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Moroccan J. of Pure and Appl. Anal. (MJPAA)
Volume 8(3), 2022, Pages 401–437
ISSN: Online 2351-8227 - Print 2605-6364
DOI: 10.2478/mjpaa-2022-0027

Convergence of a finite volume scheme for a parabolic system applied to image processing

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Abstract. We analyze a finite volume scheme for a nonlinear reaction-diffusion system applied to image processing. First, we demonstrate the existence of a solution to the finite volume scheme. Then, based on the derivation of a series of a priori estimates and the use of Kolmogorov's compactness criterion, we prove that the solution to the finite volume scheme converges to the weak solution. In the numerical experiments, we show the effectiveness of the proposed model with respect to the modified (in the sense of Catté, Lions, Morel and Coll) Perona-Malik nonlinear image selective smoothing equation in terms of preserving small details, texture, and fine structures.

Mathematics Subject Classification (2020). 65M08, 65M12, 35K57, 94A08
Key words and phrases. Image Processing, Perona-Malik Equation, Finite Volume Method, Convergence, nonlinear reaction-diffusion

Received June 18, 2022 - Accepted: September 24, 2022.
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1. Introduction

In recent years, many mathematicians have been interested in image processing and computer vision. In particular, image denoising, whose main task is to recover a noise-free image u from an observed image u_0. It is important to look for the restored image u as

$$u_0 = u + n_u, \tag{1.1}$$

where u_0 is the noisy image corrupted by the noise perturbation n_u. The problem (1.1) is usually regularized and reformulated as a minimization problem as follows

$$\text{find} \quad \overline{u} \quad \text{s.t.} \quad \overline{u} \in \operatorname{argmin}_{u \in U}\{J(u) := R(u) + \lambda\Phi(u_0, u)\}, \tag{1.2}$$

where R is the regularization term, Φ is the data fidelity term, and $\overline{u}$ is the recovered image computed by minimizing the energy functional J in the appropriate Banach space U with respect to u.

It should be noted that the various choices of the fidelity and the regularization terms are reflected in different approaches in the literature. The famous choice is the one introduced by Rudin-Osher-Fatemi [20], where $\Phi(u_0, u) = \|u_0 - u\|_{L^2(\Omega)}$ and $R(u) = \int_\Omega |\nabla u| dx$ (Total Variation (TV) seminorm [2]), which is referred to as the *ROF* model:

$$\min_{u \in BV(\Omega)} \int_\Omega |\nabla u| dx + \lambda \|u_0 - u\|^2_{L^2(\Omega)}, \tag{1.3}$$

where $\Omega \subseteq \mathbb{R}^2$ is an open and bounded domain and $BV(\Omega)$ is the space of functions of bounded variation. For a more detailed mathematical study of this model, Chambolle et al [8] have shown the existence and uniqueness of its solution. On the other hand, the problem (1.3) is formally associated to the following Euler-Lagrange equation

$$u - \frac{1}{2\lambda} div\Big(\frac{\nabla u}{|\nabla u|}\Big) = u_0 \quad \text{in } \Omega. \tag{1.4}$$

This approach can preserve the image's edges while also removing noise by searching for solutions of piecewise constant functions in the space $BV(\Omega)$. However, the ROF model is not very effective in separating the cartoon parts from the texture parts. In order to solve this problem, Meyer [16] proposed replacing the L^2-norm of $(u_0 - u)$ in the problem (1.3) with a weaker norm, which may be appropriate for presenting textured or oscillating patterns. However, as a drawback of this type of model, we are unable to directly define the corresponding Euler-Lagrange equation with respect to u. To deal with Meyer's model, Osher et al. [18] presented a model in a total variation minimization framework that replaces the L^2-norm with the H^{-1}-norm (the negative Sobolev norm)

$$\min_{u \in BV(\Omega)} \int_\Omega |\nabla u| dx + \lambda \|u_0 - u\|^2_{H^{-1}(\Omega)}. \tag{1.5}$$

Since the H^{-1}-norm is defined as $||.||^2_{H^{-1}} = \int_\Omega |\nabla \Delta^{-1}(.)|^2 dx$, the problem (1.5) is formally associated to Euler-Lagrange equation

$$w = \frac{1}{2\lambda} div\Big(\frac{\nabla u}{|\nabla u|}\Big) \quad \text{in } \Omega, \tag{1.6}$$

where $w = \Delta^{-1}(u_0 - u)$. Based on the works [1, 11, 12], the problem (1.6) is actually equivalent to the following two coupled second order equations

$$\begin{cases} -\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + 2\lambda w = 0, & \text{in } \Omega, \\ -\Delta w + (u_0 - u) = 0, & \text{in } \Omega. \end{cases} \tag{1.7}$$

To obtain the best numerical solutions, it's necessary to present the evolutionary form of the steady-state system (1.7). The time evolution will be stopped at a finite time in order to obtain the optimal tradeoff between noise removal and edge or texture preservation. While the solutions of (1.7) only happened when $t \to \infty$ in the evolutionary form, this may not give the best results. Thus, it makes more sense to consider the evolutionary problems that can be written as follows

$$\begin{cases} \frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) - 2\lambda w, & \text{in } \Omega \times (0,T), \\ \frac{\partial w}{\partial t} = \Delta w - (u_0 - u), & \text{in } \Omega \times (0,T). \end{cases} \tag{1.8}$$

In current work, we propose a new nonlinear reaction-diffusion system that performs the coherence-enhancing property. This approach takes into consideration the best of the Perona-Malik diffusion process. The proposed equation with Neumann boundary conditions is given as follows

$$\begin{cases} \dfrac{\partial u}{\partial t} - div\Big(g(|\nabla G_\sigma * u|)\nabla u\Big) + 2\lambda w = 0 & \text{in } \Omega_T := \Omega \times (0,T), \\ \dfrac{\partial w}{\partial t} = \Delta w - (u_0 - u) & \text{in } \Omega_T := \Omega \times (0,T), \\ \langle g(|\nabla G_\sigma * u|)\nabla u, \nu\rangle = 0, \quad \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \Sigma_T := \partial\Omega \times (0,T), \\ u(0) = u_0, \quad w(0) = 0 & \text{in } \Omega, \end{cases} \tag{1.9}$$

where $g(s)$ is the decreasing smooth function, $g(0) = 1, 0 < g(s) \to 0$ for $s \to \infty$,

$$G_\sigma \in C^\infty\left(\mathbb{R}^2\right) \text{ is the smoothing kernel with } \int_{\mathbb{R}^2} G_\sigma(x)dx = 1.$$

Furthermore, $\lim_{\sigma\to 0} G_\sigma = \delta_x$, where δ_x is the Dirac measure at point x, $u_0 \in L^2(\Omega)$. Also

$$(G_\sigma * u)(x) = \int_{\mathbb{R}^2} G_\sigma(\mathrm{x} - \xi)\tilde{u}(\xi)d\xi,$$

where $\tilde{u}$ is an extension of u to $\mathbb{R}^2$ (see [9]) and $G_\sigma(x)$ is the Gaussian filter, given by

$$G_\sigma(x) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{\|x\|^2}{\sigma^2}\right), \quad x \in \mathbb{R}^2.$$

Moreover, in literature, different diffusion coefficients have been proposed. For example, Perona and Malik [19] provided some examples of g. They proposed diffusion functions of the

type

$$g_1(s) = \frac{1}{1+(s/\alpha)^2} \quad \text{or} \quad g_2(s) = e^{-(s/\alpha)^2}.$$

Note that in our case, we are interested in the first one with $\alpha = 1$ as in [14, 17].

In this paper we derive an approximation scheme, corresponding to (1.9), using the so-called finite volume method (see for instance [10]). In addition, we refer the readers to [4, 5, 14, 17] for examples of typical applications of such a method. The discrete approximations in this method are piecewise constant by control volumes that correspond to the pixel structure of the discrete image, so this approach is the preferred one in image processing. Furthermore, the scheme permits us to obtain L^2 a-priori estimates for fully discrete solutions, which requires us to use the Kolmogorov's compactness theorem to establish the convergence of approximations to the weak solution of (1.9).

Such studies have already been investigated from a convergence point of view by several authors. First, in 2001, K. Mikula and N. Ramarosy [17] proved the convergence to the weak solution for the semi-implicit finite volume scheme for the numerical solution of the modified (in the sense of Catté, Lions, Morel and Coll) Perona–Malik nonlinear image selective smoothing equation (called anisotropic diffusion in image processing). Second, the explicit finite volume scheme was studied by Z. Krivă in [14], where the authors show the convergence of the explicit finite volume scheme to the weak solution of the continuous equation, which imposes some restriction on the size of the scale step, unlike the semi-implicit one.

Besides, it should be mentioned that in order to avoid the restriction on the size of the scale step. We based on the semi-implicit scheme for the first equation as in [17] and the implicit scheme for the second equation of the proposed nonlinear reaction-diffusion system (1.9).

The organization of the paper is as follows: Firstly, in Section 2, we give the numerical scheme and the main result. In Section 3, we check the existence result for the finite volume scheme. Sections 4 and 5 are devoted to extracting some a priori and space-time translation estimates, respectively, that will help us to show the convergence in Section 6. Section 7 is devoted to some numerical experiments. Finally, in the Appendix, we prove the uniqueness of the classical solution.

2. Numerical scheme and main results

Now, we describe a detailed definition of the finite volume scheme for the proposed model based on the seminal work of R. Eymard et al. [10]. Let $\mathcal{T}_h$ be a mesh of Ω. The elements of $\mathcal{T}_h$ are the so called control volumes. For every pair $(K,L) \in \mathcal{T}_h^2$ with $K \neq L$, we denote their common interface by e_{KL}, i.e. $e_{KL} = \overline{K} \cap \overline{L}$, which is supposed to be included a straight line not intersecting either K or L. Let $m(e_{KL})$ denote the measure of e_{KL}, and n_{KL} the unit vector normal to e_{KL} oriented from K to L. We denote by $\mathcal{E}$ the set of pairs of adjacent control volumes, defined by $\mathcal{E} = \left\{(K,L) \in \mathcal{T}_h^2, K \neq L, m(e_{KL}) \neq 0\right\}$. For all $K \in \mathcal{T}_h$, let by $N(K) = \{L, (K,L) \in \mathcal{E}\}$ denote the set of the neighbors of K, i.e., the set of cells of $\mathcal{T}_h$ which have a common interface with K, by $N_{int}(K)$ the set of the neighbors of K located in the interior of $\mathcal{T}_h$, by $N_{ext}(K)$ the set of edges of K on the boundary $\partial\Omega$ and by $\mathcal{F}_K$ the set of edges of control volume K, defined by

$\mathcal{F}_K = \{e_{KL}, L \in N(K)\}$. We also denote by $n_{K,e}$ is the unit normal to e directed outwards from K.

We assume that there exists family of points $(x_K)_{K\in\mathcal{T}_h}$, $x_K \in K$ for every $K \in \mathcal{T}_h$, such that for every $(K,L) \in \mathcal{E}$, $\frac{x_L - x_K}{|x_L - x_K|} = n_{KL}$. Let us denote by $\delta(K)$ the diameter of the control volume $K, m(K)$ the measure in $\mathbb{R}^2$, of the control volume K, ∂K its boundary and let

$$h = \max_{K\in\mathcal{T}_h} \delta(K).$$

We denote by $d_{KL} = |x_K - x_L|$ the Euclidean distance between x_K and x_L and by x_{KL} a point of e_{KL} intersecting the segment $x_K x_L$. Also we define $T_{KL} = \frac{m(e_{KL})}{d_{KL}}$.

We denote by $H_h(\Omega) \subset L^2(\Omega)$ the space of functions which are piecewise constant on each control volume $K \in \mathcal{T}_h$. For $(u_h, v_h) \in (H_h(\Omega))^2$, we define the inner product as follows:

$$\langle u_h, v_h \rangle_{H_h} = \frac{1}{2} \sum_{K\in\mathcal{T}_h} \sum_{L\in N(K)} T_{KL} (u_L - u_K)(v_L - v_K),$$

corresponding to Neumann boundary conditions, where u_K the constant value of u_h in K, for all $u_h \in H_h(\Omega)$. We also define a norm in $H_h(\Omega)$ by

$$\|u_h\|_{H_h(\Omega)} = \left(\langle u_h, u_h \rangle_{H_h}\right)^{1/2}.$$

We finally define $L_h(\Omega) \subset L^2(\Omega)$ as the space of functions that are piecewise constant on each control volume $K \in \mathcal{T}_h$ with the related inner product and norm

$$(u_h, v_h)_{L_h(\Omega)} = \sum_{K\in\mathcal{T}_h} m(K) u_K v_K, \quad \|u_h\|^2_{L_h(\Omega)} = \sum_{K\in\mathcal{T}_h} m(K) |u_K|^2,$$

for $(u_h, v_h) \in (L_h(\Omega))^2$.

Next, we let $K \in \mathcal{T}_h$ and $L \in N(K)$ with common vertexes $(c_{i,K,L})_{1\le i\le I}$ with $k \in \mathbb{N}^*$. Now, let $\Lambda_{K,L}$ (respectively $\Lambda^{\text{ext}}_{K,\sigma}$ for $\sigma \in N_{\text{ext}}(K)$) be the open and convex polygon with vertexes (x_K, x_L) (x_K respectively) and $(c_{i,K,L})_{1\le i\le I}$. Observe that

$$\Omega = \cup_{K\in\mathcal{T}_h} \left(\left(\cup_{L\in N(K)} \overline{\Lambda}_{K,L} \right) \cup \left(\cup_{\sigma\in N_{\text{ext}}(K)} \overline{\Lambda}^{\text{ext}}_{K,\sigma} \right) \right).$$

The approximation $\nabla_h u_h$ of ∇u is defined by

$$\nabla_h u_h(x) = \begin{cases} T_{KL}(u_L - u_K) n_{KL} & \text{if } x \in \Lambda_{K,L} \\ 0 & \text{if } x \in \Lambda^{\text{ext}}_{K,\sigma} \end{cases},$$

for all $K \in \mathcal{T}_h$.

Now, we are ready to write the finite volume scheme for solving the proposed problem (1.9):

We approximate the model as follows: Determine vectors $(u^n_K)_{K\in\mathcal{T}_h}$ and $(w^n_K)_{K\in\mathcal{T}_h}$ for $n \in \{0, \ldots, N\}$, such that for all $K \in \mathcal{T}_h$ and $n \in \{0, \ldots, N-1\}$

$$m(K)\frac{u_K^{n+1}-u_K^n}{\Delta t}-\sum_{L\in N(K)} g_{KL}^{\sigma,n}\left(\tilde{u}_h\right) T_{KL}\left(u_L^{n+1}-u_K^{n+1}\right) + 2\lambda m(K) w_K^{n+1}=0, \tag{2.1}$$

$$m(K)\frac{w_K^{n+1}-w_K^n}{\Delta t}-\sum_{L\in N(K)} T_{KL}\left(w_L^{n+1}-w_K^{n+1}\right)=m(K)(u_K^{n+1}-u_K^0), \tag{2.2}$$

$$u_K^0=\frac{1}{m(K)}\int_K u_0(x)dx, \quad g_{KL}^{\sigma,n}\left(\tilde{u}_h\right):=g\left(\left|\nabla G_\sigma * \tilde{u}_h\left(x_{KL},t_n\right)\right|\right), \tag{2.3}$$

where $\tilde{u}$ is a periodic extension of the discrete image calculated in the $n-$th scale step. For more details, see [14].

