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Existence of solutions for 4p-order PDES with Neumann boundary conditions

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Abstract. In this work, we study the existence of at least one non decreasing sequence of nonnegative eigenvalues for the problem:

$$\begin{cases} \Delta^{2p}u = \lambda m(x)u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial(\Delta u)}{\partial \nu} = ... = \frac{\partial(\Delta^{2p-1}u)}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $p \in \mathbb{N}^*$, $m \in L^\infty(\Omega)$, and $\Delta^{2p}u := \Delta(\Delta...(\Delta u))$, $2p$ times the operator Δ.

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1. Introduction

This work is interested by a 4p-order problem under Neumann boundary conditions:

$$\begin{cases} \Delta^{2p}u = \lambda m(x)u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial(\Delta u)}{\partial \nu} = ... = \frac{\partial(\Delta^{2p-1}u)}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 1$, $p \in \mathbb{N}^*$, $m \in L^\infty(\Omega)$ is a weight function with $\mu\{x \in \Omega : m(x) > 0\} \neq 0$, λ is a real parameter, $\frac{\partial u}{\partial \nu}$ is the outer normal derivative and $\Delta^{2p}u := \Delta(\Delta...(\Delta u))$, $2p$ times the operator Δ. We show the existence of at least one non decreasing sequence of nonnegative eigenvalues.

In recent decades, the study of differential equations introducing fourth order operators has been an interesting topic since this type of problems are applicable in all fields of engineering and physics such as fluid dynamique and nonlinear elasticity problems.see. [1, 4].

This paper was motivated by [2, 5] : in [5], the authors interested to the spectrum of a 4p order elliptic equation with a dirichlet boundary conditions. They proved the existence of at least one non decreasing sequence of nonnegative eigenvalues.

Our main result is the following theorem :

Theorem 1.1. *The problem (1.1) has at least one non decreasing sequence of nonnegative eigenvalues* $(\lambda_n)_n$ *defined as :*

$$\lambda_n = \inf_{K \in \Gamma_n} \sup_{u \in K} G(u),$$

and satisfying $\lambda_n \longrightarrow +\infty$ *as* $n \longrightarrow +\infty$,
where

$$G(u) := \frac{1}{2}\int_\Omega |\Delta^p u|^2 \, dx,$$

$$\Gamma_n = \{K \subset M : K \text{ is compact symetric and } \gamma(K) \geq n\}$$

and
$M = \left\{ u \in W^{2p,2}(\Omega), \forall 0 \leqslant k \leqslant 2p-1 : \frac{\partial(\Delta^k u)}{\partial \nu} = 0 \quad \text{on } \partial\Omega \text{ and } \frac{1}{2}\int_\Omega mu^2 dx = 1 \right\}$.

The proof of the Theorem 1.1 is based on the Szulkin's theorem [6] :

Theorem 1.2. *Suppose that M is a closed symmetric* C^1 *-submanifold of a real Banach space X and* 0 *does not belong to M. Suppose that* $f \in C^1(M, \mathbb{R})$ *is even and bounded below. Define*

$$c_n = \inf_{A \in \Gamma_n} \sup_{x \in A} f(x).$$

If $\Gamma_k \neq \emptyset$ *for some* $k \geq 1$ *and if f satisfies the Palais Smale condition for all* $c = c_n, n = 1, ..., k$, *then f has at least k distinct pairs of critical points.*

2. Position of problem

By using the Green formula, we have, for all $(u,v) \in (W^{2p,2}(\Omega))^2$:

$$\langle \Delta^{2p} u, v\rangle = \langle \Delta^p u, \Delta^p v\rangle + \sum_{i=1}^{p} g(\Delta^{2p-i}u, \Delta^{i-1}v), \tag{2.1}$$

with $\forall i \in \{1,...,p\}$

$$g(\Delta^{2p-i}u, \Delta^{i-1}v) = \int_{\partial\Omega} (\frac{\partial(\Delta^{2p-i}u)}{\partial\nu}\Delta^{i-1}v - \frac{\partial(\Delta^{i-1}v)}{\partial\nu}\Delta^{2p-i}u)\partial\sigma. \tag{2.2}$$

For more details, the reader can refer to [5].

