\langle \frac{\partial u_m}{\partial \tau}, u_m\theta^2\rangle + \langle Au_m, u_m\theta^2\rangle + (\lambda_1+\lambda_2)\int_{Q_m^\omega} |u_m|^2\theta^2\,dx\,dy\,dt \\ &= \lambda_1\int_{Q_m^\omega}\Big(\int_{-m}^{m} u_m(x-z,y,t)\Gamma_\delta(x-z)dz\Big)u_m\theta^2\,dx\,dy\,dt \\ &\quad +\lambda_2\int_{Q_m^\omega}\Big(\int_{-m}^{m} u_m(x,y-z,t)\Gamma_\delta(y-z)dz\Big)u_m\theta^2\,dx\,dy\,dt. \end{aligned} \tag{2.4}$$

For the first term on the left-hand side of (2.4), we have

$$\begin{aligned} \langle \frac{\partial u_m}{\partial \tau}, u_m\theta^2\rangle &= \int_{Q_m^\omega} \frac{\partial u_m}{\partial t} u_m\theta^2\,dx\,dy\,dt \\ &= \frac{1}{2}\int_{B_m}\int_0^\omega \frac{\partial (u_m\theta)^2}{\partial t}\,dx\,dy\,dt \\ &= \frac{1}{2}\int_{B_m} |u_m(\omega)|^2\theta^2\,dx\,dy - \frac{1}{2}\int_{B_m} |u_m(0)|^2\theta^2\,dx\,dy. \end{aligned} \tag{2.5}$$

Concerning the second term on the left-hand side of (2.4), we obtain

$$\begin{aligned}
\langle Au_m, u_m\theta^2\rangle &= -\frac{\sigma_1^2}{2}\int_{Q_m^\omega} e^{2(\alpha_1-1)x}\frac{\partial^2 u_m}{\partial x^2}\theta^2 u_m\,dx\,dy\,dt - \frac{\sigma_2^2}{2}\int_{Q_m^\omega} e^{2(\alpha_2-1)y}\frac{\partial^2 u_m}{\partial y^2}\theta^2 u_m\,dx\,dy\,dt\\
&\quad -\int_{Q_m^\omega}(r-\lambda_1\kappa_1-\frac{1}{2}\sigma_1^2 e^{2(\alpha_1-1)x})\frac{\partial u_m}{\partial x}\theta^2 u_m\,dx\,dy\,dt\\
&\quad -\int_{Q_m^\omega}(r-\lambda_2\kappa_2-\frac{1}{2}\sigma_2^2 e^{2(\alpha_2-1)y})\frac{\partial u_m}{\partial y}\theta^2 u_m\,dx\,dy\,dt\\
&\quad -\rho\sigma_1\sigma_2\int_{Q_m^\omega} e^{(\alpha_1-1)x}e^{(\alpha_2-1)y}\frac{\partial^2 u_m}{\partial x\partial y}\theta^2 u_m\,dx\,dy\,dt + r\int_{Q_m^\omega}|u_m|^2\theta^2\,dx\,dy\,dt\\
&= \frac{\sigma_1^2}{2}\int_{Q_m^\omega} e^{2(\alpha_1-1)x}\left|\frac{\partial u_m}{\partial x}\right|^2\theta^2\,dx\,dy\,dt + \frac{\sigma_2^2}{2}\int_{Q_m^\omega} e^{2(\alpha_2-1)y}\left|\frac{\partial u_m}{\partial y}\right|^2\theta^2\,dx\,dy\,dt\\
&\quad -\int_{Q_m^\omega}(r-\lambda_1\kappa_1-\frac{1}{2}\sigma_1^2 e^{2(\alpha_1-1)x})\frac{\partial u_m}{\partial x}\theta^2 u_m\,dx\,dy\,dt\\
&\quad -\int_{Q_m^\omega}(r-\lambda_2\kappa_2-\frac{1}{2}\sigma_2^2 e^{2(\alpha_2-1)y})\frac{\partial u_m}{\partial y}\theta^2 u_m\,dx\,dy\,dt\\
&\quad -\rho\sigma_1\sigma_2\int_{Q_m^\omega} e^{(\alpha_1-1)x}e^{(\alpha_2-1)y}\frac{\partial^2 u_m}{\partial x\partial y}\theta^2 u_m\,dx\,dy\,dt + r\int_{Q_m^\omega}|u_m|^2\theta^2\,dx\,dy\,dt\\
&= \frac{\sigma_1^2}{2}\int_{Q_m^\omega} e^{-2(\alpha_2-1)y}\left|\frac{\partial u_m}{\partial x}\right|^2 dx\,dy\,dt + \frac{\sigma_2^2}{2}\int_{Q_m^\omega} e^{-2(\alpha_1-1)x}\left|\frac{\partial u_m}{\partial y}\right|^2 dx\,dy\,dt\\
&\quad -\int_{Q_m^\omega}((r-\lambda_1\kappa_1)\theta^2-\frac{1}{2}\sigma_1^2 e^{-2(\alpha_2-1)y})\frac{\partial u_m}{\partial x}u_m\,dx\,dy\,dt\\
&\quad -\int_{Q_m^\omega}((r-\lambda_2\kappa_2)\theta^2-\frac{1}{2}\sigma_2^2 e^{-2(\alpha_1-1)x})\frac{\partial u_m}{\partial y}u_m\,dx\,dy\,dt\\
&\quad -\rho\sigma_1\sigma_2\int_{Q_m^\omega} e^{-(\alpha_1-1)x}e^{-(\alpha_2-1)y}\frac{\partial^2 u_m}{\partial x\partial y}u_m\,dx\,dy\,dt + r\int_{Q_m^\omega}|u_m|^2\theta^2\,dx\,dy\,dt.
\end{aligned} \tag{2.6}$$