For the purpose of the analysis, we introduce "piecewise constant" functions.

$$u_h(x,t)=u_K^{n+1} \qquad and \qquad w_h(x,t)=w_K^{n+1},$$

for all $(x,t)\in K\times(n\Delta t,(n+1)\Delta t]$, with $K\in\mathcal{T}_h$ and $n\in\{0,\ldots,N-1\}$.

Remark 1. *To make the notation simpler, it will be understood all the time that when $h\to 0$ then $\Delta t\to 0$. This is not a real time-step restriction, but it does allow us to write "(u_h,w_h)" instead of " $(u_{h,\Delta t},w_{h,\Delta t})$ " and "$h\to 0$" instead of " $h,\Delta t\to 0$", and so forth.*

Remark 2. *If there exists $C_0>0$ such that $\sum_{K\in\mathcal{T}_h} m(K)\left(u_K^n\right)^2<C_0$. Then, there exist C_σ, C_1 and C_2 positive constants, such that*

$$\begin{aligned}\left|\frac{\partial}{\partial x_i}G_\sigma * \tilde{u}_h\left(x_{KL},t_n\right)\right| &\le \int_{\mathbb{R}^2}\left|\frac{\partial}{\partial x_i}G_\sigma\left(x_{KL}-\xi\right)\tilde{u}\left(\xi,t_n\right)\right|d\xi \\ &\le \frac{1}{2}\int_{\mathbb{R}^2}\left|\frac{\partial}{\partial x_i}G_\sigma\left(x_{KL}-\xi\right)\right|^2 d\xi+\frac{1}{2}\int_{\mathbb{R}^2}\left|\tilde{u}\left(\xi,t_n\right)\right|^2 d\xi\le C_\sigma \\ &+C_1\int_{\Omega_\sigma}\left|\tilde{u}\left(\xi,t_n\right)\right|^2 d\xi\le C_\sigma+C_1\sum_{K\in\mathcal{T}_h} m(K)\left(u_K^n\right)^2\le C_2,\end{aligned} \tag{2.4}$$

where $\Omega_\sigma=\Omega\cup\bigcup_{x\in\partial\Omega}B_\sigma(x)$ ($B_\sigma(x)$ which is a ball centered at x with a radius σ).

It comes from (2.4) that $\left|\nabla G_\sigma * \tilde{u}_h\left(x_{KL},t_n\right)\right|<\infty$, which in turn implies that there exists a positive constant μ such that

$$g_{KL}^{\sigma,n}\left(\tilde{u}_h\right)\ge\mu>0.$$

Remark 3. *In terms of implementing the convolution included in the system of the Perona-Malik function g. To deal with the convolution term, we follow the interesting strategy used in [13]. The gradient of the convolution term in (2.1), i.e.*

$$\nabla G_\sigma * \tilde{u}_h\left(x_{KL},t_n\right)=\left(\frac{\partial}{\partial x_i}\left(G_\sigma * \tilde{u}_h\left(x_{KL},t_n\right)\right)\right)_{i=1,2},$$

where x_i are space variables, is computed using the convolution derivative property

$$\frac{\partial}{\partial x_i}\left(G_\sigma * \tilde{u}_h\left(x_{KL},t_n\right)\right)=\frac{\partial G_\sigma}{\partial x_i} * \tilde{u}_h\left(x_{KL},t_n\right).$$

Then we have

$$\begin{aligned}\frac{\partial G_\sigma}{\partial x_i} * \tilde{u}_h\left(x_{KL}, t_n\right) &= \int_{\mathbb{R}^2} \frac{\partial G_\sigma}{\partial x_i}\left(x_{KL} - s\right) \tilde{u}_h\left(s, t_n\right) ds \\ &= \sum_r \bar{u}_r^n \int_r \frac{\partial G_\sigma}{\partial x_i}\left(x_{KL} - s\right) ds\end{aligned}$$

and thus

$$\nabla G_\sigma * \tilde{u}_h\left(x_{KL}, t_n\right) = \sum_r \bar{u}_r^n \int_r \nabla G_\sigma\left(x_{KL} - s\right) ds,$$

where the sum is evaluated on control volumes $r \in \mathcal{T}_h$ and on control volumes contained in the reflexion of $\mathcal{T}_h$ through boundary of Ω which are arround x_{KL}. Hereby, the sum is restricted to control volumes intersecting $B_\sigma\left(x_{KL}\right)$, the ball centered at x_{KL} with radius σ.

Moreover, before we give our finite main result, we provide a pertinent definition of a weak solution for the proposed model.

Definition 4. A weak solution of Equation (1.9) is a double (u, w) such that $u, w \in L^2\left(0, T; H^1(\Omega)\right)$ and satisfying the following weak formulation:

$$\begin{aligned}&-\int_\Omega u(0)\phi_1(0)\, \mathrm{d}x - \int_{\Omega_T} u\partial_t\phi_1\, \mathrm{d}x\, \mathrm{d}t + \int_{\Omega_T} g(|\nabla G_\sigma * u|)\nabla u \cdot \nabla\phi_1\, \mathrm{d}x\, \mathrm{d}t \\ &\qquad + 2\lambda \int_{\Omega_T} w\phi_1\, \mathrm{d}x\, \mathrm{d}t = 0, \\ &-\int_{\Omega_T} w\partial_t\phi_2\, \mathrm{d}x\, \mathrm{d}t + \int_{\Omega_T} \nabla w \cdot \nabla\phi_2\, \mathrm{d}x\, \mathrm{d}t + \int_{\Omega_T} \left(u_0 - u\right)\phi_2\, \mathrm{d}x\, \mathrm{d}t = 0,\end{aligned}$$

for all $\phi_i \in \Phi$, $i = 1, 2$, where the function space

$$\Phi = \left\{\phi \in L^2\left(0, T; H^1(\Omega)\right), \partial_t\phi \in L^2\left(0, T; \left(H^1(\Omega)\right)'\right), \phi(., T) = 0\right\},$$

and $\left(H^1(\Omega)\right)'$ denotes the dual space of $H^1(\Omega)$.

Our main result is

Theorem 2.1. Assume that $u_0 \in L^2(\Omega)$. Then the finite volume solution (u_h, w_h), converges along a subsequence to (u, w) as $h \to 0$, where (u, w) is a weak solution of (1.9). The convergence is in the following sense:

$$\begin{aligned}(u_h, w_h) &\to (u, w) \text{ strongly in } L^2\left(\Omega_T\right) \text{ and a.e. in } \Omega_T, \\ (\nabla_h u_h, \nabla_h w_h) &\to (\nabla u, \nabla w) \text{ weakly in } \left(L^2\left(\Omega_T\right)\right)^2.\end{aligned}$$

3. Existence of the finite volume scheme

The existence of a solution to the finite volume scheme (2.1)-(2.3) will be achieved by the following result, proved in [15, 21].

Lemma 5. *Let $\mathcal{F}$ be a finite dimensional Hilbert space with scalar product $[\cdot,\cdot]$ and norm $\|\cdot\|$, and let $\mathcal{L}$ be a continuous mapping from $\mathcal{F}$ into itself such that*

$$[\mathcal{L}(\zeta),\zeta] > 0 \text{ for } \|\zeta\| = \alpha > 0.$$

Then there exists $\zeta \in \mathcal{F}$ with $\|\zeta\| \leq \alpha$ such that

$$\mathcal{L}(\zeta) = 0.$$

The existence of a solution to the finite volume scheme is given in

Proposition 6. *Let $\mathcal{D}$ be an admissible discretization of Ω_T. Then the problem (2.1) -(2.3) admits at least one solution $(u_K^n, w_K^n)_{(K,n)\in\mathcal{T}_h\times\{0,\dots,N\}}$.*

Proof. First we introduce the Hilbert spaces $E_h = (H_h(\Omega)\cap L_h(\Omega))^2$, with the norm

$$\|(u_h,w_h)\|_{E_h}^2 := \|u_h\|_{H_h(\Omega)}^2 + \|w_h\|_{H_h(\Omega)}^2 + \sum_{K\in\mathcal{T}_h} m(K)\left(|u_K|^2+|w_K|^2\right),$$

Now, let $(\varphi_{1,h},\varphi_{2,h})\in E_h$ and define the discrete bilinear forms

$$T_h\left((u_h^n,w_h^n),(\varphi_{1,h},\varphi_{2,h})\right) = \frac{1}{2\lambda}\left(u_h^n,\varphi_{1,h}\right) + \left(w_h^n,\varphi_{2,h}\right),$$

$$b_h\left((u_h^{n+1},w_h^{n+1}),(\varphi_{1,h},\varphi_{2,h})\right) = \sum_{K\in\mathcal{T}_h} m(K)\left(w_K^{n+1}\varphi_{1,K} - (u_K^{n+1}-u_K^0)\varphi_{2,K}\right),$$

$$a_h\left((u_h^{n+1},w_h^{n+1}),(\varphi_{1,h},\varphi_{2,h})\right) = \frac{1}{2\lambda}\sum_{K\in\mathcal{T}_h}\sum_{L\in N(K)} g_{KL}^{\sigma,n}\left(\tilde{u}_h\right) T_{KL}\left(u_L^{n+1}-u_K^{n+1}\right)$$
$$\times\left(\varphi_{1,L}^{n+1}-\varphi_{1,K}^{n+1}\right) + \sum_{K\in\mathcal{T}_h}\sum_{L\in N(K)} T_{KL}\left(w_L^{n+1}-w_K^{n+1}\right)\left(\varphi_{2,L}^{n+1}-\varphi_{2,K}^{n+1}\right).$$

Multiplying (2.1) and (2.2) by $\varphi_{1,K}$, $\varphi_{2,K}$, respectively, we get

$$\frac{1}{\Delta t}\left(T_h\left((u_h^{n+1},w_h^{n+1}),(\varphi_{1,h},\varphi_{2,h})\right) - T_h\left((u_h^n,w_h^n),(\varphi_{1,h},\varphi_{2,h})\right)\right)$$
$$+ a_h\left((u_h^{n+1},w_h^{n+1}),(\varphi_{1,h},\varphi_{2,h})\right) + b_h\left((u_h^{n+1},w_h^{n+1}),(\varphi_{1,h},\varphi_{2,h})\right) = 0.$$

We now rely on the result of the Lemma (5) to demonstrate the existence of (u_h^{n+1},w_h^{n+1}) for all $K\in\mathcal{T}_h$ and $n\in\{0,\dots,N\}$. We define the mapping $\mathcal{L}$ from E_h into itself

$$\left[\mathcal{L}\left((u_h^{n+1},w_h^{n+1})\right),(\varphi_{1,h},\varphi_{2,h})\right] = \frac{1}{\Delta t}\left[T_h\left((u_h^{n+1},w_h^{n+1}),(\varphi_{1,h},\varphi_{2,h})\right)\right.$$
$$\left.-T_h\left((u_h^n,w_h^n),(\varphi_{1,h},\varphi_{2,h})\right)\right] + a_h\left((u_h^{n+1},w_h^{n+1}),(\varphi_{1,h},\varphi_{2,h})\right)$$
$$+ b_h\left((u_h^{n+1},w_h^{n+1}),(\varphi_{1,h},\varphi_{2,h})\right),$$

for all $(\varphi_{1,h}, \varphi_{2,h}) \in E_h$. Using the discrete Hölder inequality, it is easy to obtain the following bounds :

$$\begin{aligned}
a_h\left((u_h, w_h), (\varphi_{1,h}, \varphi_{2,h})\right) &\leq C \left\|(u_h, w_h)\right\|_{E_h} \left\|(\varphi_{1,h}, \varphi_{2,h})\right\|_{E_h}, \\
T_h\left((u_h, w_h), (\varphi_{1,h}, \varphi_{2,h})\right) &\leq C \left\|(u_h, w_h)\right\|_{E_h} \left\|(\varphi_{1,h}, \varphi_{2,h})\right\|_{E_h}, \\
b_h\left((u_h, w_h), (\varphi_{1,h}, \varphi_{2,h})\right) &\leq C \left\|(u_h, w_h)\right\|_{E_h} \left\|(\varphi_{1,h}, \varphi_{2,h})\right\|_{E_h},
\end{aligned}$$

for all (u_h, w_h) and $(\varphi_{1,h}, \varphi_{2,h})$ in E_h. This implies that the mapping $\mathcal{L}$ is continuous. The continuity follows from the continuity of the discrete forms $a_h(\cdot,\cdot)$, $T_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$.

Our goal now is to prove that

$$\left[\mathcal{L}\left((u_h^{n+1}, w_h^{n+1})\right), (u_h^{n+1}, w_h^{n+1})\right] > 0 \quad \text{for } \left\|(u_h^{n+1}, w_h^{n+1})\right\|_{E_h} = \alpha > 0, \tag{3.1}$$

for a sufficiently large α. We observe that

$$\begin{aligned}
\left[\mathcal{L}\left((u_h^{n+1}, w_h^{n+1})\right), (u_h^{n+1}, w_h^{n+1})\right] &= \frac{1}{2\lambda\Delta t} \sum_{K\in\mathcal{T}_h} m(K) \left|u_K^{n+1}\right|^2 \\
&+ \frac{1}{\Delta t} \sum_{K\in\mathcal{T}_h} m(K) \left|w_K^{n+1}\right|^2 + a_h\left((u_h^{n+1}, w_h^{n+1}), (u_h^{n+1}, w_h^{n+1})\right) \\
&+ b_h\left((u_h^{n+1}, w_h^{n+1}), (u_h^{n+1}, w_h^{n+1})\right) - \frac{1}{2\lambda\Delta t} \sum_{K\in\mathcal{T}_h} m(K) u_K^n u_K^{n+1} \\
&- \frac{1}{\Delta t} \sum_{K\in\mathcal{T}_h} m(K) w_K^n w_K^{n+1}.
\end{aligned}$$