Here, we consider the space $Y = \bigcap_{k=0}^{p-1} \left\{ v \in W^{2p,2}(\Omega) : \frac{\partial}{\partial\nu}(\Delta^k v) = 0 \quad \text{on} \quad \partial\Omega \right\}$

So, for $(u,v) \in Y^2$ and u verifies the conditions of the problem (1.1):

$$\langle \Delta^{2p} u, v\rangle = \langle \Delta^p u, \Delta^p v\rangle \tag{2.3}$$

Proposition 2.1. *The set Y is a closed subspace of $W^{2p,2}(\Omega)$.*

Proof of proposition 2.1. Put $Y_k = \left\{ v \in W^{2p,2}(\Omega) : \frac{\partial}{\partial\nu}(\Delta^k v) = 0 \quad \text{on} \quad \partial\Omega \right\}$.

We must show that ($\forall 0 \le k \le p-1$) the operator

$$\begin{array}{rccc} T^k : & W^{2p,2}(\Omega) & \to & L^2(\partial\Omega) \\ & u & \longmapsto & (\frac{\partial}{\partial\nu}(\Delta^k u))|_{\partial\Omega} \end{array}$$

is continuous from $\left(W^{2p,2}(\Omega), \|.\|_{W^{2p,2}}\right)$ to $\left(L^2(\partial\Omega), |.|_{L^2(\partial\Omega)}\right)$.

First, the operator

$$\begin{array}{rccc} \Delta^k : & W^{2p,2}(\Omega) & \to & W^{2,2}(\Omega) \\ & u & \longmapsto & \Delta^k u \end{array}$$

is linear and continuous from $\left(W^{2p,2}(\Omega), \|.\|_{W^{2p,2}}\right)$ to $\left(W^{2,2}(\Omega), \|.\|_{W^{2,2}}\right)$.
Indeed, we have :

$$\begin{aligned} \left\|\Delta^k u\right\|_{W^{2,2}} &= \sum_{|\alpha|\le 2} \left\|D^\alpha(\Delta^k u)\right\|_{L^2} \\ &\le \sum_{|\beta|\le 2p} \left\|D^\beta(u)\right\|_{L^2} = \|u\|_{W^{2p,2}} \end{aligned}$$

On the other hand, the operator

$$\begin{array}{rccc} D : & W^{2,2}(\Omega) & \to & L^2(\partial\Omega) \\ & u & \longmapsto & \frac{\partial u}{\partial\nu}|_{\partial\Omega} \end{array}$$

is continuous from $\left(W^{2,2}(\Omega), \|.\|_{W^{2,2}}\right)$ to $\left(L^2(\partial\Omega), |.|_{L^2(\partial\Omega)}\right)$. We can refer to [3]
Finally, $\forall 0 \le k \le p-1$, the operator $T^k = D \circ \Delta^k$ is continuous, and since

$$Y_k = (T^k)^{-1}\{0\},$$

the set Y_k is closed. Consequently, Y is a closed subspace of $W^{2p,2}(\Omega)$.
Hence, the proof of proposition 2.1 is completed.

Remark 2.1. *Note that the subspace Y is reflexive, separable and uniformly convex.*

We say that u is a solution of the problem (1.1) if:

$$\forall v \in W^{2p,2}(\Omega) : \langle \Delta^{2p} u, v \rangle = \lambda \int_\Omega m(x) uv dx. \tag{2.4}$$

But $u \longmapsto \|\Delta^p u\|_{L^2} := \left(\int_\Omega |\Delta^p u|^2 \, dx \right)^{\frac{1}{2}}$ is not a norm in Y. So, we consider a perturbation of the problem (1.1) as follows:

$$(P_\varepsilon) \begin{cases} \Delta^{2p} u + \varepsilon u = \lambda m(x) u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial (\Delta u)}{\partial \nu} = ... = \frac{\partial (\Delta^{2p-1} u)}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

with $\varepsilon \in]0,1[$ is enough small. u is a solution of (P_ε) if

$$\forall v \in Y : \langle \Delta^p u, \Delta^p v \rangle + \varepsilon \int_\Omega uv \, dx = \lambda \int_\Omega m(x) uv \, dx.$$

