

Sums and products of periodic functions

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Dedicated to the memory of Mohammed Zarrabi

ABSTRACT. There exist two real valued periodic functions on the real line such that, for every $x \in \mathbb{R}$, $f_1(x) + f_2(x) = x$, but it is impossible to find two real valued periodic functions on the real line such that, for every $x \in \mathbb{R}$, $f_1(x) + f_2(x) = x^2$. The purpose of this note is to prove this result and also to study the possibility of decomposing more general polynomials into sum of periodic functions.

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1. Introduction.

Given a function f from \mathbb{R} into \mathbb{R} , is it possible to find two periodic functions $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$ such that $f = f_1 + f_2$? We will see that it is indeed possible whenever $f(x) = x$ for all x . However, M. Laczkovich and S. G. Révész obtained in [3] that in this case, f_1 and f_2 are not measurable (see also [2]). On the other hand, if, for some $n \geq 2$, $f(x) = x^n$ for all $x \in \mathbb{R}$, the question has a negative answer. More generally, if Δ_T denotes the linear operator defined by $\Delta_T f(x) = f(x + T) - f(x)$, Mortola and R. Peirone [4] have shown that if T_1, \dots, T_n are rationally independent, then $\Delta_{T_1} \cdots \Delta_{T_n} f = 0$ if and only if f is the sum of n functions f_k ($1 \leq k \leq n$) with f_k T_k -periodic (Wierdl [5] proved before one implication). This result is extended in [1] and [2] in more general settings. It implies in particular that if, for some integer $n \geq 1$, $f(x) = x^n$ for all $x \in \mathbb{R}$, then f can be expressed as the sum of $n + 1$ periodic functions,

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but cannot be written as the sum of n periodic functions. We shall prove this last result below. Our contribution is to provide simple proofs of the above results. We shall obtain also new results : First, we prove that the exponential function is the product of two periodic functions, but is not the sum of two periodic functions. As a consequence, we show that the solutions of an homogeneous linear differential equation of order n on \mathbb{R} with constant coefficients are linear combinations of periodic functions and of products of at most three periodic functions.

2. The sum of two periodic functions.

The following proposition shows that not every function f can be expressed as the sum of two periodic functions.

Proposition 2.1. *If, for all $x \in \mathbb{R}$, $f_1(x) + f_2(x) = f(x)$ and if f_1 is T_1 -periodic and f_2 is T_2 -periodic, then for all $x \in \mathbb{R}$,*

$$f(x + T_1 + T_2) + f(x) = f(x + T_1) + f(x + T_2) \quad (1)$$

Indeed both members of this equality are equal to $f_1(x + T_2) + f_2(x + T_1) + f_1(x) + f_2(x)$.

Corollary 2.2. *If $f(x) = x^n$ with $n \geq 2$, or if $f(x) = e^x$, then f cannot be written as the sum of two periodic functions.*

Indeed, condition (1) is not satisfied whenever $f(x) = x^n$ with $n \geq 2$ or whenever $f(x) = e^x$, thus these functions cannot be written as the sum of two periodic functions. However, the following surprising result holds true.

Theorem 2.3. *There exists two periodic functions $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$ such that for all $x \in \mathbb{R}$, $f_1(x) + f_2(x) = x$.*

Remark 2.4. *It is obvious that there exists two periodic functions $f_1, f_2 : \mathbb{C} \rightarrow \mathbb{C}$ such that for all $z \in \mathbb{C}$, $f_1(z) + f_2(z) = z$. It is enough to define, for $z = x + iy$, $f_1(z) = x$ and $f_2(z) = iy$: f_1 is i -periodic and f_2 is 1-periodic.*

The following proposition shows that the construction of the functions f_1 and f_2 cannot be performed without the axiom of choice.

Proposition 2.5. *Let $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$ such that f_1 is T_1 -periodic, f_2 is T_2 -periodic, with $T_1 > 0$ and $T_2 > 0$, and $\lim_{|x| \rightarrow +\infty} |f_1(x) + f_2(x)| = +\infty$ then :*

- 1) T_1/T_2 est irrational.
- 2) For each non empty open interval I of \mathbb{R} , f_1 and f_2 are unbounded on I .
- 3) f_1 and f_2 are not measurable.