It follows that from Young's inequality that

$$\begin{aligned}
&\left[\mathcal{L}\left((u_h^{n+1}, w_h^{n+1})\right), (u_h^{n+1}, w_h^{n+1})\right] \\
&\geq \frac{1}{2\lambda\Delta t} \sum_{K\in\mathcal{T}_h} m(K) \left|u_K^{n+1}\right|^2 + \frac{1}{\Delta t} \sum_{K\in\mathcal{T}_h} m(K) \left|w_K^{n+1}\right|^2 + \mu \left\|u_h^{n+1}\right\|_{H_h(\Omega)}^2 \\
&+ \left\|w_h^{n+1}\right\|_{H_h(\Omega)}^2 - \frac{1}{2\lambda\Delta t} \sum_{K\in\mathcal{T}_h} m(K) u_K^n u_K^{n+1} - \frac{1}{\Delta t} \sum_{K\in\mathcal{T}_h} m(K) w_K^n w_K^{n+1} \\
&+ \sum_{K\in\mathcal{T}_h} m(K) u_K^0 w_K^{n+1}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\left[\mathcal{L}\left((u_h^{n+1}, w_h^{n+1})\right), (u_h^{n+1}, w_h^{n+1})\right] \\
&\geq \frac{1}{2\lambda\Delta t}\sum_{K\in\mathcal{T}_h} m(K)\left|u_K^{n+1}\right|^2 + \frac{1}{\Delta t}\sum_{K\in\mathcal{T}_h} m(K)\left|w_K^{n+1}\right|^2 + \mu\left\|u_h^{n+1}\right\|^2_{H_h(\Omega)} \\
&+ \left\|w_h^{n+1}\right\|^2_{H_h(\Omega)} - \frac{1}{4\lambda\Delta t}\sum_{K\in\mathcal{T}_h} m(K)\left|u_K^{n+1}\right|^2 - \frac{1}{4\lambda\Delta t}\sum_{K\in\mathcal{T}_h} m(K)\left|u_K^{n}\right|^2 \\
&- \frac{1}{4\Delta t}\sum_{K\in\mathcal{T}_h} m(K)\left|w_K^{n+1}\right|^2 - \frac{1}{\Delta t}\sum_{K\in\mathcal{T}_h} m(K)\left|w_K^{n}\right|^2 - \frac{1}{4\Delta t}\sum_{K\in\mathcal{T}_h} m(K)\left|w_K^{n+1}\right|^2 \\
&\qquad - \Delta t\sum_{K\in\mathcal{T}_h} m(K)\left|u_K^{0}\right|^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\left[\mathcal{L}\left((u_h^{n+1}, w_h^{n+1})\right), (u_h^{n+1}, w_h^{n+1})\right] \\
&\geq \frac{1}{2\lambda\Delta t}\sum_{K\in\mathcal{T}_h} m(K)\left|u_K^{n+1}\right|^2 + \frac{1}{2\Delta t}\sum_{K\in\mathcal{T}_h} m(K)\left|w_K^{n+1}\right|^2 + \mu\left\|u_h^{n+1}\right\|^2_{H_h(\Omega)} \\
&+ \left\|w_h^{n+1}\right\|^2_{H_h(\Omega)} - \frac{1}{4\lambda\Delta t}\sum_{K\in\mathcal{T}_h} m(K)\left|u_K^{n}\right|^2 - \frac{1}{\Delta t}\sum_{K\in\mathcal{T}_h} m(K)\left|w_K^{n}\right|^2 \\
&\quad - \Delta t\sum_{K\in\mathcal{T}_h} m(K)\left|u_K^{0}\right|^2 \\
&\geq \min\left\{\frac{1}{2\lambda\Delta t}, \frac{1}{2\Delta t}, \mu\right\}\left\|(u_h^{n+1}, w_h^{n+1})\right\|^2_{E_h} \\
&\quad - \frac{1}{4\lambda\Delta t}\sum_{K\in\mathcal{T}_h} m(K)\left|u_K^{n}\right|^2 - \frac{1}{\Delta t}\sum_{K\in\mathcal{T}_h} m(K)\left|w_K^{n}\right|^2 - \Delta t\sum_{K\in\mathcal{T}_h} m(K)\left|u_K^{0}\right|^2.
\end{aligned} \tag{3.2}$$

Finally, for given u_h^n, u_h^0 and w_h^n, we deduce from (3.2) that (3.1) holds for α large enough. Consequently, we achieve the existence of at least one solution to the scheme (2.1)-(2.3).

4. A priori estimates

The aim is to establish several a priori estimates for the finite volume scheme, which will ultimately imply the required convergence results.

Lemma 7. *Let $(u_K^n, w_K^n)_{K\in\mathcal{T}_h, n\in\{0,\dots,N\}}$, be a solution of the finite volume scheme (2.1) -(2.2). Then we have*

(i) $\max_{n\in\{1,\dots,N\}}\sum_{K\in\mathcal{T}_h} m(K)\left|u_K^n\right|^2 \leq C,$

(ii) $\max_{n\in\{1,\dots,N\}}\sum_{K\in\mathcal{T}_h} m(K)\left|w_K^n\right|^2 \leq C,$

(iii) $\frac{1}{2}\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}_h}\sum_{L\in N(K)} T_{KL}\left|u_K^{n+1} - u_L^{n+1}\right|^2 \leq C,$

(iv) $\frac{1}{2}\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}_h}\sum_{L\in N(K)}T_{KL}\left|w_K^{n+1}-w_L^{n+1}\right|^2\le C.$

where the constant C is independent of n.

Proof. Let us consider n such that $0\le n<N$. We multiply the scheme (2.1) and (2.2) by $\Delta t u_K^{n+1}$ and $\Delta t w_K^{n+1}$, respectively, to obtain

$$\begin{aligned} m(K)\left(u_K^{n+1}-u_K^n\right)u_K^{n+1} =&\Delta t\sum_{L\in N(K)}g_{KL}^{\sigma,n}\left(\tilde{u}_h\right)T_{KL}\left(u_L^{n+1}-u_K^{n+1}\right)u_K^{n+1}\\ &-2\lambda\Delta t m(K)w_K^{n+1}u_K^{n+1}, \end{aligned} \tag{4.1}$$

$$\begin{aligned} m(K)\left(w_K^{n+1}-w_K^n\right)w_K^{n+1} =&\Delta t\sum_{L\in N(K)}T_{KL}\left(w_L^{n+1}-w_K^{n+1}\right)w_K^{n+1}\\ &+\Delta t m(K)(u_K^{n+1}-u_K^0)w_K^{n+1}. \end{aligned} \tag{4.2}$$

Using the property $a(a-b)=\frac{1}{2}a^2-\frac{1}{2}b^2+\frac{1}{2}(a-b)^2$ and summing over $K\in\mathcal{T}_h$, we have that

$$\begin{aligned} &\frac{1}{2}\sum_{K\in\mathcal{T}_h}m(K)\left(u_K^{n+1}\right)^2-\frac{1}{2}\sum_{K\in\mathcal{T}_h}(u_K^n)^2+\frac{1}{2}\sum_{K\in\mathcal{T}_h}m(K)\left(u_K^{n+1}-u_K^n\right)^2\\ &\qquad=\Delta t\sum_{K\in\mathcal{T}_h}\sum_{L\in N(K)}g_{KL}^{\sigma,n}\left(\tilde{u}_h\right)T_{KL}\left(u_L^{n+1}-u_K^{n+1}\right)u_K^{n+1}\\ &\qquad\quad-2\lambda\Delta t\sum_{K\in\mathcal{T}_h}m(K)w_K^{n+1}u_K^{n+1}, \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\frac{1}{2}\sum_{K\in\mathcal{T}_h}m(K)\left(w_K^{n+1}\right)^2-\frac{1}{2}\sum_{K\in\mathcal{T}_h}(w_K^n)^2+\frac{1}{2}\sum_{K\in\mathcal{T}_h}m(K)\left(w_K^{n+1}-w_K^n\right)^2\\ &\qquad=\Delta t\sum_{K\in\mathcal{T}_h}\sum_{L\in N(K)}T_{KL}\left(w_L^{n+1}-w_K^{n+1}\right)w_K^{n+1}\\ &\qquad\quad+\Delta t\sum_{K\in\mathcal{T}_h}m(K)(u_K^{n+1}-u_K^0)w_K^{n+1}, \end{aligned} \tag{4.4}$$

Since $g_{KL}^{\sigma,n}\left(\tilde{u}_h\right)T_{KL}=g_{LK}^{\sigma,n}\left(\tilde{u}_h\right)T_{LK}$, we can rearrange the summation of the first term of the right hand side of (4.3) and (4.4) to obtain

$$\begin{aligned} &\sum_{K\in\mathcal{T}_h}\sum_{L\in N(K)}\left[g_{KL}^{\sigma,n}\left(\tilde{u}_h\right)\,T_{KL}\left(u_L^{n+1}-u_K^{n+1}\right)u_K^{n+1}\right]\\ &\qquad=-\frac{1}{2}\sum_{(K,L)\in\mathcal{E}}\left[g_{KL}^{\sigma,n}\left(\tilde{u}_h\right)T_{KL}\left(u_K^{n+1}-u_L^{n+1}\right)^2\right], \end{aligned}$$

$$\sum_{K\in\mathcal{T}_h}\sum_{L\in N(K)}T_{KL}\left(w_L^{n+1}-w_K^{n+1}\right)w_K^{n+1}=-\frac{1}{2}\sum_{(K,L)\in\mathcal{E}}T_{KL}\left(w_K^{n+1}-w_L^{n+1}\right)^2.$$

Summing the resulting equation over n and multiplying (4.2) by 2λ then adding together the outcomes yields, we have

$$\begin{aligned}
&\frac{1}{2}\sum_{K\in\mathcal{T}_h} m(K)\left(u_K^N\right)^2+\lambda\sum_{K\in\mathcal{T}_h} m(K)\left(w_K^N\right)^2+\frac{1}{2}\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_h} m(K)\left(u_K^{n+1}-u_K^n\right)^2\\
&+\lambda\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_h} m(K)\left(w_K^{n+1}-w_K^n\right)^2+\frac{1}{2}\sum_{n=0}^{N-1}\Delta t\sum_{(K,L)\in\mathcal{E}}\Big[g_{KL}^{\sigma,n}\left(\tilde{u}_h\right)T_{KL}\\
&\times\left(u_K^{n+1}-u_L^{n+1}\right)^2\Big]+\lambda\sum_{n=0}^{N-1}\Delta t\sum_{(K,L)\in\mathcal{E}} T_{KL}\left(w_K^{n+1}-w_L^{n+1}\right)^2\\
&=\frac{1}{2}\sum_{K\in\mathcal{T}_h} m(K)\left(u_K^0\right)^2-2\lambda\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}_h} m(K)u_K^0 w_K^{n+1}.
\end{aligned}\tag{4.5}$$

Then we use Young's inequality on the right-hand side to obtain

$$\begin{aligned}
\frac{1}{2}\sum_{K\in\mathcal{T}_h} m(K)\left(u_K^0\right)^2-2\lambda\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}_h} m(K)u_K^0 w_K^{n+1}&\le\frac{1}{2}\sum_{K\in\mathcal{T}_h} m(K)\left(u_K^0\right)^2\\
+\lambda\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}_h} m(K)(u_K^0)^2+\lambda\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}_h} m(K)(w_K^{n+1})^2.&
\end{aligned}$$

Since $u_0\in L^2(\Omega)$, there exists $C_1>0$ such that

$$\frac{1}{2}\sum_{K\in\mathcal{T}_h} m(K)\left(u_K^0\right)^2\le C_1,\tag{4.6}$$

and one can show that there exists a positive constant C_2 such that

$$\begin{aligned}
&\frac{1}{2}\sum_{K\in\mathcal{T}_h} m(K)\left(u_K^N\right)^2+\lambda\sum_{K\in\mathcal{T}_h} m(K)\left(w_K^N\right)^2+\frac{1}{2}\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}_h} m(K)\times\left(u_K^{n+1}-u_K^n\right)^2\\
&+\lambda\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_h} m(K)\left(w_K^{n+1}-w_K^n\right)^2+\frac{1}{2}\sum_{n=0}^{N-1}\Delta t\sum_{(K,L)\in\mathcal{E}}\left[g_{KL}^{\sigma,n}\left(\tilde{u}_h\right)T_{KL}\left(u_K^{n+1}-u_L^{n+1}\right)^2\right]\\
&+\lambda\sum_{n=0}^{N-1}\Delta t\sum_{(K,L)\in\mathcal{E}} T_{KL}\left(w_K^{n+1}-w_L^{n+1}\right)^2\le C_2+\lambda\sum_{n=0}^{N}\Delta t\sum_{K\in\mathcal{T}_h} m(K)(w_K^n)^2.
\end{aligned}\tag{4.7}$$

Using the discrete Gronwall lemma, there exists a positive constant C such that

$$\sum_{K\in\mathcal{T}_h} m(K)\left(w_K^n\right)^2\le C,\ \text{for all } 0\le n\le N,\tag{4.8}$$

hold for every Δt sufficiently small where the constant C does not depend on $h,\Delta t$. Using (2.4) and (4.8) in (4.7), one can deduce assertions of the lemma.

5. Space and time translation estimates

In this section, we derive estimates on differences in space and time translates of the function v_h which implies that the sequence v_h is relatively compact in $L^2(\Omega_T)$.

5.1. Space translation estimates.

Lemma 8. *For all vector $\varrho \in \mathbb{R}^2$, there exist positive constants C_1 and C_2 such that*

$$\int_{\Omega_\varrho \times (0,T)} (u_h(x+\varrho,t) - u_h(x,t))^2 \, dxdt \leq C_1 |\varrho|(|\varrho| + 2h),$$

$$\int_{\Omega_\varrho \times (0,T)} (w_h(x+\varrho,t) - w_h(x,t))^2 \, dxdt \leq C_2 |\varrho|(|\varrho| + 2h),$$

where $\Omega_\varrho = \{x \in \Omega, [x, x+\varrho] \in \Omega\}$.

Proof. Let $\varrho \in \mathbb{R}^2$ be a given vector and for all $x \in \Omega_\varrho$, we denote by $\chi(x,K,L)$ the function defined as follows

$$\chi(x,K,L) = \begin{cases} 1 \text{ if the segment } [x, x+\varrho] \text{ intersects } e_{KL}, K \text{ and } L; \text{ and } \varrho_{KL} > 0 \\ 0 \text{ otherwise} \end{cases},$$

where ϱ_{KL} is the following value $\varrho_{KL} = \frac{\varrho}{|\varrho|} \cdot n_{KL}$. For any $t \in (0,T)$ there exists $n \in \mathbb{N}$ such that $(n-1)\Delta t < t \leq n\Delta t$. Then, we can see that

$$u_h(x+\varrho,t) - u_h(x,t) = u^n_{K(x+\varrho)} - u^n_{K(x)} = \sum_{(K,L)\in\mathcal{E}} \chi(x,K,L)\,(u^n_L - u^n_K)\,, \tag{5.1}$$

$$w_h(x+\varrho,t) - w_h(x,t) = w^n_{K(x+\varrho)} - w^n_{K(x)} = \sum_{(K,L)\in\mathcal{E}} \chi(x,K,L)\,(w^n_L - w^n_K)\,, \tag{5.2}$$

where $K(x)$ is the volume $K \in \mathcal{T}_h$ where $x \in K$. Using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} (u_h(x+\varrho,t) - u_h(x,t))^2 \leq & \left(\sum_{(K,L)\in\mathcal{E}} \chi(x,K,L)\varrho_{KL} d_{KL} \right) \\ & \times \left(\sum_{(K,L)\in\mathcal{E}} \chi(x,K,L) \frac{(u^n_L - u^n_K)^2}{\varrho_{KL} d_{KL}} \right), \end{aligned} \tag{5.3}$$

$$\begin{aligned} (w_h(x+\varrho,t) - w_h(x,t))^2 \leq & \left(\sum_{(K,L)\in\mathcal{E}} \chi(x,K,L)\varrho_{KL} d_{KL} \right) \\ & \left(\sum_{(K,L)\in\mathcal{E}} \chi(x,K,L) \frac{(w^n_L - w^n_K)^2}{\varrho_{KL} d_{KL}} \right). \end{aligned} \tag{5.4}$$

Using the fact that $\varrho_{KL} d_{KL} = \frac{\varrho}{|\varrho|} n_{KL} d_{KL} = \frac{\varrho}{|\varrho|} (x_L - x_K)$, we have that

$$\sum_{(K,L)\in\mathcal{E}} \chi(x,K,L)\varrho_{KL} d_{KL} = \frac{\varrho}{|\varrho|}\left(x_{K(x+\varrho)} - x_{K(x)}\right). \tag{5.5}$$