This means that :

$$\forall v \in Y : \int_\Omega \Delta^p u \Delta^p v \, dx + \varepsilon \int_\Omega uv \, dx = \lambda \int_\Omega m(x) uv \, dx.$$

Let be the functionel

$$\Phi_\varepsilon(u) = \frac{1}{2} \left(\int_\Omega |\Delta^p u|^2 \, dx + \varepsilon \int_\Omega |u|^2 \, dx - \lambda \int_\Omega m(x) u^2 dx \right).$$

Φ_ε is the energy functional associated to the problem(P_ε) (2.5). We have:

$$\forall (u,v) \in Y^2 : \langle \Phi_\varepsilon'(u), v \rangle = \langle \Delta^p u, \Delta^p v \rangle + \varepsilon \int_\Omega uv \, dx - \lambda \int_\Omega m(x) uv \, dx.$$

3. Existence result

We consider the functionals :

$$G_\varepsilon(u) := \frac{1}{2} (\int_\Omega |\Delta^p u|^2 \, dx + \varepsilon \int_\Omega |u|^2 \, dx)$$

and

$$F(u) := \frac{1}{2} \int_\Omega m u^2 dx.$$

So, we have:

$$\Phi_\varepsilon(u) = G_\varepsilon(u) - \lambda F(u).$$

Theorem 3.1. *The problem* (P_ε) *(2.5) has at least one non decreasing sequence of nonnegative eigenvalues* $(\lambda_{\varepsilon,n})_n$ *given by :*

$$\lambda_{\varepsilon,n} = \inf_{K\in\Gamma_n} \sup_{u\in K} G_\varepsilon(u),$$

and satisfying $\lambda_{\varepsilon,n} \longrightarrow +\infty$ *as* $n \longrightarrow +\infty$,

In order to apply the A.Szulkin's theorem, we should show the following proposition:

Proposition 3.1. *The functionals* G_ε *and* F *satisfies the following properties :*

(i) G_ε *and* F *are* C^1 *functionals.*
(ii) $F^{'}$ *is completely continuous.*
(iii) $G_\varepsilon^{'}$ *is of* (S^+) *type and* G_ε *is an even functional and bounded below.*
(iv) G_ε *satisfies the* (PS) *condition on* $M = \left\{u \in X_p : F(u) = 1\right\} \neq \emptyset$.
(v) $\forall n \in \mathbb{N}^* : \Gamma_n \neq \emptyset$ *with*

$$\Gamma_n = \left\{K \subset M : K \text{ is compact symetric and } \gamma(K) \geq n\right\}.$$

First, we will prove that $\varphi_\varepsilon : v \longmapsto (2G_\varepsilon(v))^{\frac{1}{2}}$ is a norm in Y:

(1) For $v \in Y : \varphi_\varepsilon(v) = 0 \Leftrightarrow G_\varepsilon(v) = 0$.
So,

$$\|\Delta^p u\|_{L^2} = 0 \text{ and } \|u\|_{L^2} = 0$$

Then $u = 0$.

(2) It's clear that, $\forall u \in Y, \forall \alpha \in \mathbb{R}$:

$$\varphi_\varepsilon(\alpha u) = (\|\Delta^p(\alpha u)\|_{L^2}^2 + \varepsilon\|\alpha u\|_{L^2}^2)^{\frac{1}{2}} = |\alpha|\varphi_\varepsilon(u)$$

(3) For $(u, v) \in Y^2$:

$$\begin{aligned}
\varphi_\varepsilon(u+v)^2 &= 2G_\varepsilon(u+v) = \|\Delta^p u + \Delta^p v\|_{L^2}^2 + \varepsilon\|u+v\|_{L^2}^2 \\
&= \|\Delta^p u\|_{L^2}^2 + \|\Delta^p v\|_{L^2}^2 + 2\int_\Omega \Delta^p u \Delta^p v \, dx + \varepsilon\left(\|u\|_{L^2}^2 + \|v\|_{L^2}^2 + 2\int_\Omega uv \, dx\right) \\
&= 2G_\varepsilon(u) + 2G_\varepsilon(v) + 2\left(\int_\Omega \Delta^p u \Delta^p v \, dx + \varepsilon\int_\Omega uv \, dx\right)
\end{aligned}$$