Concerning the third and fourth terms on the right-hand side of (2.6), since $u_m(x,y,t)=0$ on $\partial B_m\times[0,t]$, in view of Green Formulate we have

$$\frac{1}{2}\sigma_1^2\int_{Q_m^\omega} e^{-2(\alpha_2-1)y}\frac{\partial u_m}{\partial x}u_m\,dx\,dy\,dt = 0. \tag{2.7}$$

It follows that

$$
\begin{aligned}
&-\int_{Q_m^\omega}((r-\lambda_1\kappa_1)\theta^2-\frac{1}{2}\sigma_1^2e^{-2(\alpha_2-1)y})\frac{\partial u_m}{\partial x}u_m\,dx\,dy\,dt\\
&=-(r-\lambda_1\kappa_1)\int_{Q_m^\omega}\theta^2\frac{\partial u_m}{\partial x}u_m\,dx\,dy\,dt+\frac{1}{2}\sigma_1^2\int_{Q_m^\omega}e^{-2(\alpha_2-1)y}\frac{\partial u_m}{\partial x}u_m\,dx\,dy\,dt\\
&=-(r-\lambda_1\kappa_1)\int_{Q_m^\omega}\theta^2\frac{\partial u_m}{\partial x}u_m\,dx\,dy\,dt\\
&=(r-\lambda_1\kappa_1)\int_{Q_m^\omega}e^{-2(\alpha_1-1)x}e^{-2(\alpha_2-1)y}\frac{\partial u_m}{\partial x}u_m\,dx\,dy\,dt\\
&=(r-\lambda_1\kappa_1)\int_{Q_m^\omega}e^{-2(\alpha_2-1)y}u_m\frac{\partial}{\partial x}\Big(u_me^{-2(\alpha_1-1)x}\Big)\,dx\,dy\,dt\\
&=(r-\lambda_1\kappa_1)\int_{Q_m^\omega}e^{-2(\alpha_2-1)y}u_m\Big(\frac{\partial u_m}{\partial x}e^{-2(\alpha_1-1)x}-2(\alpha_1-1)u_me^{-2(\alpha_1-1)x}\Big)\,dx\,dy\,dt\\
&=(r-\lambda_1\kappa_1)\Big(\int_{Q_m^\omega}\theta^2\frac{\partial u_m}{\partial x}u_m\,dx\,dy\,dt-2(\alpha_1-1)\int_{Q_m^\omega}\theta^2|u_m|^2\,dx\,dy\,dt\Big).
\end{aligned}
\tag{2.8}
$$