Next, we add and subtract $x+\varrho$ to obtain

$$\begin{aligned}\left|x_{K(x+\varrho)} - x_{K(x)}\right| &= \left|\left(x_{K(x+\varrho)} - (x+\varrho)\right) - \left(x_{K(x)} - x\right) + \varrho\right| \\ &\leq \left|x_{K(x+\varrho)} - (x+\varrho)\right| + \left|x_{K(x)} - x\right| + |\varrho| \leq 2h + |\varrho|,\end{aligned}$$

since $(x+\varrho) \in K(x+\varrho)$ and $x \in K(x)$. This result implies that

$$\sum_{(K,L)\in\mathcal{E}} \chi(x,K,L)\varrho_{KL} d_{KL} \leq 2h + |\varrho|. \tag{5.6}$$

Now, we integrate (5.3) and (5.4) on $\Omega_\varrho \times (0,T)$ and use (5.6) to obtain

$$\begin{aligned}&\int_{\Omega_\varrho\times(0,T)} \left(u_h(x+\varrho,t) - u_h(x,t)\right)^2 dxdt \\ &\qquad \leq (2h+|\varrho|) \sum_{n=0}^{N} \Delta t \sum_{(K,L)\in\mathcal{E}} \frac{(u_L^n - u_K^n)^2}{\varrho_{KL} d_{KL}} \int_{\Omega_\varrho} \chi(x,K,L)dx,\end{aligned} \tag{5.7}$$

$$\begin{aligned}&\int_{\Omega_\varrho\times(0,T)} \left(w_h(x+\varrho,t) - w_h(x,t)\right)^2 dxdt \\ &\qquad \leq (2h+|\varrho|) \sum_{n=0}^{N} \Delta t \sum_{(K,L)\in\mathcal{E}} \frac{(u_L^n - u_K^n)^2}{\varrho_{KL} d_{KL}} \int_{\Omega_\varrho} \chi(x,K,L)dx,\end{aligned} \tag{5.8}$$

since u_h and w_h are piecewise constant for each interval $(n\Delta t, (n+1)\Delta t)$. We observe that

$$\int_{\Omega_\varrho} \chi(x,K,L)dx \leq m\left(e_{KL}\right)|\varrho|\varrho_{KL},$$

and using this result in (5.7) and (5.8) we obtain

$$\begin{aligned}&\int_{\Omega_\varrho\times(0,T)} \left(u_h(x+\varrho,t) - u_h(x,t)\right)^2 dxdt \\ &\qquad \leq (2h+|\varrho|)|\varrho| \sum_{n=0}^{N} \Delta t \sum_{(K,L)\in\mathcal{E}} T_{KL}\left(u_L^n - u_K^n\right)^2,\end{aligned} \tag{5.9}$$

$$\begin{aligned}&\int_{\Omega_\varrho\times(0,T)} \left(w_h(x+\varrho,t) - w_h(x,t)\right)^2 dxdt \\ &\qquad \leq (2h+|\varrho|)|\varrho| \sum_{n=0}^{N} \Delta t \sum_{(K,L)\in\mathcal{E}} T_{KL}\left(w_L^n - w_K^n\right)^2.\end{aligned} \tag{5.10}$$

Finally, using the discrete a priori estimate of the lemma (7), we complete the proof.

5.2. Time translation estimates.

Lemma 9. *There exists a positive constant C such that*

$$\int_{\Omega\times(0,T-s)} \left(u_h(x,t+s)-u_h(x,t)\right)^2 dxdt \le Cs,$$

$$\int_{\Omega\times(0,T-s)} \left(w_h(x,t+s)-w_h(x,t)\right)^2 dxdt \le Cs,$$

for all $s \in (0,T)$.

Proof. Let $s \in (0,T)$. We define the following functions of time t

$$A(t)=\int_\Omega \left(u_h(x,t+s)-u_h(x,t)\right)^2 dx \text{ and } B(t)=\int_\Omega \left(w_h(x,t+s)-w_h(x,t)\right)^2 dx.$$

Set $n_t = E(t/\Delta t)$ and $n_{t+s} = E((t+s)/\Delta t)$, where $E(.)$ means the upper integer part of a positive real number. We get

$$A(t)=\sum_{K\in\mathcal{T}_h} m(K)\left(u_K^{n_{t+s}}-u_K^{n_t}\right)^2 \quad \text{and} \quad B(t)=\sum_{K\in\mathcal{T}_h} m(K)\left(w_K^{n_{t+s}}-w_K^{n_t}\right)^2,$$

This implies

$$A(t)=\sum_{K\in\mathcal{T}_h} m(K)\left(u_K^{n_{t+s}}-u_K^{n_t}\right)\sum_{t\le n\Delta t<t+s}\left(u_K^{n+1}-u_K^n\right), \tag{5.11}$$

$$B(t)=\sum_{K\in\mathcal{T}_h} m(K)\left(w_K^{n_{t+s}}-w_K^{n_t}\right)\sum_{t\le n\Delta t<t+s}\left(w_K^{n+1}-w_K^n\right). \tag{5.12}$$

Using the scheme (2.1-2.2), we have

$$\begin{aligned} A(t)=&\sum_{t\le n\Delta t<t+s}\Delta t\sum_{K\in\mathcal{T}_h}\left(\left(u_K^{n_{t+s}}-u_K^{n_t}\right)\times\sum_{L\in N(K)}\left[g_{KL}^{\sigma,n}\left(\tilde{u}_h\right)T_{KL}\left(u_L^{n+1}-u_K^{n+1}\right)\right]\right)\\ &-2\lambda\sum_{t\le n\Delta t<t+s}\Delta t\sum_{K\in\mathcal{T}_h} m(K)\left(\left(u_K^{n_{t+s}}-u_K^{n_t}\right)w_K^{n+1}\right), \end{aligned} \tag{5.13}$$

$$\begin{aligned} B(t)=&\sum_{t\le n\Delta t<t+s}\Delta t\sum_{K\in\mathcal{T}_h}\left(\left(w_K^{n_{t+s}}-w_K^{n_t}\right)\times\sum_{L\in N(K)}T_{KL}\left(w_L^{n+1}-w_K^{n+1}\right)\right)\\ &+\sum_{t\le n\Delta t<t+s}\Delta t\sum_{K\in\mathcal{T}_h} m(K)\left(\left(w_K^{n_{t+s}}-w_K^{n_t}\right)\left(u_K^{n+1}-u_K^0\right)\right). \end{aligned} \tag{5.14}$$

We observe that we can rewrite (5.13-5.14) as

$$\begin{aligned} A(t)=&\sum_{t\le n\Delta t<t+s}\frac{\Delta t}{2}\sum_{(K,L)\in\mathcal{E}}\left(\left(u_K^{n_{t+s}}-u_K^{n_t}-u_L^{n_{t+s}}+u_L^{n_t}\right)\right.\\ &\left.\times\left[g_{KL}^{\sigma,n}\left(\tilde{u}_h\right)T_{KL}\left(u_L^{n+1}-u_K^{n+1}\right)\right]\right)\\ &-2\lambda\sum_{t\le n\Delta t<t+s}\Delta t\sum_{K\in\mathcal{T}_h} m(K)\left(\left(u_K^{n_{t+s}}-u_K^{n_t}\right)w_K^{n+1}\right), \end{aligned} \tag{5.15}$$

$$B(t) = \sum_{t\le n\Delta t<t+s} \frac{\Delta t}{2} \sum_{(K,L)\in\mathcal{E}} \left((w_K^{n_{t+s}} - w_K^{n_t} - w_L^{n_{t+s}} + w_L^{n_t}) \times T_{KL} \left(w_L^{n+1} - w_K^{n+1} \right) \right) + \sum_{t\le n\Delta t<t+s} \Delta t \sum_{K\in\mathcal{T}_h} m(K) \left((w_K^{n_{t+s}} - w_K^{n_t}) \left(u_K^{n+1} - u_K^0 \right) \right). \tag{5.16}$$

Using the relation "$ab \le \frac{a^2}{2} + \frac{b^2}{2}$" yield

$$\begin{aligned} A(t) &\le \frac{1}{2}A_0(t) + \frac{1}{2}A_1(t) + \frac{1}{4}A_2(t) + A_3(t), \\ B(t) &\le \frac{1}{2}B_0(t) + \frac{1}{2}B_1(t) + \frac{1}{4}B_2(t) + B_3(t), \end{aligned} \tag{5.17}$$

where

$$\begin{aligned} A_0(t) &= \sum_{t\le n\Delta t<t+s} \Delta t \sum_{(K,L)\in\mathcal{E}} T_{KL} \left(u_L^{n_t} - u_K^{n_t} \right)^2, \\ B_0(t) &= \sum_{t\le n\Delta t<t+s} \Delta t \sum_{(K,L)\in\mathcal{E}} T_{KL} \left(w_L^{n_t} - w_K^{n_t} \right)^2, \end{aligned} \tag{5.18}$$

$$\begin{aligned} A_1(t) &= \sum_{t\le n\Delta t<t+s} \Delta t \sum_{(K,L)\in\mathcal{E}} T_{KL} \left(u_L^{n_{t+s}} - u_K^{n_{t+s}} \right)^2, \\ B_1(t) &= \sum_{t\le n\Delta t<t+s} \Delta t \sum_{(K,L)\in\mathcal{E}} T_{KL} \left(w_L^{n_{t+s}} - w_K^{n_{t+s}} \right)^2, \end{aligned} \tag{5.19}$$

$$\begin{aligned} A_2(t) &= \sum_{t\le n\Delta t<t+s} \Delta t \sum_{(K,L)\in\mathcal{E}} \left(g_{KL}^{\sigma,n}(u_h) \right)^2 T_{KL} \left(u_L^{n+1} - u_K^{n+1} \right)^2, \\ B_2(t) &= \sum_{t\le n\Delta t<t+s} \Delta t \sum_{(K,L)\in\mathcal{E}} T_{KL} \left(w_L^{n+1} - w_K^{n+1} \right)^2, \end{aligned} \tag{5.20}$$

$$\begin{aligned} A_3(t) &= -2\lambda \sum_{t\le n\Delta t<t+s} \Delta t \sum_{K\in\mathcal{T}_h} m(K) \left((u_K^{n_{t+s}} - u_K^{n_t}) w_K^{n+1} \right), \\ B_3(t) &= \sum_{t\le n\Delta t<t+s} \Delta t \sum_{K\in\mathcal{T}_h} m(K) \left((w_K^{n_{t+s}} - w_K^{n_t}) \left(u_K^{n+1} - u_K^0 \right) \right). \end{aligned} \tag{5.21}$$

We integrate (5.17) according to the time variable to obtain

$$\begin{aligned} \int_0^{T-s} A(t)dt &\le \frac{1}{2}\int_0^{T-s} A_0(t)dt + \frac{1}{2}\int_0^{T-s} A_1(t)dt + \frac{1}{4}\int_0^{T-s} A_2(t)dt + \int_0^{T-s} A_3(t)dt, \\ \int_0^{T-s} B(t)dt &\le \frac{1}{2}\int_0^{T-s} B_0(t)dt + \frac{1}{2}\int_0^{T-s} B_1(t)dt + \frac{1}{4}\int_0^{T-s} B_2(t)dt + \int_0^{T-s} B_3(t)dt. \end{aligned} \tag{5.22}$$

The next step is to provide a bound for each term. We observe that

$$\chi_{\{t\le n\Delta t<t+s\}} = \chi_{\{n\Delta t-s<t\le n\Delta t\}}, \tag{5.23}$$

then, we get

$$\begin{aligned}\int_0^{T-s} A_0(t)dt &= \int_0^{T-s} \sum_{t\le n\Delta t<t+s} \Delta t \sum_{(K,L)\in\mathcal{E}} T_{KL}\left(u_L^{n_t}-u_K^{n_t}\right)^2 dt \\ &= \int_0^{T-s} \Delta t \sum_{(K,L)\in\mathcal{E}} T_{KL}\left(u_L^{n_t}-u_K^{n_t}\right)^2 \sum_{n\in\mathbb{N}} \chi_{\{t\le n\Delta t<t+s\}} dt \\ &\le \sum_{n_t=0}^{N-1} \Delta t \sum_{(K,L)\in\mathcal{E}} T_{KL}\left(u_L^{n_t}-u_K^{n_t}\right)^2 \times \int_{n_t\Delta t}^{(n_t+1)\Delta t} \sum_{n\in\mathbb{N}} \chi_{\{n\Delta t-s<t\le n\Delta t\}} dt.\end{aligned} \tag{5.24}$$

Moreover, we have that

$$I = \int_{n_t\Delta t}^{(n_t+1)\Delta t} \sum_{n\in\mathbb{N}} \chi_{\{n\Delta t-s<t\le n\Delta t\}} dt = \sum_{n\in\mathbb{N}} \int_{n_t\Delta t}^{(n_t+1)\Delta t} \chi_{\{n\Delta t-s<t\le n\Delta t\}} dt,$$

and if we the change of variable $z = t - n\Delta t + s$, we have that

$$I = \sum_{n\in\mathbb{N}} \int_{n_t\Delta t-n\Delta t+s}^{(n_t+1)\Delta t-n\Delta t+s} \chi_{\{0<z\le s\}} dz = s.$$

We deduce that

$$\int_0^{T-s} A_0(t)dt \le s \sum_{n_t=0}^{N} \Delta t \sum_{(K,L)\in\mathcal{E}} T_{KL}\left(u_L^{n_t}-u_K^{n_t}\right)^2.$$

Using the estimates of the lemma (7) gives

$$\int_0^{T-s} A_0(t)dt \le Cs \text{ and } \int_0^{T-s} B_0(t)dt \le Cs. \tag{5.25}$$

Similarly, by only changing n_t into n_{t+s}, it is easy to prove that

$$\int_0^{T-s} A_1(t)dt \le Cs \text{ and } \int_0^{T-s} B_1(t)dt \le Cs. \tag{5.26}$$

Using (5.20), we have

$$\begin{aligned}\int_0^{T-s} A_2(t)dt &= \int_0^{T-s} \sum_{n\in\mathbb{N}} \Delta t \sum_{(K,L)\in\mathcal{E}} \left(g_{KL}^{\sigma,n}\left(\tilde{u}_h\right)\right)^2 \\ &\qquad \times T_{KL}\left(u_L^{n+1}-u_K^{n+1}\right)^2 \chi_{\{t\le n\Delta t<t+s\}} dt, \\ \int_0^{T-s} B_2(t)dt &= \int_0^{T-s} \sum_{n\in\mathbb{N}} \Delta t \sum_{(K,L)\in\mathcal{E}} T_{KL}\left(w_L^{n+1}-w_K^{n+1}\right)^2 \chi_{\{t\le n\Delta t<t+s\}} dt.\end{aligned} \tag{5.27}$$

This implies

$$\int_0^{T-s} A_2(t)dt \leq \sum_{n=0}^{N-1} \Delta t \sum_{(K,L)\in\mathcal{E}} \left(g_{KL}^{\sigma,n}\left(\tilde{u}_h\right)\right)^2 T_{KL}\left(u_L^{n+1}-u_K^{n+1}\right)^2 \int_0^{T-s} \chi_{\{n\Delta t-s<t\leq n\Delta t\}}dt, \tag{5.28}$$

$$\int_0^{T-s} B_2(t)dt \leq \sum_{n=0}^{N-1} \Delta t \sum_{(K,L)\in\mathcal{E}} T_{KL}\left(w_L^{n+1}-w_K^{n+1}\right)^2 \int_0^{T-s} \chi_{\{n\Delta t-s<t\leq n\Delta t\}}dt.$$