Using the Cauchy-Schwarz inequality , we find this inequalities:

$$\int_\Omega \Delta^p u \Delta^p v \, dx \leqslant \|\Delta^p u\|_{L^2}\|\Delta^p v\|_{L^2}$$

and

$$\int_\Omega uv \, dx \leqslant \|u\|_{L^2}\|v\|_{L^2}$$

Thus, we conclude that :

$$\varphi_\varepsilon(u+v)^2 \leqslant \varphi_\varepsilon(u)^2 + \varphi_\varepsilon(v)^2 + 2(\|\Delta^p u\|_{L^2}\|\Delta^p v\|_{L^2} + \varepsilon\|u\|_{L^2}\|v\|_{L^2}) \tag{3.1}$$

In addition to this, we remark that:

$$\left(\int_\Omega \Delta^p u \Delta^p v \, dx + \varepsilon \int_\Omega uv \, dx\right)^2 \leqslant 4G_\varepsilon(u)G_\varepsilon(v). \tag{3.2}$$

Indeed,

$$\begin{aligned}
\left(\int_\Omega \Delta^p u \Delta^p v \, dx + \varepsilon \int_\Omega uv \, dx\right)^2 &\leqslant (\|\Delta^p u\|_{L^2}\|\Delta^p v\|_{L^2} + \varepsilon\|u\|_{L^2}\|v\|_{L^2})^2 \\
&= \|\Delta^p u\|_{L^2}^2\|\Delta^p v\|_{L^2}^2 + (\varepsilon\|u\|_{L^2}^2)(\varepsilon\|v\|_{L^2}^2) \\
&+ \varepsilon\,(2\|\Delta^p u\|_{L^2}\|v\|_{L^2}\|\Delta^p v\|_{L^2}\|u\|_{L^2}) \\
&\leqslant \|\Delta^p u\|_{L^2}^2\|\Delta^p v\|_{L^2}^2 + (\varepsilon\|u\|_{L^2}^2)(\varepsilon\|v\|_{L^2}^2) \\
&+ \varepsilon\left(\|\Delta^p u\|_{L^2}^2\|v\|_{L^2}^2 + \|\Delta^p v\|_{L^2}^2\|u\|_{L^2}^2\right) \\
&\leqslant \left(\|\Delta^p u\|_{L^2}^2 + \varepsilon\|u\|_{L^2}^2\right)\left(\|\Delta^p v\|_{L^2}^2 + \varepsilon\|v\|_{L^2}^2\right) \\
&= \varphi_\varepsilon(u)^2\varphi_\varepsilon(v)^2
\end{aligned}$$

So, the inequality (3.1) becomes :

$$\varphi_\varepsilon(u+v)^2 \leqslant \varphi_\varepsilon(u)^2 + \varphi_\varepsilon(v)^2 + 2\varphi_\varepsilon(u)\varphi_\varepsilon(v) = (\varphi_\varepsilon(u) + \varphi_\varepsilon(v))^2$$

Then, $\varphi_\varepsilon(u+v) \leqslant \varphi_\varepsilon(u) + \varphi_\varepsilon(v)$.

Finally, $\forall \varepsilon > 0$ the functional φ_ε is a norm in Y.
Put $\forall \varepsilon > 0, \forall u \in Y : \|u\|_{\varepsilon,Y} = \varphi_\varepsilon(u)$

Proof of proposition 3.1. (i) G_ε and F are C^1 functionals: It is clear that G_ε and F are continuous and derivable functionals on Y and $\forall (u,v) \in Y^2$:

$$\left\langle G_\varepsilon'(u), v \right\rangle = \int_\Omega \Delta^p u \; \Delta^p v \, dx + \varepsilon \int_\Omega uv \, dx. \tag{3.3}$$

According to (3.2), we have:

$$|\left\langle G_\varepsilon'(u), v \right\rangle| \leqslant \|u\|_{\varepsilon,Y}\|v\|_{\varepsilon,Y}$$