Thus, we deduce that

$$
-\int_{Q_m^\omega}((r-\lambda_1\kappa_1)\theta^2-\frac{1}{2}\sigma_1^2e^{-2(\alpha_2-1)y})\frac{\partial u_m}{\partial x}u_m\,dx\,dy\,dt=-(r-\lambda_1\kappa_1)(\alpha_1-1)\int_{Q_m^\omega}\theta^2|u_m|^2\,dx\,dy\,dt.
\tag{2.9}
$$

Similarly, we show that

$$
-\int_{Q_m^\omega}((r-\lambda_2\kappa_2)\theta^2-\frac{1}{2}\sigma_2^2e^{2(\alpha_1-1)x})\frac{\partial u_m}{\partial y}u_m\,dx\,dy\,dt=-(r-\lambda_2\kappa_2)(\alpha_2-1)\int_{Q_m^\omega}\theta^2|u_m|^2\,dx\,dy\,dt.
\tag{2.10}
$$

For the last term on the right-hand side of (2.6), in view of Green Formulate we have

$$
\begin{aligned}
&-\rho\sigma_1\sigma_2\int_{Q_m^\omega}e^{-(\alpha_1-1)x}e^{-(\alpha_2-1)y}\frac{\partial^2u_m}{\partial x\partial y}u_m\,dx\,dy\,dt\\
&=\rho\sigma_1\sigma_2\int_{Q_m^\omega}e^{-(\alpha_2-1)y}\frac{\partial u_m}{\partial y}\frac{\partial}{\partial x}\Big(e^{-(\alpha_1-1)x}u_m\Big)\,dx\,dy\,dt\\
&=\rho\sigma_1\sigma_2\int_{Q_m^\omega}e^{-(\alpha_2-1)y}\frac{\partial u_m}{\partial y}\Big(\frac{\partial u_m}{\partial x}e^{-(\alpha_1-1)x}-(\alpha_1-1)e^{-(\alpha_1-1)x}u_m\Big)\,dx\,dy\,dt\\
&=\rho\sigma_1\sigma_2\int_{Q_m^\omega}\theta\frac{\partial u_m}{\partial y}\frac{\partial u_m}{\partial x}\,dx\,dy\,dt-\rho\sigma_1\sigma_2(\alpha_1-1)\int_{Q_m^\omega}\frac{\partial u_m}{\partial y}e^{-(\alpha_2-1)y}e^{-(\alpha_1-1)x}u_m\,dx\,dy\,dt\\
&=\rho\sigma_1\sigma_2\int_{Q_m^\omega}\theta\frac{\partial u_m}{\partial y}\frac{\partial u_m}{\partial x}\,dx\,dy\,dt-\rho\sigma_1\sigma_2\frac{(\alpha_1-1)(\alpha_2-1)}{2}\int_{Q_m^\omega}\theta|u_m|^2\,dx\,dy\,dt.
\end{aligned}
\tag{2.11}
$$

Since $\theta \le \theta^2$, thus by combining (2.6) and (2.9) $-$ (2.11) we conclude that