Moreover, one can observe that

$$\int_0^{T-s} \chi_{\{n\Delta t-s<t\leq n\Delta t\}}dt = \min(T-s, n\Delta t) - \max(0, n\Delta t - s) \leq s. \tag{5.29}$$

Using the estimates of the lemma (7) and (5.28) in (5.29) yields

$$\int_0^{T-s} A_2(t)dt \leq Cs \text{ and } \int_0^{T-s} B_2(t)dt \leq Cs. \tag{5.30}$$

For the terms $A_3(t)$ and $A_3(t)$. Using Young's inequality and the relation $\frac{(a-b)^2}{2} \leq a^2+b^2$ in (5.21), we get

$$\begin{aligned} A_3(t) \leq & 2\lambda \sum_{t\leq n\Delta t<t+s} \Delta t \sum_{K\in\mathcal{T}_h} m(K)\left(\left(u_K^{n_{t+s}}\right)^2+\left(u_K^{n_t}\right)^2+\frac{(w_K^{n+1})^2}{2}\right), \\ B_3(t) \leq & \sum_{t\leq n\Delta t<t+s} \Delta t \sum_{K\in\mathcal{T}_h} m(K)\left(\left(w_K^{n_{t+s}}\right)^2+\left(w_K^{n_t}\right)^2+\left(u_K^{n+1}\right)^2+\left(u_K^0\right)^2\right). \end{aligned} \tag{5.31}$$

Now we integrate it over $(0, T-s)$, we obtain

$$\begin{aligned} \int_0^{T-s} A_3(t) \leq & 2\lambda A_3^1(t) + 2\lambda A_3^2(t) + \lambda A_3^3(t), \\ \int_0^{T-s} B_3(t) \leq & B_3^1(t) + B_3^2(t) + B_3^3(t) + B_3^4(t), \end{aligned} \tag{5.32}$$

where $A_3^i, i=1,\cdots,3$ and $B_3^j, j=1,\cdots,4$ correspond to

$$\begin{aligned} A_3^1 &= \int_0^{T-s} \sum_{t\leq n\Delta t<t+s} \Delta t \sum_{K\in\mathcal{T}_h} m(K)\left(u_K^{n_{t+s}}\right)^2 dt, \\ B_3^1 &= \int_0^{T-s} \sum_{t\leq n\Delta t<t+s} \Delta t \sum_{K\in\mathcal{T}_h} m(K)\left(w_K^{n_{t+s}}\right)^2 dt, \end{aligned} \tag{5.33}$$

$$\begin{aligned} A_3^2 &= \int_0^{T-s} \sum_{t\leq n\Delta t<t+s} \Delta t \sum_{K\in\mathcal{T}_h} m(K)\left(u_K^{n_t}\right)^2 dt, \\ B_3^2 &= \int_0^{T-s} \sum_{t\leq n\Delta t<t+s} \Delta t \sum_{K\in\mathcal{T}_h} m(K)\left(w_K^{n_t}\right)^2 dt, \end{aligned} \tag{5.34}$$

$$A_3^3 = \int_0^{T-s} \sum_{t\le n\Delta t<t+s} \Delta t \sum_{K\in\mathcal{T}_h} m(K)\left(w_K^{n+1}\right)^2 dt,$$
$$B_3^3 = \int_0^{T-s} \sum_{t\le n\Delta t<t+s} \Delta t \sum_{K\in\mathcal{T}_h} m(K)\left(u_K^{n+1}\right)^2 dt, \tag{5.35}$$

$$B_3^4 = \int_0^{T-s} \sum_{t\le n\Delta t<t+s} \Delta t \sum_{K\in\mathcal{T}_h} m(K)\left(u_K^{0}\right)^2 dt. \tag{5.36}$$

By using the same argument as in the estimates of $A_0(t)$ and $B_0(t)$ and applying the estimates of lemma (7), for some all $k = 1, 2, 3$, we get

$$A_3^k \le CTs, \quad \textit{and} \quad B_3^k \le CTs. \tag{5.37}$$

for some positive constant C . For B_3^4, we use the estimate (4.6) and (5.23) to have

$$B_3^4 \le C_1 \sum_{n=0}^{N-1} \Delta t \int_0^{T-s} \chi_{\{n\Delta t-s<t\le n\Delta t\}} dt, \tag{5.38}$$

which with (5.29) implies that

$$B_3^4 \le C_1 Ts. \tag{5.39}$$

Thus, using (5.37) and (5.39), we can deduce that

$$\int_0^{T-s} A_3(t)dt \le 5\lambda CTs, \quad \textit{and} \quad \int_0^{T-s} B_3(t)dt \le (3C + C_1)Ts. \tag{5.40}$$

Finally, applying (5.25), (5.26), (5.30) and (5.40) in (5.22). This concludes the proof of the lemma.

6. Convergence Results

The following lemma is a consequence of previously established a priori estimates (lemmas (8) and (9)) and Kolmogorov's compactness criterion (see, e.g., [6], Theorem IV.25).

Lemma 10. *There exists a subsequence of (u_h, w_h), such that, as $h \to 0$,*

(i) $(u_h, w_h) \to (u, w)$ *strongly in* $L^2(\Omega_T)$ *and a.e. in* Ω_T,

(ii) $(\nabla_h u_h, \nabla_h w_h) \to (\nabla u, \nabla w)$ *weakly in* $\left(L^2(\Omega_T)\right)^2$.

Proof. (i) By using the estimate (i) of Lemma (7), the time and space translation estimates given in Lemmas (8-9) and Kolmogorov's compactness criterion, one can deduce the claim.

(ii) This claim will be omitted as it is similar to that of Lemma (4.4) of [7].

The last step is to prove that u and w constructed in the lemma (10) is the weak solution of (1.9). First, let us define the set of functions

$$\Psi = \left\{ \phi \in C^{2,1}(\overline{\Omega} \times [0,T]), \nabla\phi \cdot \nu = 0 \text{ on } \partial\Omega \times (0,T), \phi(.,T) = 0 \right\},$$

which is dense in the set Φ defined in (4).

Let $\phi_1, \phi_2 \in \Psi$ be given. We multiply the scheme (2.1-2.2) by $\phi_1(x_K, t_n)\,\Delta t$ and $\phi_2(x_K, t_n)\,\Delta t$ respectively, to have a discrete analogy of the weak solution identity (4). Then we sum over all $K \in \mathcal{T}_h$ and $n = 0, \ldots, N-1$. It holds

$$\sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}_h} m(K) \frac{\left(u_K^{n+1} - u_K^n\right)}{\Delta t} \phi_1(x_K, t_n) - \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}_h} \phi_1(x_K, t_n)$$
$$\times \sum_{L\in N(K)} \left[g_{KL}^{\sigma,n}(\tilde{u}_h)\, T_{KL} \left(u_L^{n+1} - u_K^{n+1} \right) \right] + 2\lambda \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}_h} m(K) \phi_1(x_K, t_n)\, w_K^{n+1} = 0,$$

$$\sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}_h} m(K) \frac{\left(w_K^{n+1} - w_K^n\right)}{\Delta t} \phi_2(x_K, t_n) - \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}_h} \phi_2(x_K, t_n)$$
$$\times \sum_{L\in N(K)} T_{KL} \left(w_L^{n+1} - w_K^{n+1} \right) - \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}_h} m(K) \phi_2(x_K, t_n) \left(u_K^{n+1} - u_K^0 \right) = 0.$$

Making a discrete integration by part of each term, keeping in mind that $\phi_1(x_K, T) = 0$ and $\phi_2(x_K, T) = 0$ for all $K \in \mathcal{T}_h$, we obtain

$$\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_h} m(K) \left(u_K^{n+1} - u_K^n \right) \phi_1(x_K, t_n) = - \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_h} m(K) u_K^{n+1}$$
$$\times \left(\phi_1(x_K, t_{n+1}) - \phi_1(x_K, t_n) \right) - \sum_{K\in\mathcal{T}_h} m(K) u_K^0 \phi_1(x_K, 0),$$

$$\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_h} m(K) \left(w_K^{n+1} - w_K^n \right) \phi_2(x_K, t_n) = - \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_h} m(K) w_K^{n+1}$$
$$\times \left(\phi_2(x_K, t_{n+1}) - \phi_2(x_K, t_n) \right).$$

For the second term, we gather the sum over $K \in \mathcal{T}_h$ and over $L \in N(K)$ to obtain

$$- \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}_h} \phi_1(x_K, t_n) \sum_{L\in N(K} \left[g_{KL}^{\sigma,n}(\tilde{u}_h)\, T_{KL} \left(u_L^{n+1} - u_K^{n+1} \right) \right]$$
$$= \frac{1}{2} \sum_{n=0}^{N-1} \Delta t \sum_{(K,L)\in\mathcal{E}} g_{KL}^{\sigma,n}(\tilde{u}_h)\, T_{KL} \left(u_L^{n+1} - u_K^{n+1} \right)$$
$$\times \left(\phi_1(x_L, t_n) - \phi_1(x_K, t_n) \right),$$

since $g_{KL}^{\sigma,n}(\tilde{u}_h)\, T_{KL} = g_{LK}^{\sigma,n}(\tilde{u}_h)\, T_{LK}$. And we have also

$$
\begin{aligned}
&- \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}_h} \phi_2 (x_K, t_n) \sum_{L \in N(K} T_{KL} \left(w_L^{n+1} - w_K^{n+1} \right) \\
&\qquad = \frac{1}{2} \sum_{n=0}^{N-1} \Delta t \sum_{(K,L) \in \mathcal{E}} T_{KL} \left(w_L^{n+1} - w_K^{n+1} \right) \\
&\qquad\qquad \times \left(\phi_2 (x_L, t_n) - \phi_2 (x_K, t_n) \right).
\end{aligned}
$$

Then we can rewrite the scheme as follows

$$
\begin{aligned}
&\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_h} m(K) u_K^{n+1} \left(\phi_1 (x_K, t_{n+1}) - \phi_1 (x_K, t_n) \right) + \sum_{K \in \mathcal{T}_h} m(K) u_K^0 \phi_1 (x_K, 0) \\
&- \frac{1}{2} \sum_{n=0}^{N-1} \Delta t \sum_{(K,L) \in \mathcal{E}} g_{KL}^{\sigma,n} (\tilde{u}_h) T_{KL} \left(u_L^{n+1} - u_K^{n+1} \right) \times \left(\phi_1 (x_L, t_n) - \phi_1 (x_K, t_n) \right) \\
&\qquad - 2\lambda \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}_h} m(K) \phi_1 (x_K, t_n) w_K^{n+1} = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_h} m(K) w_K^{n+1} \left(\phi_2 (x_K, t_{n+1}) - \phi_2 (x_K, t_n) \right) \\
&- \frac{1}{2} \sum_{n=0}^{N-1} \Delta t \sum_{(K,L) \in \mathcal{E}} T_{KL} \left(w_L^{n+1} - w_K^{n+1} \right) \times \left(\phi_2 (x_L, t_n) - \phi_2 (x_K, t_n) \right) \\
&\qquad + \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}_h} m(K) \phi_2 (x_K, t_n) \left(u_K^{n+1} - u_K^0 \right) = 0.
\end{aligned}
$$

In the following, we show the convergence of each term to its continuous analogue for all test functions in Ψ.

Lemma 11. *We have that*

$$
\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_h} m(K) u_K^{n+1} \phi_1 (x_K, t_{n+1}) - \phi_1 (x_K, t_n) \to \int_0^T \int_\Omega u \frac{\partial \phi_1}{\partial t} (x,t) dx dt,
$$

$$
\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_h} m(K) w_K^{n+1} \phi_2 (x_K, t_{n+1}) - \phi_2 (x_K, t_n) \to \int_0^T \int_\Omega w \frac{\partial \phi_2}{\partial t} (x,t) dx dt,
$$

as $h \to 0$, *for all* $\phi_1, \phi_2 \in \Psi$.

Proof. Give $\phi_1, \phi_2 \in \Psi$. We set the difference between the discrete and continuous terms as follows

$$T_{u1} = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_h} \Big[m(K) u_K^{n+1} \left(\phi_1 \left(x_K, t_{n+1} \right) - \phi_2 \left(x_K, t_n \right) \right) - \int_{t_n}^{t_{n+1}} \int_K u \frac{\partial \phi_1}{\partial t}(x,t) dx dt \Big],$$

$$T_{w2} = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_h} \Big[m(K) w_K^{n+1} \left(\phi_2 \left(x_K, t_{n+1} \right) - \phi_2 \left(x_K, t_n \right) \right) - \int_{t_n}^{t_{n+1}} \int_K w \frac{\partial \phi_2}{\partial t}(x,t) dx dt \Big].$$

We add and substract $u_K^{n+1} \int_{t_n}^{t_{n+1}} \int_K \frac{\partial \phi_1}{\partial t}(x,t) dx dt$ and $w_K^{n+1} \int_{t_n}^{t_{n+1}} \int_K \frac{\partial \phi_2}{\partial t}(x,t) dx dt$ in the summation to get

$$|T_{u1}| \leq \left| \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_h} u_K^{n+1} \int_{t_n}^{t_{n+1}} \int_K \left(\frac{\partial \phi_1}{\partial t} \left(x_K, t \right) - \frac{\partial \phi_1}{\partial t}(x,t) \right) dx dt \right| + \left| \int_0^T \int_\Omega \left(u_h - u \right) \frac{\partial \phi_1}{\partial t}(x,t) dx dt \right|,$$

$$|T_{w1}| \leq \left| \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_h} w_K^{n+1} \int_{t_n}^{t_{n+1}} \int_K \left(\frac{\partial \phi_2}{\partial t} \left(x_K, t \right) - \frac{\partial \phi_2}{\partial t}(x,t) \right) dx dt \right| + \left| \int_0^T \int_\Omega \left(w_h - w \right) \frac{\partial \phi_2}{\partial t}(x,t) dx dt \right|.$$

In order to separate u_K^{n+1}, w_K^{n+1} and to have its discrete $L^2\left(\Omega_T\right)$ norm in the first terms of the right hand side, we apply the Cauchy-Schwartz inequality. Then using the fact that

$$\left(\frac{\partial \phi_1}{\partial t} \left(x_K, t \right) - \frac{\partial \phi_1}{\partial t}(x,t) \right) \leq hM, \qquad \left(\frac{\partial \phi_2}{\partial t} \left(x_K, t \right) - \frac{\partial \phi_2}{\partial t}(x,t) \right) \leq hM,$$

for some $M > 0$, since $\phi_1, \phi_2 \in C^{2,1}(\overline{\Omega} \times [0,T])$, we can deduce that this term tends to 0 as $h \to 0$. We have the same affirmation for the second terms on the right hand side, since we have the strong convergence (u_h, w_h) to (u, w) in $L^2(\Omega_T)$. Thus, as $h \to 0$, we showed

$$|T_{u1}| \to 0, \qquad |T_{w1}| \to 0.$$

Lemma 12. *For a given u_0 and for u_K^0 as defined in (2.3) we have that*

$$\sum_{K \in \mathcal{T}_h} m(K) u_K^0 \phi_1 \left(x_K, 0 \right) \to \int_\Omega u_0(x) \phi_1(x,0) dx,$$

as $h \to 0$ for all $\phi_1 \in \Psi$.