Therfore,

$$\begin{aligned}
\sup_{\|v\|_{\varepsilon,Y}\leqslant 1, v\neq 0} \frac{|\left\langle G_\varepsilon'(u), v \right\rangle|}{\|v\|_{\varepsilon,Y}} &\leqslant \|u\|_{\varepsilon,Y} \\
\Rightarrow \quad \|G_\varepsilon'(u)\|_{L(Y,\mathbb{R})} &\leqslant \|u\|_{\varepsilon,Y}
\end{aligned}$$

Hence, $G_\varepsilon^{'}$ is continuous and G_ε is a C^1 functional.
On the other hand, for $F^{'}$, we have :

$$\begin{aligned}
\left|\left\langle F^{'}(u), v\right\rangle\right| &\leq \|m\|_\infty \|u\|_{L^2}\|v\|_{L^2} \\
&\Rightarrow \\
\left|\left\langle F^{'}(u), v\right\rangle\right| &\leq \frac{\|m\|_\infty}{\varepsilon^2}\varphi_\varepsilon(u)\varphi_\varepsilon(v) \\
&\Rightarrow \\
\left\|F^{'}(u)\right\|_{L(Y,\mathbb{R})} &\leq \frac{\|m\|_\infty}{\varepsilon^2}\|u\|_{\varepsilon,Y}.
\end{aligned}$$

So, $F^{'}$ is continuous and finally F is a C^1 functional.

(ii) $F^{'}$ is completely continuous :
Let $(u_n)_n$ be a sequence of Y such that $u_n \rightharpoonup u$ in Y .
Let us show that :

$$\sup_{v\in Y, \|v\|_{\varepsilon,y}\leq 1}\left|\left\langle F^{'}(u_n) - F^{'}(u), v\right\rangle\right| \longrightarrow 0.$$

We have

$$\begin{aligned}
\left|\left\langle F^{'}(u_n) - F^{'}(u), v\right\rangle\right| &= \left|\int_\Omega m(u_n - u)v dx\right| \\
&\leq \|m\|_\infty \|u_n - u\|_{L^2}\|v\|_{L^2} \qquad (3.4) \\
&\leq \frac{\|m\|_\infty}{\varepsilon}\|u_n - u\|_{L^2}\|v\|_{\varepsilon,Y}.
\end{aligned}$$

So,

$$\sup_{v\in Y, \|v\|_{\varepsilon,y}\leq 1}\left|\left\langle F^{'}(u_n) - F^{'}(u), v\right\rangle\right| \leqslant \frac{\|m\|_\infty}{\varepsilon}\|u_n - u\|_{L^2}.$$

Thanks to the compact embedding of $W^{2p,2}(\Omega)$ into $L^2(\Omega)$ and $Y \subset W^{2p,2}(\Omega)$, we deduct the compact embedding of Y into $L^2(\Omega)$.
Hence, it results that $u_n \longrightarrow u$ in $L^2(\Omega)$ and

$$\lim_{n\longrightarrow+\infty}\left(\sup_{v\in Y, \|v\|_{\varepsilon,y}\leq 1}\left|\left\langle F^{'}(u_n) - F^{'}(u), v\right\rangle\right|\right) = 0.$$

(iii) G_ε is an even and positive functional, so it's bounded below.
Let $(v_n)_n$ be a sequence of Y such that $v_n \rightharpoonup v$ in Y
and $\limsup_{n\longrightarrow+\infty}\left\langle G_\varepsilon^{'}(v_n), v_n - v\right\rangle \leq 0$.
We must show that $v_n \longrightarrow v$ in Y.