$$
\begin{aligned}
&\langle Au_m, u_m\theta^2\rangle \\
&= \frac{\sigma_1^2}{2}\int_{Q_m^\omega} e^{-2(\alpha_2-1)y}\left|\frac{\partial u_m}{\partial x}\right|^2 dx\,dy\,dt + \frac{\sigma_2^2}{2}\int_{Q_m^\omega} e^{-2(\alpha_1-1)x}\left|\frac{\partial u_m}{\partial y}\right|^2 dx\,dy\,dt \\
&-(r-\lambda_1\kappa_1)(\alpha_1-1)\int_{Q_m^\omega}\theta^2|u_m|^2\,dx\,dy\,dt - (r-\lambda_2\kappa_2)(\alpha_2-1)\int_{Q_m^\omega}|u_m|^2\theta^2\,dx\,dy\,dt \\
&+r\int_{Q_m^\omega}|u_m|^2\theta^2\,dx\,dy\,dt + \rho\sigma_1\sigma_2\int_{Q_m^\omega}\theta\frac{\partial u_m}{\partial y}\frac{\partial u_m}{\partial x}\,dx\,dy\,dt \\
&-\rho\sigma_1\sigma_2\frac{(\alpha_1-1)(\alpha_2-1)}{2}\int_{Q_m^\omega}\theta|u_m|^2\,dx\,dy\,dt \\
&\ge (1-\rho)\frac{\sigma_1^2}{2}\int_{Q_m^\omega} e^{-2(\alpha_2-1)y}\left|\frac{\partial u_m}{\partial x}\right|^2 dx\,dy\,dt + (1-\rho)\frac{\sigma_2^2}{2}\int_{Q_m^\omega} e^{-2(\alpha_1-1)x}\left|\frac{\partial u_m}{\partial y}\right|^2 dx\,dy\,dt \\
&+\Big(r-(1-\lambda_1\kappa_1)(\alpha_1-1)-(1-\lambda_2\kappa_2)(\alpha_2-1)\Big)\int_{Q_m^\omega}|u_m|^2\theta^2\,dx\,dy\,dt \\
&+\frac{\rho}{2}\int_{Q_m^\omega}\Big(\sigma_1 e^{-(\alpha_2-1)y}\frac{\partial u_m}{\partial y}+\sigma_2 e^{-(\alpha_1-1)x}\frac{\partial u_m}{\partial x}\Big)^2 dx\,dy\,dt \\
&-\rho\sigma_1\sigma_2\frac{(\alpha_1-1)(\alpha_2-1)}{2}\int_{Q_m^\omega}|u_m|^2\theta^2\,dx\,dy\,dt \\
&\ge (1-\rho)\frac{\sigma_1^2}{2}\int_{Q_m^\omega} e^{-2(\alpha_2-1)y}\left|\frac{\partial u_m}{\partial x}\right|^2 dx\,dy\,dt + (1-\rho)\frac{\sigma_2^2}{2}\int_{Q_m^\omega} e^{-2(\alpha_1-1)x}\left|\frac{\partial u_m}{\partial y}\right|^2 dx\,dy\,dt \\
&+\Big(r-(1-\lambda_1\kappa_1)(\alpha_1-1)-(1-\lambda_2\kappa_2)(\alpha_2-1)-\rho\sigma_1\sigma_2\frac{(\alpha_1-1)(\alpha_2-1)}{2}\Big)\int_{Q_m^\omega}|u_m|^2\theta^2\,dx\,dy\,dt.
\end{aligned}
\tag{2.12}
$$

For the first term on the right-hand side of (2.4), we have

$$
\begin{aligned}
&\lambda_1\int_{Q_m^\omega}\Big(\int_{-m}^{m}u_m(x-z,y,t)\Gamma_\delta(x-z)dz\Big)u_m\theta^2\,dx\,dy\,dt \\
&=\lambda_1\int_{Q_m^\omega}\Big(\int_{\mathbb{R}}u_m(x-z,y,t)\cdot\chi_{[-m,m]}(z)\Gamma_\delta(x-z)dz\Big)u_m e^{-2(\alpha_1-1)x}e^{-2(\alpha_2-1)y}\,dx\,dy\,dt \\
&=\lambda_1\int_{Q_m^\omega}\Big(\int_{\mathbb{R}}u_m(x-z,y,t)\cdot\chi_{[-m,m]}(z)\theta(z,y)e^{-(\alpha_1-1)(x-z)}\Gamma_\delta(x-z)dz\Big)u_m\theta(x,y)\,dx\,dy\,dt \\
&=\lambda_1\int_{Q_m^\omega}\big(u_m(x,y,t)\cdot\chi_{[-m,m]}(x)\theta(x,y)*\widehat{\Gamma}_\delta(x)\big)u_m\theta(x,y)\,dx\,dy\,dt \\
&\le\lambda_1\Big(\int_{Q_m^\omega}\big(u_m(x,y,t)\cdot\chi_{[-m,m]}(x)\theta(x,y)*\widehat{\Gamma}_\delta(x)\big)^2\,dx\,dy\,dt\Big)^{\frac{1}{2}}\Big(\int_{Q_m^\omega}|u_m|^2\theta(x,y)^2\,dx\,dy\,dt\Big)^{\frac{1}{2}} \\
&\le\lambda_1\Big(\int_{Q_m^\omega}\big(u_m(x,y,t)\cdot\chi_{[-m,m]}(x)\theta(x,y)\big)^2\,dx\,dy\,dt\Big)^{\frac{1}{2}}\Big(\int_{Q_m^\omega}\widehat{\Gamma}_\delta(x)\,dx\,dy\,dt\Big) \\
&\qquad\times\Big(\int_{Q_m^\omega}|u_m|^2\theta(x,y)^2\,dx\,dy\,dt\Big)^{\frac{1}{2}} \\
&\le\lambda_1\Big(\int_{Q_m^\omega}\widehat{\Gamma}_\delta(x)\,dx\,dy\,dt\Big)\int_{Q_m^\omega}|u_m|^2\theta(x,y)^2\,dx\,dy\,dt,
\end{aligned}
\tag{2.13}
$$