Proof. By using (2.3), we obtain that

$$\sum_{K\in\mathcal{T}_h} m(K)u_K^0\phi_1\left(x_K,0\right) - \int_\Omega u_0(x)\phi_1(x,0)dx = \sum_{K\in\mathcal{T}_h}\int_K \left(\phi_1\left(x_K,0\right)-\phi_1(x,0)\right)u_0(x)dx.$$

Since $\phi_1 \in C^{2,1}(\overline{\Omega}\times[0,T])$, there exists a positive constant M such that

$$\left|\phi_1\left(x_K,0\right)-\phi_1(x,0)\right| \le hM,$$

and we can conclude the assertion of the lemma.

Lemma 13. *We have that*

$$\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}_h} m(K)\phi_1\left(x_K,t_n\right)w_K^{n+1} \to \int_0^T\int_\Omega w(x,t)\phi_1(x,t)dxdt,$$

$$\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}_h} m(K)\phi_2\left(x_K,t_n\right)\left(u_K^{n+1}-u_K^0\right) \to \int_0^T\int_\Omega \left(u-u_0\right)\phi_2(x,t)dxdt,$$

as $h\to 0$, for all $\phi_1,\phi_2\in\Psi$.

Proof. Let $\phi_1,\phi_2\in\Psi$ be given. We set the difference between the discrete and continuous terms as follows

$$A = \sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}_h} m(K)\phi_1\left(x_K,t_n\right)w_K^{n+1} - \int_0^T\int_\Omega w(x,t)\phi_1(x,t)dxdt = A_1+A_2,$$

$$\begin{aligned} B &= \sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}_h} m(K)\phi_2\left(x_K,t_n\right)\left(u_K^{n+1}-u_K^0\right) - \int_0^T\int_\Omega \left(u-u_0\right)\phi_2(x,t)dxdt \\ &= B_1+B_2, \end{aligned}$$

where

$$A_1 = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_h}\phi_1\left(x_K,t_n\right)\int_{t_n}^{t_{n+1}}\int_K\left(w_K^{n+1}-w\right)dxdt,$$

$$A_2 = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_h}\int_{t_n}^{t_{n+1}}\int_K w\left[\phi_1\left(x_K,t_n\right)-\phi_1(x,t)\right]dxdt,$$

and

$$B_1 = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_h}\phi_2\left(x_K,t_n\right)\int_{t_n}^{t_{n+1}}\int_K\left[\left(u_K^{n+1}-u\right)-\left(u_K^0-u_0\right)\right]dxdt,$$

$$B_2 = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_h}\int_{t_n}^{t_{n+1}}\int_K\left(u-u_0\right)\left[\phi_2\left(x_K,t_n\right)-\phi_2(x,t)\right]dxdt.$$

Applying Cauchy-Schwartz inequality and the relation $\frac{1}{2}(a-b)^2\le a^2+b^2$ we get

$$|A_1| \le \left(\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}_h} m(K)\left|\phi_1\left(x_K,t_n\right)\right|^2\right)^{\frac{1}{2}}\left(\int_0^T\int_\Omega\left(w_h-w\right)^2dxdt\right)^{\frac{1}{2}},$$

$$|B_1| \leq 2 \left(\sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}_h} m(K) \left| \phi_2 \left(x_K, t_n \right) \right|^2 \right)^{\frac{1}{2}} \times \left(\int_0^T \int_\Omega \left((u_h - u)^2 + \left(u_h(x,0) - u_0 \right)^2 \right) dxdt \right)^{\frac{1}{2}},$$

from where we have

$$|A_1| \to 0 \quad and \quad |B_1| \to 0, \qquad \text{as } h \to 0.$$

From the regularity of ϕ_1, ϕ_2, one can show the existence of a positive constants C_1, C_2 that for all $x \in K$ and $t \in \left[t^n, t^{n+1} \right]$, we have

$$|\phi_1 \left(x_K, t^n \right) - \phi_1(x,t)| \leq C_1(\Delta t + h) \text{ and } |\phi_2 \left(x_K, t^n \right) - \phi_2(x,t)| \leq C_2(\Delta t + h).$$

Then it comes that

$$|A_2| \leq C_1(\Delta t + h) \int_0^T \int_\Omega |w(x,t)| \, dxdt,$$

$$|B_2| \leq C_2(\Delta t + h) \int_0^T \int_\Omega |u - u_0| \, dxdt,$$

which implies in turn that

$$|A_2| \to 0 \quad and \quad |B_2| \to 0, \qquad \text{as } h \to 0.$$

Thus, the lemma is demonstrated.

Lemma 14. *We have that*

$$\frac{1}{2} \sum_{n=0}^{N-1} \Delta t \sum_{(K,L) \in \mathcal{E}} g_{KL}^{\sigma,n} \left(\tilde{u}_h \right) T_{KL} \left(u_L^{n+1} - u_K^{n+1} \right) \left(\phi_1 \left(x_L, t_n \right) - \phi_1 \left(x_K, t_n \right) \right) \to \int_0^T \int_\Omega g \left(|\nabla G_\sigma * u| \right) \nabla u \nabla \phi_1 dxdt,$$

as $h \to 0$, for all $\phi_1 \in \Psi$.

Proof. First, we define

$$A_1 = \frac{1}{2} \sum_{n=0}^{N-1} \Delta t \sum_{(K,L) \in \mathcal{E}} \left(u_L^{n+1} - u_K^{n+1} \right) g_{KL}^{\sigma,n} \left(\tilde{u}_h \right) A_{KL}^n m \left(e_{KL} \right),$$

where A_{KL}^n represents the difference between discrete and continuous normal derivative evaluated on (x_{KL}, t_n), i.e.

$$A_{KL}^n = \left(\frac{\phi_1 \left(x_L, t_n \right) - \phi_1 \left(x_K, t_n \right)}{d_{KL}} \right) - \nabla \phi_1 \left(x_{KL}, t_n \right) n_{KL}.$$

Since $\phi_1 \in C^{2,1}(\overline{\Omega} \times [0,T])$, there exists a constant $C > 0$ such that

$$|A_{KL}^n| \leq hC.$$

Then, we have

$$|A_1| \leq h\frac{C}{2}\sum_{n=0}^{N-1}\Delta t\sum_{(K,L)\in\mathcal{E}}\left|u_L^{n+1}-u_K^{n+1}\right| m\left(e_{KL}\right). \tag{6.1}$$

We multiply and divide by $\sqrt{d_{KL}}$ the right-hand side of (6.1) and we use the Cauchy-Schwartz inequality to get

$$|A_1| \leq h\frac{C}{2}\left(\sum_{n=0}^{N-1}\Delta t\sum_{(K,L)\in\mathcal{E}}T_{KL}\left(u_L^{n+1}-u_K^{n+1}\right)^2\right)^{\frac{1}{2}}\times\left(\sum_{n=0}^{N-1}\Delta t\sum_{(K,L)\in\mathcal{E}}d_{KL}m\left(e_{KL}\right)\right)^{\frac{1}{2}}.$$

We observe that there exists a positive constant $C^{'}$ such that

$$\sum_{(K,L)\in\mathcal{E}}d_{KL}m\left(e_{KL}\right)\leq C^{'}|\Omega|. \tag{6.2}$$

Then, combined with the estimate (iii) of Lemma 7, it implies that

$$|A_1| \leq h\frac{C\left(C^{'}|\Omega|TC_{2u}\right)^{\frac{1}{2}}}{2},$$

and we can therefore conclude that

$$|A_1|\to 0, \qquad \text{as } h\to 0.$$

Next, we set

$$A_2=\frac{1}{2}\sum_{n=0}^{N-1}\sum_{(K,L)\in\mathcal{E}}\left(u_L^{n+1}-u_K^{n+1}\right)g_{KL}^{\sigma,n}\left(\tilde{u}_h\right)\int_{t_n}^{t_{n+1}}\int_{e_{KL}}\overline{A}_{KL}^n dxdt, \tag{6.3}$$

where

$$\overline{A}_{KL}^n=\left(\nabla\phi_1\left(x_{KL},t_n\right)-\nabla\phi_1(x,t)\right)n_{KL}.$$

Due to the regularity of ϕ_1, for any $x\in e_{KL}$, we have

$$\left|\overline{A}_{KL}^n\right|\leq(\Delta t+h)C,$$

with a positive constant C depending only on ϕ_1. We use this result to replace $\overline{A}_{KL}^n$ in (6.3) and by the same argument as for the estimation of the term A_1 we deduce that

$$|A_2|\to 0, \qquad \text{as } h\to 0.$$

Now, we define the third term as follows

$$A_3=\frac{1}{2}\sum_{n=0}^{N-1}\sum_{(K,L)\in\mathcal{E}}\left(u_L^{n+1}-u_K^{n+1}\right)\int_{t_n}^{t_{n+1}}\int_{e_{KL}}G_{KL}^n\nabla\phi_1 n_{KL}dxdt,$$

where $G_{KL}^n=g\left(|\nabla G_\sigma*\tilde{u}_h\left(x_{KL},t_n\right)|\right)-g\left(|\nabla G_\sigma*\tilde{u}_h(x,t)|\right)$. We should bound G_{KL}^n to prove the convergence of A_3 to 0. For that purpose, we use the fact that g is Lipschitz continuous. Then we have

$$|G_{KL}^n|\leq L_g||\nabla G_\sigma*\tilde{u}_h\left(x_{KL},t_n\right)|-|\nabla G_\sigma*\tilde{u}_h(x,t)||,$$

where L_g is the Lipschitz constant of g, and we can use the triangular inequality to get

$$|G_{KL}^n| \leq L_g \left|\nabla G_\sigma * \tilde{u}_h\left(x_{KL}, t_n\right) - \nabla G_\sigma * \tilde{u}_h(x,t)\right|.$$

Using the form of the convolution and as $t \in (t_n, t_{n+1})$, we can show that for any $x \in e_{KL}$ holds

$$|G_{KL}^n| \leq L_g \sum_r |u_K^n| \int_r \left|\nabla G_\sigma\left(x_{KL} - s\right) - \nabla G_\sigma(x-s)\right| ds,$$

where the sum is evaluated only on control volumes $r \in \mathcal{T}_h$ and r in reflexion of $\mathcal{T}_h$ through the boundary of Ω intersecting $B_\sigma\left(x_{KL}\right)$, the ball centered at x_{KL} with radius σ. Since $G_\sigma \in C^\infty\left(\mathbb{R}^2\right)$, the Cauchy Schwarz inequality and the estimate (i) of Lemma (7) we obtain

$$|G_{KL}^n| \leq hC,$$

with a positive constant C. Since $\nabla\phi_1$ is a continuous function,
$S = \sup_{\Omega_T} |\nabla\phi_1| < \infty$, we have that

$$|A_3| \leq h\frac{CS}{2} \sum_{n=0}^{N-1} \Delta t \sum_{(K,L)\in\mathcal{E}} \left|u_L^{n+1} - u_K^{n+1}\right| m\left(e_{KL}\right),$$

which, together with the Cauchy-Schwarz inequality and (6.2) leads to

$$|A_3| \to 0, \qquad \text{as } h \to 0.$$

The fourth term will be given as follows

$$A_4 = \frac{1}{2} \sum_{n=0}^{N-1} \sum_{(K,L)\in\mathcal{E}} \left(u_L^{n+1} - u_K^{n+1}\right) \times \int_{t_n}^{t_{n+1}} \int_{e_{KL}} \left(g\left(|\nabla G_\sigma * \tilde{u}_h|\right) - g\left(|\nabla G_\sigma * u|\right)\right) \nabla\phi_1 n_{KL} dx dt.$$

The Lipschitz continuity of g and the triangular inequality give that

$$\left|g\left(|\nabla G_\sigma * \tilde{u}_h|\right) - g\left(|\nabla G_\sigma * u|\right)\right| \leq L_g \left|\nabla G_\sigma * \left(\tilde{u}_h - u\right)\right|.$$

Using Cauchy-Schwartz inequality in the convolution term leads to the following result

$$\left|g\left(|\nabla G_\sigma * \tilde{u}_h|\right) - g\left(|\nabla G_\sigma * u|\right)\right| \leq C \left(\int_{\mathbb{R}^2} |\nabla G_\sigma(x-s)|^2 ds\right)^{\frac{1}{2}} \|\bar{u}_h - u\|_{L^2(Q_T)}.$$

Since G_σ is $C^\infty(\mathbb{R}^2)$, and G_σ has a compact support, we can prove that there exists a constant $C > 0$ such that

$$\left(\int_{\mathbb{R}^2} |\nabla G_\sigma(x-s)|^2 ds\right)^{\frac{1}{2}} \leq C.$$

Then, we use the same technique as for the estimate of A_1 to see that there exists a positive constant $C^{'}$ such that

$$|A_4| \leq C^{'} \|u_h - u\|_{L^2(\Omega_T)},$$

and since u_h converges to u strongly in $L^2\left(\Omega_T\right)$, we conclude that

$$|A_4| \to 0, \qquad \text{as } h \to 0.$$

The last term is defined by

$$A_5 = \int_0^T \int_\Omega div\left(g\left(|\nabla G_\sigma * u|\right)\nabla\phi_1\right)(u - u_h)\,dxdt.$$

By the hypotheses, $g(\sqrt{\cdot}) \in C^\infty(\mathbb{R}^+)$, and convolution property state that $\nabla G_\sigma * u \in C^\infty\left(\mathbb{R}^2\right)$ since $G_\sigma \in C^\infty(\mathbb{R}^2)$. Then $div\left(g\left(|\nabla G_\sigma * u|\right)\nabla\phi_1\right) \in L^2\left(\Omega_T\right)$ and it comes from the strong convergence of u_h to u that

$$|A_5| \to 0, \qquad \text{as } h \to 0.$$

Finally, since

$$\frac{1}{2}\sum_{n=0}^{N-1}\Delta t \sum_{(K,L)\in\mathcal{E}} g_{KL}^{\sigma,n}\left(\tilde{u}_h\right) T_{KL}\left(u_L^{n+1} - u_K^{n+1}\right)\left(\phi_1\left(x_L,t_n\right) - \phi_1\left(x_K,t_n\right)\right)$$
$$- \int_0^T \int_\Omega g\left(|\nabla G_\sigma * u|\right)\nabla u\nabla\phi_1 dxdt = \sum_{i=1}^{5} A_i.$$

We can conclude that

$$\frac{1}{2}\sum_{n=0}^{N-1}\Delta t \sum_{(K,L)\in\mathcal{E}} g_{KL}^{\sigma,n}\left(\tilde{u}_h\right) T_{KL}\left(u_L^{n+1} - u_K^{n+1}\right)\left(\phi_1\left(x_L,t_n\right) - \phi_1\left(x_K,t_n\right)\right)$$
$$\to \int_0^T \int_\Omega g\left(|\nabla G_\sigma * u|\right)\nabla u\nabla\phi_1 dxdt,$$

as $h \to 0$. This ends the proof of Lemma.