Proof of Proposition 2.5 : Let us denote $f = f_1 + f_2$.

- 1) If T_1/T_2 irrational, there exists $T > 0$ such that f_1 et f_2 are T -periodic. Therefore $f = f_1 + f_2$ is also T -periodic, which is absurd.
- 2) Let us fix a bounded non empty open interval I of \mathbb{R} . Since $T_1\mathbb{Z} + T_2\mathbb{Z}$ is a dense subgroup of \mathbb{R} , there exists a sequence (a_n, b_n) of distincts elements of \mathbb{Z}^2 such that for all n , $a_n T_1 + b_n T_2 \in I$. Since for all relative integer a , the set $\{a T_1 + b T_2, b \in \mathbb{Z}\} \cap I$ is finite (because I is bounded), we get that

$\lim_{n \rightarrow +\infty} |a_n| = +\infty$. Since $f_2(a_n T_1 + b_n T_2) = f_2(a_n T_1) = f_1(a_n T_1) - f_1(0) + f_2(a_n T_1) = f(a_n T_1) - f_1(0)$, f_2 is unbounded on I . Similarly, f_1 is unbounded on I . In particular, f_1 and f_2 cannot be continuous at any point of \mathbb{R} .

3) In order to get a contradiction, let us assume that f_1 is measurable and let us denote μ the Lebesgue measure on \mathbb{R} . Without loss of generality, we can assume that $T_1 > T_2 > 0$. Set

$$A_n = \{x \in [0, 4T_1]; |f_1(x)| \leq n \text{ and } |f_2(x)| \leq n\}$$

The sets A_n are measurable subsets of \mathbb{R} , $\cup A_n = [0, 4T_1]$, and the sequence (A_n) is non decreasing. Since $\mu([0, 4T_1]) = 4T_1$, there exists n_0 such that $\mu(A_{n_0}) \geq 3T_1$. From now on, we set $A = A_{n_0}$. Let us fix n such that $|f(x)| > 2n_0$ for all $x \geq nT_1$, and then define $p = \min\{q \in \mathbb{N}; qT_2 \geq nT_1\}$. Since μ is translation invariant, $\mu(nT_1 + A) = \mu(pT_2 + A) = \mu(A) \geq 3T_1$. Moreover, $nT_1 + A \subset [nT_1, (n+5)T_1]$, and since $0 < pT_2 - nT_1 < T_1$, we also have $pT_2 + A \subset [nT_1, (n+5)T_1]$. Since $\mu([nT_1, (n+5)T_1]) = 5T_1$, we deduce that $\mu((nT_1 + A) \cap (pT_2 + A)) \geq T_1 > 0$. Let $x \in (nT_1 + A) \cap (pT_2 + A)$. There exist $y, z \in A$ such that $x = y + nT_1 = z + pT_2$, and so $f(x) = f_1(x) + f_2(x) = f_1(y) + f_2(z)$. Hence $2n_0 < |f(x)| \leq |f_1(y)| + |f_2(z)| \leq 2n_0$, which is absurd. Therefore f_1 is not measurable, and similarly, f_2 is not measurable.

Proof of Theorem 2.3 :

Let us denote $\alpha = \sqrt{2}$. If r, s are rationals, we define $h_1(r + s\alpha) = r$ and $h_2(r + s\alpha) = s\alpha$. h_1 et h_2 are two well defined functions from $\mathbb{Q}[\alpha]$ into \mathbb{R} , and for all $a \in \mathbb{Q}[\alpha]$, $h_1(a) + h_2(a) = a$, h_1 is α -periodic and h_2 is 1-periodic. \mathbb{R} is a vector space over the field $\mathbb{Q}[\alpha]$, let $(e_i)_{i \in I}$ be a basis of \mathbb{R} over $\mathbb{Q}[\alpha]$ such that there exists $i_0 \in I$ with $e_{i_0} = 1$ (the existence of such a basis relies on the axiom of choice). If $x \in \mathbb{R}$, there exists elements $a_i \in \mathbb{Q}[\alpha]$, which are all equal to zero except for a finite number of $i \in I$, such that $x = \sum_{i \in I} a_i e_i$. We then define

$$f_1(x) = \sum_{i \in I} h_1(a_i) e_i \quad \text{and} \quad f_2(x) = \sum_{i \in I} h_2(a_i) e_i.$$