with $\widehat{\Gamma}_\delta(s) = e^{-(\alpha_1-1)s}\Gamma_\delta(s)$. Similarly, we have

$$\begin{aligned}&\lambda_2 \int_{Q_m^\omega} \Big(\int_{-m}^{m} u_m(x, y-z, t)\Gamma_\delta(y-z)dz\Big) u_m\theta^2 \, dx\, dy\, dt \\ &\leq \lambda_2 \Big(\int_{Q_m^\omega} \widehat{\Gamma}_\delta(y)\, dx\, dy\, dt\Big) \int_{Q_m^\omega} |u_m|^2\theta(x,y)^2 \, dx\, dy\, dt.\end{aligned} \tag{2.14}$$

By combining (2.4), (2.5) (2.12) $-$ (2.14), we conclude that

$$\begin{aligned}&\frac{1}{2}\int_{B_m} |u_m(\omega)|^2\theta^2 \, dx\, dy + (1-\rho)\frac{\sigma_1^2}{2}\int_{Q_m^\omega} e^{-2(\alpha_2-1)y}\Big|\frac{\partial u_m}{\partial x}\Big|^2 \, dx\, dy\, dt \\ &\quad +(1-\rho)\frac{\sigma_2^2}{2}\int_{Q_m^\omega} e^{-2(\alpha_1-1)x}\Big|\frac{\partial u_m}{\partial y}\Big|^2 \, dx\, dy\, dt \\ &\leq \frac{1}{2}\int_{B_m} |u_m(0)|^2\theta^2 \, dx\, dy + \lambda_1\Big(\int_{Q_m^\omega} \widehat{\Gamma}_\delta(x)\, dx\, dy\, dt\Big)\int_{Q_m^\omega} |u_m|^2\theta^2 \, dx\, dy\, dt \\ &+\lambda_2\Big(\int_{Q_m^\omega} \widehat{\Gamma}_\delta(x)\, dx\, dy\, dt\Big)\int_{Q_m^\omega} |u_m|^2\theta^2 \, dx\, dy\, dt - \Big(r+\lambda_1+\lambda_2-(1-\lambda_1\kappa_1)(\alpha_1-1) \\ &-(1-\lambda_2\kappa_2)(\alpha_2-1) - \rho\sigma_1\sigma_2\frac{(\alpha_1-1)(\alpha_2-1)}{2}\Big)\int_{Q_m^\omega} |u_m|^2\theta^2 \, dx\, dy\, dt.\end{aligned} \tag{2.15}$$

It follows that

$$\begin{aligned}&\frac{1}{2}\int_{B_m} |u_m(\omega)|^2\theta^2 \, dx\, dy + (1-\rho)\int_{Q_m^\omega}\Big(\frac{\sigma_1^2}{2}e^{2(\alpha_1-1)x}\theta^2\Big|\frac{\partial u_m}{\partial x}\Big|^2 + \frac{\sigma_2^2}{2}e^{2(\alpha_2-1)y}\theta^2\Big|\frac{\partial u_m}{\partial y}\Big|^2\Big) \, dx\, dy\, dt \\ &\leq \int_{B_m} |u_m(0)|^2\theta^2 \, dx\, dy + |D|\int_{Q_m^\omega} |u_m|^2\theta^2 \, dx\, dy\, dt\end{aligned} \tag{2.16}$$