Using the formula (3.3), we have:

$$\begin{aligned}
\left\langle G_\varepsilon'(v_n) - G_\varepsilon'(v), v_n - v \right\rangle &= \left\langle G_\varepsilon'(v_n), v_n - v \right\rangle - \left\langle G_\varepsilon'(v), v_n - v \right\rangle \\
&= \left(\int_\Omega |\Delta^p v_n|^2 \, dx - \int_\Omega \Delta^p v_n \Delta^p v \, dx + \varepsilon \int_\Omega |v_n|^2 \, dx - \varepsilon \int_\Omega v_n v \, dx\right) \\
&- \left(\int_\Omega \Delta^p v \Delta^p v_n \, dx - \int_\Omega |\Delta^p v|^2 \, dx + \varepsilon \int_\Omega v v_n \, dx - \varepsilon \int_\Omega |v|^2 \, dx\right) \\
&= \left(\|\Delta^p v_n\|_{L^2}^2 + \|\Delta^p v\|_{L^2}^2 - 2\int_\Omega \Delta^p v_n \Delta^p v \, dx\right) \\
&+ \varepsilon\left(\|v_n\|_{L^2}^2 + \|v\|_{L^2}^2 - 2\int_\Omega v_n v \, dx\right) \\
&= \|v_n - v\|_{\varepsilon,Y}^2 .
\end{aligned}$$

Then, $\left\langle G_\varepsilon'(v_n) - G_\varepsilon'(v), v_n - v \right\rangle = \|v_n - v\|_{\varepsilon,Y}^2 \geq 0$

Note that $v_n \rightharpoonup v$ in Y implies that $\lim\limits_{n \longrightarrow +\infty} \left\langle G_\varepsilon'(v), v_n - v \right\rangle = 0.$

So,

$$\limsup_{n \longrightarrow +\infty} \left\langle G_\varepsilon'(v_n) - G_\varepsilon'(v), v_n - v \right\rangle = \limsup_{n \longrightarrow +\infty} \left\langle G_\varepsilon'(v_n), v_n - v \right\rangle \leq 0.$$

It means that :

$$\limsup_{n \longrightarrow +\infty} \|v_n - v\|_{\varepsilon,Y}^2 = 0.$$

Therefore, the result follows.

Consequently, G_ε' satisfies the (S^+) condition.

(iv) G_ε satisfies the (PS) condition on $M = \{u \in Y : F(u) = 1\} \neq \emptyset$:

Let $(v_n)_n$ be a Palais Smale (P.S) sequence of M for the functional G_ε : $G_\varepsilon(v_n)$ is bounded and $G_\varepsilon'(v_n) \longrightarrow 0$.

Let us show that there exists a subsequence of (v_n) which strongly converges in Y.

Since $(v_n) \subset M$, so $\int_\Omega m(v_n)^2 dx = 2$. Furthermore, $G_\varepsilon(v_n)$ is bounded assure that $(\|v_n\|_{\varepsilon,Y}^2)_n$ is bounded: $\exists C > 0, \forall n : \|v_n\|_{\varepsilon,Y} \leqslant C$.

Note that :

$$\left\langle G_\varepsilon'(v_n), v_n - v \right\rangle = \left\langle G_\varepsilon'(v_n), v_n \right\rangle - \left\langle G_\varepsilon'(v_n), v \right\rangle .$$

We have :

$$\left|\left\langle G_\varepsilon'(v_n), v \right\rangle\right| \leq \left\|G_\varepsilon'(v_n)\right\|_{L(Y,\mathbb{R})} \|v\|_{\varepsilon,Y}$$

and

$$\begin{aligned}
\left|\left\langle G_\varepsilon'(v_n), v_n \right\rangle\right| &\leq \left\|G_\varepsilon'(v_n)\right\|_{L(Y,\mathbb{R})} \|v_n\|_{\varepsilon,Y} \\
&\leq C \left\|G_\varepsilon'(v_n)\right\|_{L(Y,\mathbb{R})}
\end{aligned}$$

It follows by the condition : $G_\varepsilon^{'}(v_n) \longrightarrow 0$, that :
$\lim_{n\longrightarrow+\infty} \left\langle G_\varepsilon^{'}(v_n), v \right\rangle = 0$ and $\lim_{n\longrightarrow+\infty} \left\langle G_\varepsilon^{'}(v_n), v_n \right\rangle = 0.$
Consequently,

$$\lim_{n\longrightarrow+\infty} \left\langle G_\varepsilon^{'}(v_n), v_n - v \right\rangle = 0 \leq 0.$$

On the other hand, the space Y is a Banach reflexive, then there exists a subsequence noted also by (v_n) which weekly converges in $Y : v_n \rightharpoonup v$ in Y.