The function f_1 is α -periodic since

$$\begin{aligned} f_1(x + \alpha) &= f_1(a_{i_0} + \alpha + \sum_{i \neq i_0} a_i e_i) = h_1(a_{i_0} + \alpha) + \sum_{i \neq i_0} h_1(a_i) e_i \\ &= h_1(a_{i_0}) + \sum_{i \neq i_0} h_1(a_i) e_i = f_1(x). \end{aligned}$$

Similarly, f_2 is 1-periodic and, for all $x \in \mathbb{R}$,

$$f_1(x) + f_2(x) = \sum_{i \in I} (h_1(a_i) + h_2(a_i)) e_i = \sum_{i \in I} a_i e_i = x.$$

3. The sum of n periodic functions.

We now extend Theorem 1 in the following way :

Theorem 3.1. *If P is a polynomial of degree exactly $n \geq 2$ with real coefficients, then :*

1) *There do not exist any periodic functions $f_1, \dots, f_n : \mathbb{R} \rightarrow \mathbb{R}$ such that for all $x \in \mathbb{R}$, $P(x) = f_1(x) + \dots + f_n(x)$.*

2) *There exist periodic functions $f_1, \dots, f_n, f_{n+1} : \mathbb{R} \rightarrow \mathbb{R}$ such that for all $x \in \mathbb{R}$, $P(x) = f_1(x) + \dots + f_n(x) + f_{n+1}(x)$.*

Proof of Theorem 3.1: 1) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $T \in \mathbb{R}$, $T \neq 0$. Let us recall that the linear operator Δ_T is defined by $\Delta_T f(x) = f(x+T) - f(x)$. If f is T periodic, then $\Delta_T(f) = 0$. Let us fix T_1, \dots, T_n non zero real numbers. It is easy to see that the operators Δ_{T_k} commute. So, if $1 \leq k \leq n$ and f_k is T_k -periodic, $\Delta_{T_1} \cdots \Delta_{T_n} f_k(x) = 0$, and, if $f = f_1 + \dots + f_n$ with f_k T_k -periodic for all $1 \leq k \leq n$, the linearity of the operators Δ_{T_k} implies $\Delta_{T_1} \cdots \Delta_{T_n} f(x) = 0$. On the other hand,

$$\Delta_{T_1} \Delta_{T_2} f(x) = f(x + T_1 + T_2) - f(x + T_1) - f(x + T_2) + f(x),$$

and more generally,

$$\Delta_{T_1} \cdots \Delta_{T_n} f(x) = \sum_{k=0}^n (-1)^k \sum_{I \subset \{1, \dots, n\}, |I|=n-k} f\left(x + \sum_{i \in I} T_i\right).$$

If $k < n$, $\Delta_{T_1} \cdots \Delta_{T_n} x^k = 0$ and $\Delta_{T_1} \cdots \Delta_{T_n} x^n = T_1 T_2 \cdots T_n$. By linearity, if $P(x) = \sum_{k=0}^n a_k x^k$ with $a_n \neq 0$, then $\Delta_{T_1} \cdots \Delta_{T_n} P(x) = a_n T_1 T_2 \cdots T_n \neq 0$. This shows that P cannot be written as the sum of n periodic functions.