Lemma 15. *We have that*

$$\frac{1}{2}\sum_{n=0}^{N-1}\Delta t \sum_{(K,L)\in\mathcal{E}} T_{KL}\left(w_L^{n+1} - w_K^{n+1}\right)\left(\phi_2\left(x_L,t_n\right) - \phi_2\left(x_K,t_n\right)\right)$$
$$\to \int_0^T \int_\Omega \nabla w\nabla\phi_2 dxdt,$$

as $h \to 0$, for all $\phi_2 \in \Psi$.

Proof. We set the difference between the discrete and continuous terms of the lemma as follows

$$A = \frac{1}{2}\sum_{n=0}^{N-1}\Delta t \sum_{(K,L)\in\mathcal{E}} T_{KL}\left(w_L^{n+1} - w_K^{n+1}\right)\left(\phi_2\left(x_L,t_n\right) - \phi_2\left(x_K,t_n\right)\right)$$
$$- \int_0^T \int_\Omega \nabla w\nabla\phi_2 dxdt.$$

Next, we define I and A_1^* by

$$I = \int_0^T \int_\Omega \nabla w\nabla\phi_2 dxdt, \qquad A_1^* = \int_0^T \int_\Omega w_h\Delta\phi_2 dxdt.$$

Integration-by-parts yields

$$I = -\int_0^T \int_\Omega w\Delta\phi_2 dxdt.$$

On the other hand, using the convergence results, it is easy to prove that $\int_\Omega w_h\Delta\phi_2 dx \to -\int_\Omega \nabla w\nabla\phi_2 dx$ weakly in $L^2(0,T)$, as $h \to 0$. Thus, it holds

$$A_1^* \to -I, \quad \text{as } h \to 0.$$

Note that

$$\begin{aligned} A_1^* &= \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_h} \sum_{L\in N(K)} w_K^{n+1} \int_{t_n}^{t_{n+1}} \int_{\sigma_{K,L}} \nabla\phi_2 \cdot n_{KL} d\gamma \\ &= -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{(K,L)\in\mathcal{E}} \left(w_L^{n+1} - w_K^{n+1}\right) \int_{t_n}^{t_{n+1}} \int_{\sigma_{K,L}} \nabla\phi_2 \cdot n_{KL} d\gamma. \end{aligned}$$

Hence

$$\begin{aligned} A = \frac{1}{2} \sum_{n=0}^{N-1} \sum_{(K,L)\in\mathcal{E}} m\left(\sigma_{K,L}\right) \left(w_L^{n+1} - w_K^{n+1}\right) \Bigg(\int_{t_n}^{t_{n+1}} \frac{\phi_2\left(x_L, t^n\right) - \phi_2\left(x_K, t^n\right)}{d_{KL}} \\ - \frac{1}{m\left(\sigma_{K,L}\right)} \int_{t_n}^{t_{n+1}} \int_{\sigma_{K,L}} \nabla\phi_2 \cdot n_{KL} d\gamma \Bigg). \end{aligned}$$

This implies from the regularity of ϕ_2 that

$$\frac{\phi_2\left(x_L, t^n\right) - \phi_2\left(x_K, t^n\right)}{d(K,L)} = \nabla\phi_2\left(x, t^n\right) \cdot n_{KL},$$

with x between x_K and x_L, and so

$$\left| \int_{t_n}^{t_{n+1}} \frac{\phi_2\left(x_L, t^n\right) - \phi_2\left(x_K, t^n\right)}{d(K,L)} - \frac{1}{m\left(\sigma_{K,L}\right)} \int_{t_n}^{t_{n+1}} \int_{\sigma_{K,L}} \nabla\phi_2 \cdot n_{KL} d\gamma \right| \le C\Delta t h,$$

for some constant $C > 0$, which implies that $|A| \to 0$, as $h \to 0$. Thus, the lemma is demonstrated.

7. Numerical experiments

In this section, we present the numerical results of the proposed model. By (2.1-2.3) we have, for each $K \in \mathcal{T}_h$

$$\begin{aligned}\left(\frac{m(K)}{\Delta t}+\sum_{L\in N(K)} g_{KL}^{\sigma,n}\left(\tilde{u}_h\right) T_{KL}\right) u_K^{n+1}-\sum_{L\in N(K)} g_{KL}^{\sigma,n}\left(\tilde{u}_h\right) T_{KL} u_L^{n+1} \\ =\frac{m(K)}{\Delta t} u_K^n-2\lambda m(K) w_K^{n+1}, \\ \left(\frac{m(K)}{\Delta t}+\sum_{L\in N(K)} T_{KL}\right) w_K^{n+1}-\sum_{L\in N(K)} T_{KL} w_L^{n+1} \\ =\frac{m(K)}{\Delta t} w_K^n+m(K)\left(u_K^{n+1}-u_K^0\right).\end{aligned} \tag{7.1}$$

Now, let us define the function ϱ as follows

$$\begin{aligned}\varrho : \mathcal{T}_h &\longrightarrow \mathbb{N}\cap[1,\xi] \\ K &\longmapsto \varrho(K)\end{aligned}$$

which numbers each volume K, where $\xi=\operatorname{card}\left(\mathcal{T}_h\right)$. Then we can construct $\mathbb{R}^{\xi\times\xi}$-matrices $A^n=\left(A_{ij}^n\right)_{\xi\times\xi}$ and $B=\left(B_{ij}\right)_{\xi\times\xi}$ where

$$A_{\varrho(K),\varrho(K)}^n=\frac{m(K)}{\Delta t}+\sum_{L\in N(K)} g_{KL}^{\sigma,n}\left(\tilde{u}_h\right) T_{KL},\quad B_{\varrho(K),\varrho(K)}=\frac{m(K)}{\Delta t}+\sum_{L\in N(K)} T_{KL},$$

and

$$A_{\varrho(K),\varrho(L)}^n=-g_{KL}^{\sigma,n}\left(\tilde{u}_h\right) T_{KL},\quad B_{\varrho(K),\varrho(L)}=-T_{KL},\quad \text{for all } L\in N(K),$$

and otherwise

$$A_{ij}^n=0,\qquad B_{ij}=0.$$

Let $\left(\mathbb{R}^\xi\right)^2$-vector $\left(U^{n+1},W^{n+1}\right)$ correspond to the discrete solution, i.e.

$$\left(\left(U^{n+1},W^{n+1}\right)\right)_{\varrho(K)}=\left(u_K^{n+1},w_K^{n+1}\right),$$

and set $\left(\mathbb{R}^\xi\right)^2$-vector $\left(E^{n+1},F^{n+1}\right)$ to

$$\begin{aligned}\left(E^{n+1}\right)_{\varrho(K)}&=\frac{m(K)}{\Delta t} u_K^n-2\lambda m(K) w_K^{n+1}, \\ \left(F^{n+1}\right)_{\varrho(K)}&=\frac{m(K)}{\Delta t} w_K^n+m(K)\left(u_K^{n+1}-u_K^0\right).\end{aligned} \tag{7.2}$$

By using the above definitions, we can rewrite (7.1) as a linear system in the form of matrices

$$A^nU^{n+1}=E^{n+1}\qquad \textit{and}\qquad BW^{n+1}=F^{n+1}. \tag{7.3}$$

Finally, the main algorithm of our model is summarized as follows:

Algorithm 1.

(1) **Input:** Given a noisy image U^0.
(2) **Initialize:** $\lambda, \sigma, tol, n = 0, m = 0$ and $Niter$.
(3) **Result**: The restored image U
(4) **While** $n \leq Niter$ **do**
(5) Compute A^n and B
(6) Set $W^{n+1,0} = W^n$ and $U^{n+1,0} = U^n$.
(7) Set $E^{n+1,0} = E^{n+1}$ and $F^{n+1,0} = F^{n+1}$
(8) **do**
(9) Compute $A^n U^{n+1,m+1} = E^{n+1,m}$
(10) Compute $BW^{n+1,m+1} = F^{n+1,m}$.
(11) Set $Err1 = \| W^{n+1,m+1} - W^{n+1,m} \|_2$ and $Err2 = \| U^{n+1,m+1} - U^{n+1,m} \|_2$
(12) Set $m = m + 1$
(13) **while** ($Err1 > tol$ and $Err2 > tol$)
(14) Set $W^{n+1} = W^{n+1,m+1}$ and $U^{n+1} = W^{n+1,m+1}$
(15) Set $n = n + 1$
(16) **end**

Using the above linear systems, we can obtain (U^1, W^1) by (U^0, W^0). The process will terminate when it has reached its goal. Most parameters of the algorithms are selected in a heuristic way to make the algorithm perform at its best.

In this section, we first start with the denoising test to show the performance of our proposed model (cf. Figs. 1 and 2). As a second test, we will see how effective our algorithm is at denoising images of different noises (cf. Figures 3 and 4). Finally, A comparison test will be done to show the advantages of using our proposed model instead of the existing models (cf. Fig. 5). The experiments work with an image size of 256×256 with an initial number of pixels of 65536.

In the first experiment, we use the spatial step $h = 1$, time step $\Delta t = 0.01$, $\sigma = 0.5$, $\lambda = 0.01$.



We applied a statistical measure to evaluate the quality of the recovery results of our model. In **Table 1**, we present the peak signal to noise ratio (PSNR) values and the associated structural similarity (SSIM) values for different tests and using different noise levels.

Table 1: Noise performance parameters PSNR and SSIM for *Moon*, *Trui* and *Scan* images with different Gaussian noise levels.

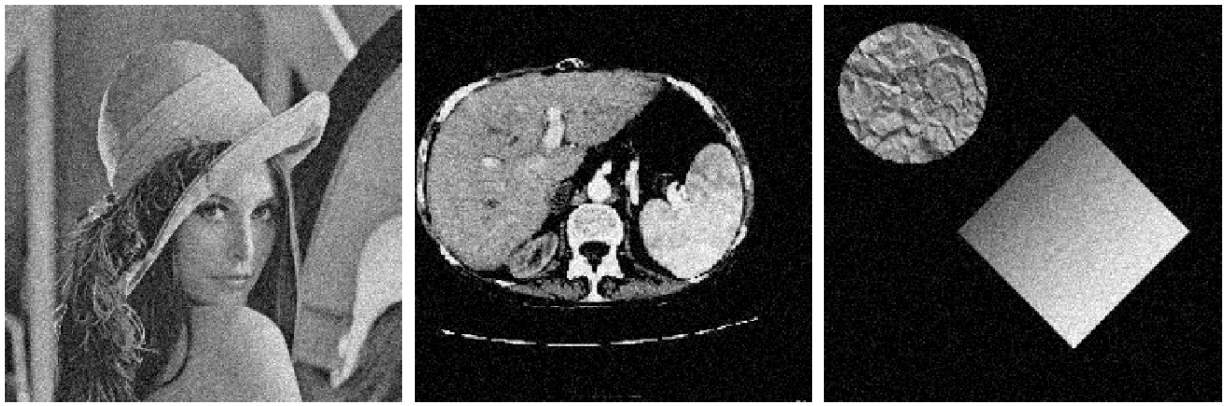


FIGURE 1. Images corrupted by Gaussian noise with zero mean and variance $s^2 = 0.07$

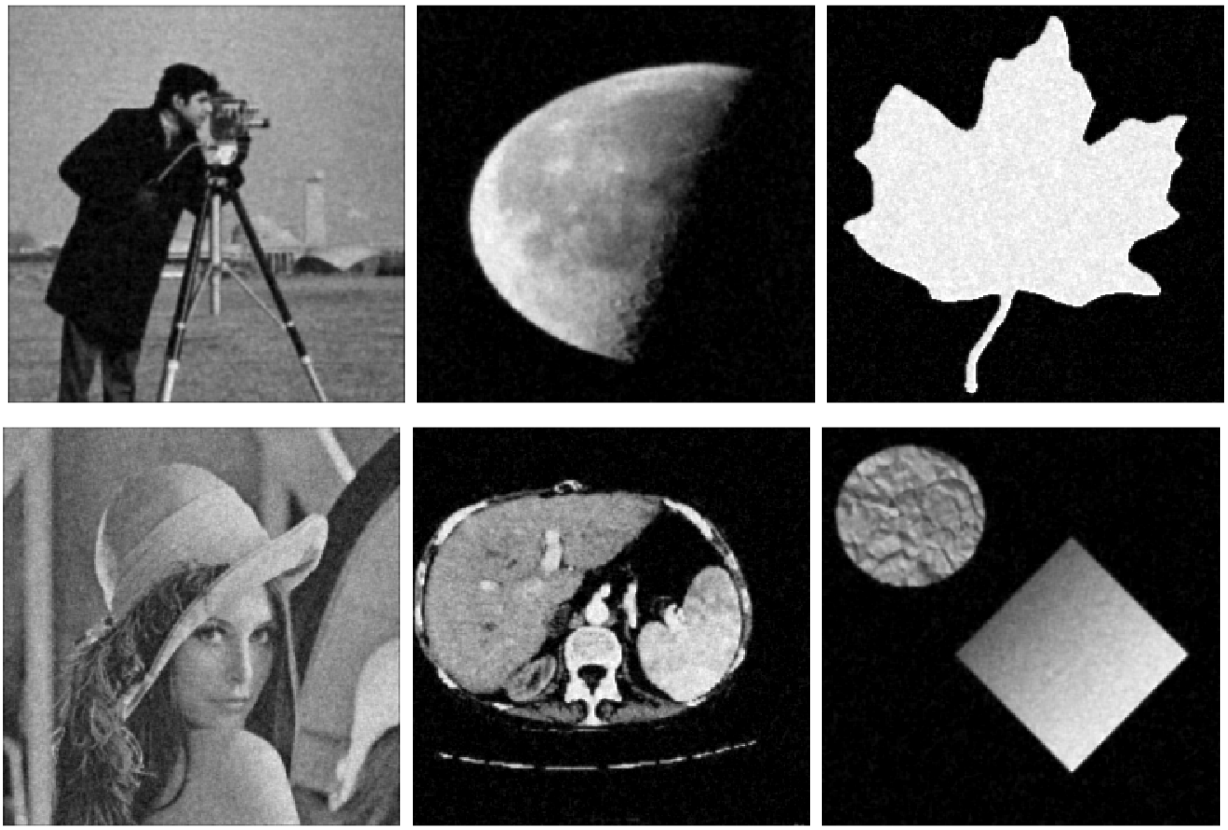


FIGURE 2. Restored image with the proposed algorithm.