Thus, thanks to the (S^+) condition, we deduce that : $v_n \longrightarrow v$ in Y.

(v) $\forall n \in \mathbb{N}^* : \Gamma_n \neq \emptyset$
Note that $\forall n \in \mathbb{N}^* : \emptyset \in \Gamma_n$. So, $\forall n \in \mathbb{N}^* : \Gamma_n \neq \emptyset$.
This acheives the proof of proposition 3.1.

Proof of theorem 3.1

By theorem 1.2 and proposition 3.1, we conclude that G_ε has n critical points given by:

$$\lambda_{\varepsilon,n} = \inf_{K\in\Gamma_n} \sup_{u\in K} G_\varepsilon(u) \tag{3.5}$$

It's easy to see that, $\forall n \in \mathbb{N}^*, \lambda_{\varepsilon,n}$ is an eigenvalue of problem (P_ε) (2.5).
Now, we must show that $\lambda_{\varepsilon,n} \longrightarrow +\infty$. We proceed in the same way as in Szulkin [6]. Since Y is separable, ther exists a biorthogonal system $(e_n, e_m^*)_{n,m}$ such that $e_n \in Y$ are linearly dense in Y and $e_m^* \in Y^{'}$ are total for $Y^{'}$.
For $n \in \mathbb{N}^*$, set

$$F_n = span\{e_1, e_2, ..., e_n\} \quad F_n^{\perp} = span\{e_{n+1}, e_{n+2}, ...\}$$

and

$$l_n := \inf_{K\in\Gamma_n} \sup_{u\in K\cap F_{n-1}^{\perp}} G_\varepsilon(u).$$

Suppose that there exists $N > 0$ such that for every $n \in \mathbb{N}^*$ there exists $u_n \in F_{n-1}^{\perp}$ which verifies $F(u_n) = 1$ and $l_n \leqslant G_\varepsilon(u_n) \leqslant N$ this implies that $(u_n)_n$ is bounded in Y. For a subsequence still denoted $(u_n)_n$, we can assume that $u_n \rightharpoonup u$ in Y and $u_n \to u$ in L^2.
However, for all $m < n$, $\langle e_m^*, e_n \rangle = 0$, then for all m, $\langle e_m^*, u \rangle = 0$ so, $u = 0$.
This contradicts the fact that $F(u_n) = 1, \forall n$ .It follows that:

$$\lim_{n\to+\infty} l_n = +\infty$$

Since $\lambda_{\varepsilon,n} \geqslant l_n$ we obtain $\lim_{n\to+\infty} \lambda_{\varepsilon,n} = +\infty$.

Lemma 3.1.

$$\lim_{\varepsilon\to 0} \lambda_{\varepsilon,n} = \lambda_n$$

Proof. Let $\eta > 0$, from the definition of λ_n, there exists $K(\eta) \in \Gamma_n$ such that

$$\lambda_n \leqslant \sup_{u \in K(\eta)} G(u) < \lambda_n + \eta$$

On the other hand, we have

$$\lambda_n \leqslant \lambda_{\varepsilon,n} \leqslant \sup_{u \in K(\eta)} G_\varepsilon(u) \leqslant \sup_{u \in K(\eta)} G(u) + \frac{\varepsilon}{2} \sup_{u \in K(\eta)} \|u\|_{L^2}^2$$

It's clear that

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{2} \sup_{u \in K(\eta)} \|u\|_{L^2}^2 = 0$$

then, for $\beta := \lambda_n + \eta - \sup_{u \in K(\eta)} G(u) > 0$,

$$\exists \alpha > 0 : \forall 0 < \varepsilon < \alpha : \quad \frac{\varepsilon}{2} \sup_{u \in K(\eta)} \|u\|_{L^2}^2 < \beta$$

So,

$$\forall 0 < \varepsilon < \alpha : \quad \lambda_n \leqslant \lambda_{\varepsilon,n} < \lambda_n + \eta.$$

Finally,

$$\forall \eta > 0, \exists \alpha > 0 : \forall 0 < \varepsilon < \alpha : \quad \lambda_n \leqslant \lambda_{\varepsilon,n} < \lambda_n + \eta.$$

This completes the proof of lemma.

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