2) We consider $n+1$ real numbers e_0, e_1, \dots, e_n which are linearly independent over \mathbb{Q} . \mathbb{R} is a vector space over \mathbb{Q} , according to the incomplete basis Theorem, we can extend the free system (e_0, e_1, \dots, e_n) into a basis $(e_i)_{i \in I}$ of \mathbb{R} over \mathbb{Q} . Thus, the set I is the union of two disjoint sets, $\{0, 1, \dots, n\}$ and J . If $x \in \mathbb{R}$, there exists scalars $a_i \in \mathbb{Q}$, which are all equal to zero except for a finite number of indices, such that $x = \sum_{i \in I} a_i e_i$. So, we have $P(x) = P\left(\sum_{i \in I} a_i e_i\right)$, and, if we expand this expression, we obtain a polynomial of degree $\leq n$ with respect to each of the $n+1$ variables a_i , $0 \leq i \leq n$. Therefore, this polynomial is the sum of $n+1$ functions f_0, f_1, \dots, f_n , in such a way that, for all $0 \leq i \leq n$, the term a_i does not appear in the expression of $f_i(x)$. The function f_i is then e_i -periodic, since

$$\begin{aligned} f_i(x + e_i) &= f_i\left((a_i + 1)e_i + \sum_{k \in I, k \neq i} a_k e_k\right) \\ &= f_i\left(\sum_{k \in I, k \neq i} a_k e_k\right) = f_i\left(a_i e_i + \sum_{k \in I, k \neq i} a_k e_k\right) = f_i(x). \end{aligned}$$

Remark 3.2. *Actually, one can prove that in the decomposition of P given in theorem 3.1 2), the functions f_1, \dots, f_n, f_{n+1} are necessarily non measurable. Indeed, it is proved in [2] that if $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $\lim_{x \rightarrow \infty} f(x) = \infty$, then there is no decomposition $f = f_1 + f_2 + \dots + f_n$ with $n \geq 2$ and f_1, f_2, \dots, f_n measurable and periodic. The proof is similar to the proof of proposition 2.5 where we consider the case $n = 2$.*

4. Sums and products of periodic functions.

Let us now consider the problem of the decomposition of a function from \mathbb{R} into \mathbb{R} as a product of periodic functions. First, let us notice that if $f = f_1 f_2$, if f_1 is T_1 -periodic and f_2 is T_2 -periodic, then for all $x \in \mathbb{R}$,

$$f(x + T_1 + T_2)f(x) = f(x + T_1)f(x + T_2) \quad (2)$$

Indeed, the two sides of this equality coincide with $f_1(x+T_2)f_2(x+T_1)f_1(x)f_2(x)$. The negation of condition (2) is a sufficient condition that can be checked easily in order to show that a function cannot be written as a product of two periodic functions. For instance, if $f(x) = x$, the function f cannot be written as a product of two periodic functions.

Theorem 4.1. *There exists two periodic functions $g_1, g_2 : \mathbb{R} \rightarrow \mathbb{R}$ such that, for all $x \in \mathbb{R}$, $g_1(x)g_2(x) = e^x$.*

Proof : Let f_1 and f_2 be given by Theorem 1, we define

$$g_1(x) = e^{f_1(x)} \quad \text{and} \quad g_2(x) = e^{f_2(x)}.$$

g_1 is α -periodic, g_2 is 1-periodic, and, for all $x \in \mathbb{R}$,

$$g_1(x)g_2(x) = e^{f_1(x)} \cdot e^{f_2(x)} = e^{f_1(x)+f_2(x)} = e^x.$$

Corollary 4.2. *The solutions of the linear differential equation of order n on \mathbb{R} (E) : $y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y = 0$ are linear combinations of periodic functions and of products of at most three periodic functions.*

Indeed, the solutions of this differential equation are linear combinations of functions of the form $x^k e^{\lambda x} \cos(\omega x)$ or $x^k e^{\lambda x} \sin(\omega x)$ (with $k \in \mathbb{N}$, $0 \leq k < n$, $\omega \in \mathbb{R}$ and $\lambda \in \mathbb{R}$), and these functions are either periodic, or sum of periodic functions, or linear combinations of products of at most three periodic functions.

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