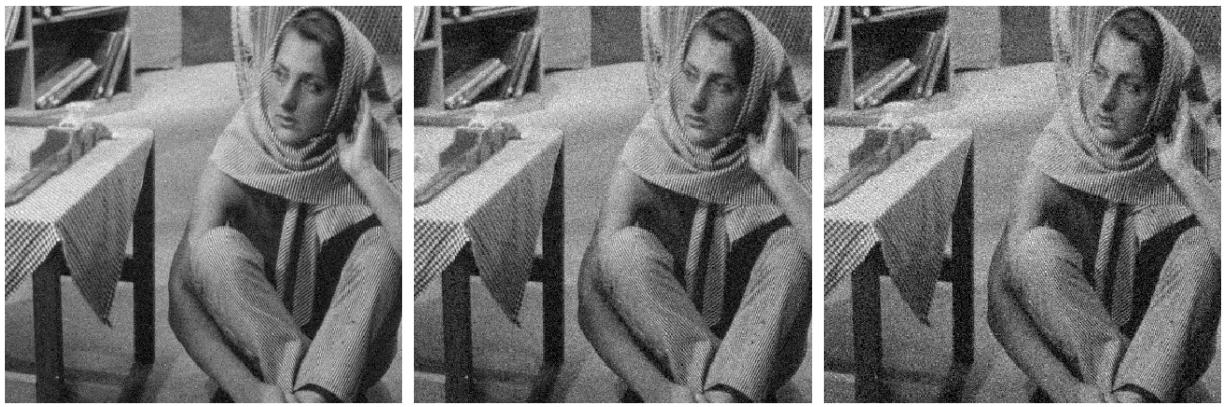
Images	Noise Level s^2	PSNR		SSIM	
		Noisy Image	Restored Image	Noisy Image	Restored Image
Moon	0.05	25.8796	**28.5515**	0.3779	**0.7649**
	0.07	23.1355	**27.7117**	0.2622	**0.6937**
	0.09	21.0815	**26.7833**	0.1941	**0.6174**
Trui	0.05	25.2058	**27.6016**	0.5393	**0.8492**
	0.07	22.4549	**25.8790**	0.4160	**0.7912**
	0.09	20.4365	**24.4669**	0.3299	**0.7385**
Scan	0.05	25.8155	**26.8509**	0.5305	**0.5995**
	0.07	23.0405	**24.5610**	0.4421	**0.5415**
	0.09	21.0138	**23.0591**	0.3827	**0.4930**

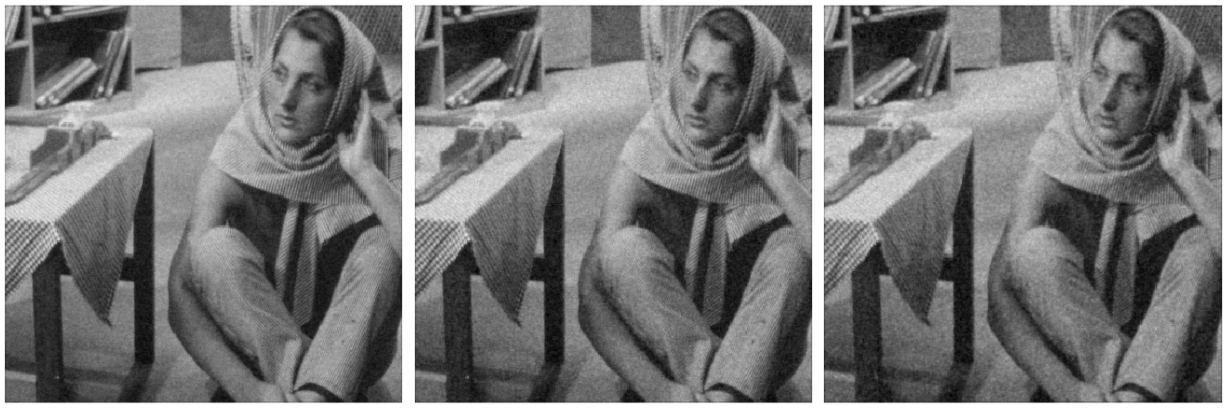
As a second test, we will use our algorithm to denoise images with different noise levels (see Figs.3 - 4).



$s^2 = 0.05$ $s^2 = 0.07$ $s^2 = 0.09$

FIGURE 3. First row: Noisy *Trui* image with noise at zero mean and different variances. Second row: Restored images using our proposed algorithm.





$s^2 = 0.05$ $s^2 = 0.07$ $s^2 = 0.09$

FIGURE 4. First row: Noisy Textured *Barbara* image with noise at zero mean and different variances. Second row: Restored images using our proposed algorithm.

In the next experiment (see Fig.5), we illustrate another experiment result of the proposed algorithm compared with the semi-implicit scheme-based algorithm , introduced by Mikula and Ramarosy in [17], by taking the same parameters $\Delta t = 0.01$, $\sigma = 0.5$, $\lambda = 0.01$

Table 2: The PSNR values of the denoising performance comparison using different values of Gaussian noise.

Images	Noise Level s^2	Denoising method	
		M.R. [17]	Our
Moon	0.03	32.5333	**33.4962**
	0.05	28.0666	**28.5515**
Trui	0.03	29.4413	**30.8295**
	0.05	26.9457	**27.6016**
Scan	0.03	27.3869	**30.7551**
	0.05	26.3807	**26.8509**

The numerical results obtained by our scheme are really encouraging and have higher PSNR values than those of Mikula and Ramarosy in [17], as shown in **Table 2**.



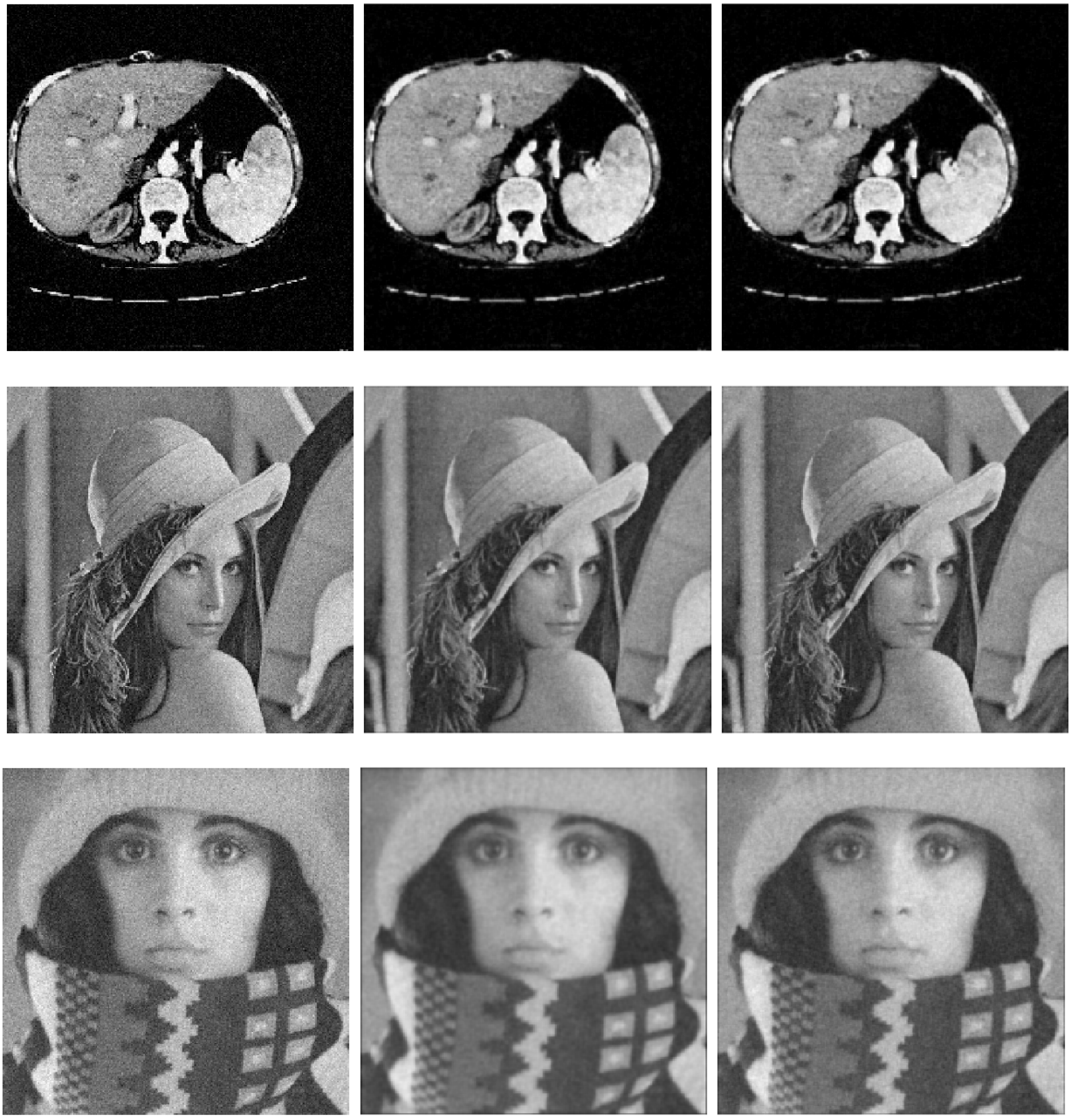


FIGURE 5. **Left column:** Top to Bottom: Noisy Moon image, Noisy Scan image, Noisy Lena image, and Noisy Trui image. **Middle column:** Restored images M.R. [17] and **Right column:** Restored images by our proposed model, where the noise level is $s^2 = 0.05$.

8. Conclusion

In this paper, we have introduced a finite volume approximation scheme for a reaction-diffusion system applied to image processing. We also proved the existence of its discrete

solution. Furthermore, we have proved that it converges to the weak solution. The main ideas of this proof are based on the a-priori estimates of L^2, and the Kolmogorov compactness theorem. Finally, in the numerical part, we observed that our experimental results show that the quality of the images obtained by the proposed model is quite efficient, especially when they are compared to the results of the modified (in the sense of Catté, Lions, Morel and Coll) Perona-Malik problem.

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Appendix 1. Let us assume that (u_1, w_1) and (u_2, w_2) are the any two weak solutions of the problem (1.9). Set $(u_1 - u_2, w_1 - w_2)$ as a test function in the definition of weak solution, then we have

$$\begin{aligned}\int_\Omega (u_1-u_2)_t(u_1-u_2)dx + \int_\Omega \left(g\left(|\nabla(u_1)_\sigma|\right)\nabla u_1 - g\left(|\nabla(u_2)_\sigma|\right)\nabla u_2\right)\\ \times\left(\nabla u_1 - \nabla u_2\right)dx + 2\lambda\int_\Omega \left(w_1-w_2\right)\left(u_1-u_2\right)dx = 0,\end{aligned} \tag{8.1}$$

$$\begin{aligned}\int_\Omega (w_1-w_2)_t(w_1-w_2)dx + \int_\Omega \left|\nabla\left(w_1-w_2\right)\right|^2 dx\\ + \int_\Omega \left(u_2-u_1\right)\left(w_1-w_2\right)dx = 0.\end{aligned} \tag{8.2}$$

Hence, adding $2\lambda \times$ (8.2) to (8.1), we get

$$\begin{aligned}\int_\Omega (u_1-u_2)_t(u_1-u_2)dx + \int_\Omega \left(g_1(t)\nabla u_1 - g_2(t)\nabla u_2\right)\left(\nabla u_1 - \nabla u_2\right)dx\\ +2\lambda\int_\Omega (w_1-w_2)_t(w_1-w_2)dx + 2\lambda\int_\Omega \left|\nabla\left(w_1-w_2\right)\right|^2 dxd = 0,\end{aligned} \tag{8.3}$$

where $g_1(t) = g\left(|\nabla(u_1(t))_\sigma|\right)$ and $g_2(t) = g\left(|\nabla(u_2(t))_\sigma|\right)$. Then, we have

$$\begin{aligned}\int_\Omega (u_1-u_2)_t(u_1-u_2)dx + \int_\Omega g_1(t)\nabla\left(u_1-u_2\right)\nabla\left(u_1-u_2\right)dx\\ +2\lambda\int_\Omega (w_1-w_2)_t(w_1-w_2)dx + 2\lambda\int_\Omega \left|\nabla\left(w_1-w_2\right)\right|^2 dxd\\ = \int_\Omega (g_1(t)-g_2(t))\nabla u_2\nabla\left(u_2-u_1\right)dx.\end{aligned} \tag{8.4}$$

Since $u_1 \in L^\infty\left(0,T;L^2(\Omega)\right)$ and g, G are C^∞, hence $g\left(|\nabla(u_1)_\sigma|\right) \in L^\infty\left(0,T;C^\infty(\Omega)\right)$. Thus, since g is decreasing, there exists a constant $\mu = \mu(g, G, \|u_0\|_{L^2(\Omega)}) > 0$ such that

$$g\left(|\nabla G_\sigma * u_1|\right) \geq \mu > 0 \quad a.e.in \quad \Omega\times(0,T).$$

Then, by using Hölder's inequality, we find

$$\begin{aligned}\int_\Omega (u_1-u_2)_t(u_1-u_2)dx + \mu\int_\Omega \left|\nabla\left(u_1-u_2\right)\right|^2 dx\\ +2\lambda\int_\Omega (w_1-w_2)_t(w_1-w_2)dx + 2\lambda\int_\Omega \left|\nabla\left(w_1-w_2\right)\right|^2 dx\\ = \int_\Omega (g_1(t)-g_2(t))\nabla u_2\nabla\left(u_2-u_1\right)dx.\end{aligned} \tag{8.5}$$

Then, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_1-u_2\|^2_{L^2(\Omega)} + \mu\|\nabla\left(u_1-u_2\right)\|^2_{L^2(\Omega)} + \lambda\frac{d}{dt}\|w_1-w_2\|^2_{L^2(\Omega)}\\ + 2\lambda\|\nabla\left(w_1-w_2\right)\|^2_{L^2(\Omega)} \leq \int_\Omega (g_1(t)-g_2(t))\nabla u_2\nabla\left(u_2-u_1\right)dx.\end{aligned} \tag{8.6}$$

Hence, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_1-u_2\|^2_{L^2(\Omega)}+\mu\|\nabla(u_1-u_2)\|^2_{L^2(\Omega)}\\&\leq\|g_1(t)-g_2(t)\|_{L^\infty(\Omega)}\|\nabla u_2(t)\|_{L^2(\Omega)}\|\nabla u_1(t)-\nabla u_2(t)\|_{L^2(\Omega)}.\end{aligned}\tag{8.7}$$

Moreover, since g and G are C^∞, we have

$$\|g_1(t)-g_2(t)\|_{L^\infty(\Omega)}\leq C\|u_1(t)-u_2(t)\|_{L^2(\Omega)},\tag{8.8}$$

where $C=C(g,G,u_0)$ is a positive constant. By using (8.7) and (8.8), we can write

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_1-u_2\|^2_{L^2(\Omega)}+\mu\|\nabla(u_1-u_2)\|^2_{L^2(\Omega)}\\&\quad\leq\frac{C}{2\mu}\|u_1(t)-u_2(t)\|^2_{L^2(\Omega)}\|\nabla u_2(t)\|^2_{L^2(\Omega)}+\frac{\mu}{2}\|\nabla u_1(t)-\nabla u_2(t)\|^2_{L^2(\Omega)}\end{aligned}$$

and so

$$\frac{d}{dt}\|u_1(t)-u_2(t)\|^2_{L^2(\Omega)}\leq\frac{C}{\mu}\|u_1(t)-u_2(t)\|^2_{L^2(\Omega)}\|\nabla u_2(t)\|^2_{L^2(\Omega)}.\tag{8.9}$$

Since $u_1(0)=u_2(0)=u_0$, by using (8.9) and Gronwall's inequality [3]. Then $u_1(x,t)=u_2(x,t)$ a.e. in Ω_T. On the other hand, by (8.6), we have

$$\frac{d}{dt}\|w_1-w_2\|^2_{L^2(\Omega)}+2\|\nabla(w_1-w_2)\|^2_{L^2(\Omega)}\leq 0.\tag{8.10}$$

Since $w_1(0)=w_2(0)=0$, then $w_1(x,t)=w_2(x,t)$ a.e. in Ω_T. This completes the proof.