with

$$\begin{aligned}D \;&= r+\lambda_1+\lambda_2-(1-\lambda_1\kappa_1)(\alpha_1-1)-(1-\lambda_2\kappa_2)(\alpha_2-1) \\ &\quad -\rho\sigma_1\sigma_2\frac{(\alpha_1-1)(\alpha_2-1)}{2} - \lambda_1\Big(\int_{Q_m^\omega} \widehat{\Gamma}_\delta(x)\, dx\, dy\, dt\Big) - \lambda_2\Big(\int_{Q_m^\omega} \widehat{\Gamma}_\delta(x)\, dx\, dy\, dt\Big).\end{aligned}$$

Thus, we conclude that

$$\int_{B_m} |u_m(\omega)|^2\theta^2 \, dx\, dy \leq \int_{B_m} |u_m(0)|^2\theta^2 \, dx\, dy + 2|D|\int_{Q_m^\omega} |u_m|^2\theta^2 \, dx\, dy\, dt. \tag{2.17}$$

We set $G_m(\omega) = \displaystyle\int_{B_m} |u_m(\omega)|^2\theta^2 \, dx\, dy$ then we have

$$G_m(\omega) \leq C_0 + 2|D|\int_0^\omega G_m(\omega)\, dt. \tag{2.18}$$

By applying the Gronwall's Lemma [14], we get

$$G_m(\omega) \leq C_1 e^{2|D|\omega} \leq C_1 e^{2|D|T}. \tag{2.19}$$

Moreover, in view of (2.16) we conclude that

$$\int_{Q_m^\omega} e^{2(\alpha_1-1)x}\theta^2\Big|\frac{\partial u_m}{\partial x}\Big|^2 \, dx\, dy\, dt + \int_{Q_m^\omega} e^{2(\alpha_1-1)y}\theta^2\Big|\frac{\partial u_m}{\partial y}\Big|^2 \, dx\, dy\, dt \leq C_2 + C_3|D|e^{2|D|T}T. \tag{2.20}$$

We deduce that the sequence $(u_m)_{m\in\mathbb{N}^*}$ is bounded in $L^2(0,T;H^1(B_\varrho))$ for any $m \geq \varrho$. Therefore, there exists a subsequence still denoted (u_m) such that

$$u_m \rightharpoonup u \quad \text{weakly in} \quad L^2(0,T;H^1(B_\varrho)). \tag{2.21}$$

Step 3: The sequence $(\frac{\partial u_m}{\partial \tau})_m$ *is uniformly bounded in* $L^2(0,T;(H^1(B_\varrho))')$.
By taking $\varphi(x,y,\tau) \in \mathcal{D}(\overline{B_\varrho \times (0,T)})$ as a test function in the approximate problem (2.3), we have

$$\langle \frac{\partial u_m}{\partial \tau}, \varphi\rangle + \langle Au_m, \varphi\rangle + (\lambda_1+\lambda_2)\int_{Q_m^\omega} u_m\varphi\, dx\, dy\, dt = \langle I_m u_m, \varphi\rangle, \tag{2.22}$$

it follows that

$$\begin{aligned}\Big|\langle \frac{\partial u_m}{\partial \tau}, \varphi\rangle\Big| &\leq \Big|\langle Au_m, \varphi\rangle\Big| + (\lambda_1+\lambda_2)\int_{Q_\rho} |u_m|\,|\varphi|\, dx\, dy\, dt \\ &\quad +\lambda_1\int_{Q_m^\omega}\Big(\int_{-m}^{m} u_m(x-z,y,t)\Gamma_\delta(x-z)dz\Big)\varphi(x,y,t)\, dx\, dy\, dt \\ &\quad +\lambda_2\int_{Q_m^\omega}\Big(\int_{-m}^{m} u_m(x,y-z,t)\Gamma_\delta(y-z)dz\Big)\varphi(x,y,t)\, dx\, dy\, dt \\ &\leq C_\varrho\|\varphi\|_{L^2(0,T;H^1(B_\varrho))},\end{aligned}$$

Having in mind the density of the space $\mathcal{D}(\overline{B_\varrho \times (0,T)})$ in $L^2(0,T;H^1(B_\varrho))$, we conclude that

$$\Big|\langle \frac{\partial u_m}{\partial \tau}, \varphi\rangle\Big| \leq C_\varrho\|\varphi\|_{L^2(0,T;H^1(B_\varrho))} \qquad \text{for any} \quad \varphi \in L^2(0,T;H^1(B_\varrho)). \tag{2.23}$$

Thus, the sequence $\Big(\frac{\partial u_m}{\partial \tau}\Big)_m$ is uniformly bounded in $L^2(0,T;(H^1(B_\varrho))')$. Therefore, in view of Aubin's Lemma, $(u_m)_m$ is relatively compact in $L^2(B_\varrho \times (0,T))$, then converges almost everywhere in $B_\varrho \times (0,T)$. We conclude that

$$\frac{\partial u_m}{\partial \tau} \rightharpoonup \frac{\partial u}{\partial \tau} \qquad \text{weakly in} \quad L^2(0,T;(H^1(B_\varrho))'). \tag{2.24}$$

Step 4 : Passage to the limit.
By taking $v \in D(B_\varrho \times (0,T))$ as a test function for the approximate problem (2.3), we obtain

$$\langle \frac{\partial u_m}{\partial \tau}, v\rangle + \langle Au_m, v\rangle + (\lambda_1+\lambda_2)\int_{Q_\varrho} u_m v\, dx\, dy\, dt = \langle I_m u_m, v\rangle, \tag{2.25}$$

For the first term on the right-hand side of (2.25), in view of (2.24) we have $\frac{\partial u_m}{\partial \tau} \rightharpoonup \frac{\partial u}{\partial t}$ weakly in $L^2(0,T;(H^1(B_\varrho)'))$, and since v belongs to $L^2(0,T;H^1(B_\varrho))$, it follows that

$$\langle \frac{\partial u_m}{\partial \tau}, v\rangle \longrightarrow \langle \frac{\partial u}{\partial \tau}, v\rangle \qquad \text{as} \quad m \to \infty. \tag{2.26}$$

in view of (2.21), we have $u_m \rightharpoonup u$ weakly in $L^2(0,T;H^1(B_\varrho))$ for any $m > \varrho$ then

$$\langle Au_m, v\rangle \longrightarrow \langle Au, v\rangle \qquad \text{as} \quad m \to \infty. \tag{2.27}$$

and

$$\int_{Q_\varrho} u_m v \, dx \, dy \, dt \longrightarrow \int_{Q_\varrho} uv \, dx \, dy \, dt \qquad \text{as} \quad m \to \infty. \tag{2.28}$$

$$\langle I_m u_m, v\rangle \longrightarrow \langle Iu, v\rangle \qquad \text{as} \quad m \to \infty. \tag{2.29}$$

Thus, by combining (2.25) and (2.26) − (2.29), we deduce that : for any $\varrho > 0$ we have

$$\langle \frac{\partial u}{\partial \tau}, v\rangle + \langle Au, v\rangle + (\lambda_1 + \lambda_2) \int_{Q_\varrho} uv \, dx \, dy \, dt = \langle Iu, v\rangle \quad \text{for any} \quad v \in D(B_\varrho \times (0,T)). \tag{2.30}$$

By the density of the space $\mathcal{D}(\overline{B_\varrho \times (0,T)})$ in $L^2(0,T;H^1(B_\varrho))$, and letting ϱ tends to infinity, we conclude that

$$\langle \frac{\partial u}{\partial \tau}, v\rangle + \langle Au, v\rangle + (\lambda_1 + \lambda_2) \int_{Q_{\mathbb{R}}} uv \, dx \, dy \, dt = \langle Iu, v\rangle \quad \text{for any} \quad v \in L^2(0,T;H^1(B_{\mathbb{R}})), \tag{2.31}$$

which completes the proof of theorem 2.1.

Conclusion

In this paper, we have presented an existence result for a type of partial integro-differential equations(PIDE) in finance modeling a process with jumps and the stochastic volatility of type constant elasticity of variance (CEV). This PIDE permit to capture the leverage effect and the impact of the jumps in the financial markets.

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