sciendo

CrossMark

Moroccan J. of Pure and Appl. Anal. (MJPAA)
Volume 8(3), 2022, Pages 364–374
ISSN: Online 2351-8227 - Print 2605-6364
DOI: 10.2478/mjpaa-2022-0025

A γ-power stochastic Lundqvist-Korf diffusion process: Computational aspects and simulation

El Azri Abdenbi[1], Nafidi Ahmed[2]

Abstract. In this paper, we introduce a new family of stochastic Lundqvist-Korf diffusion process, defined from a γ-power of the Lundqvist-Korf diffusion process. First, we determine the probabilistic characteristics of the process, such as its analytic expression, the transition probability density function from the corresponding Itô stochastic differential equation and obtain the conditional and non-conditional mean functions. We then study the statistical inference in this process. The parameters of this process are estimated by using the maximum likelihood estimation method with discrete sampling, thus we obtain a nonlinear equation, which is achieved via the simulated annealing algorithm. Finally, the results of the paper are applied to simulated data.

Mathematics Subject Classification (2020). 62M86, 62M20, 60H30, 60H35, 65C30.
Keywords and phrases. Stochastic Lundqvist-Korf diffusion process, Maximum likelihood estimation, Simulated annealing method, Statistical inference in diffusion process, Simulation.

1. Introduction

Stochastic diffusion processes related to deterministic models is of great interest to investigators in many disciplines, such as biological phenomena, energy consumption, economy

Received August 13, 2022 - Accepted: September 24, 2022.
© The Author(s) 2022. This article is published with open access by Sidi Mohamed Ben Abdallah University.

[1]*University Hassan 1st of Settat, National School of Applied Sciences, Department of Mathematics and Computer Science, Laboratory of Systems Modelization and Analysis for Decision Support, B.P. 218, 26103, Berrechid, Morocco.*
e-mail: a.elazri@uhp.ac.ma (Corresponding Author).

[2]*University Hassan 1st of Settat, National School of Applied Sciences, Department of Mathematics and Computer Science, Laboratory of Systems Modelization and Analysis for Decision Support, B.P. 218, 26103, Berrechid, Morocco.*
e-mail: ahmed.nafidi@uhp.ac.ma .

and finance, and environmental phenomena, see, Gutiérrez et al. [2, 3], Román-Román and Torres-Ruiz [11], and Nafidi et al. [7], the process is generally defined by means of stochastic differential equations. Ramos-Ábalos et al. [9] introduce a new family of Gompertz processes defined by the power of the homogeneous Gompertz diffusion model. In this stage, we also find the work of Ramos-Ábalos et al. [8] who modeled the natural gas pattern in Spain by a new stochastic gamma diffusion process of power γ.

The Lundqvist-Korf model has been treated in several works, such as the work of Sghaier et al. [13] who introduced a model of six generalized algebraic difference equations derived from basic log-logistic models, Bertalanffy-Richards and Lundqvist-Korf were used to develop a site index model for Pinus pinea plantations in northwestern Tunisia and concluded a generalized algebraic difference equation derived from the base model of Lundqvist-Korf realized the best compromise between biological and statistical constraints, producing the most adequate site index curves. Moreover the work of Amaro et al. [1] which proposed the different forms of the Richards and Lundqvist-Korf growth equations and the work of Sánchez-González et al. [12] which treated and adjusted using the generalized least squares regression method the differential forms of the growth functions of Lundqvist-Korf, Richards and McDill-Amateis.

The question of estimating parameters of the drift coefficient has received considerable attention of researchers in recent years, both when the process is observed continuously and when it is discrete. Various authors address this question, and many papers have been published on this subject, focusing on the simulated annealing (SA) algorithm for the problem of maximum likelihood (ML) estimation, such as Nafidi and El Azri [6], Nafidi et al. [7] and Román-Román et al. [10].

In the present study, we define and examine a new extension of the stochastic Lundqvist-Korf diffusion process. The rest of this study is organized as follows: The second section, presents an overview of the stochastic Lundqvist-Korf diffusion process and we define the γ-power of the stochastic Lundqvist-Korf diffusion process (γ-SLKDP) by the powers of the stochastic Lundqvist-Korf diffusion process. Them we define the proposed model as the solution to a stochastic differential equation (SDE). From this, we get the explicit expression of the process, the probability transition density function (ptdf), the moments of different orders and, in particular, the conditional and unconditional mean functions (MFs) of the process. The third section, deals with parameter estimation by the ML method, using discrete sampling. In this situation, ML estimators cannot be given in closed form because the system of likelihood equations does not have an exact solution. Therefore, numerical methods are required. The SA is proposed, either continued until convergence or interrupted after reaching a given stopping criterion, to compute or approximate the resulting ML estimator. The fourth section, presents the results of the simulation cases. Finally, we summarise the main conclusions.

2. The proposed model

2.1. An overview of the stochastic Lundqvist-Korf model. Let $\{X(t);\, t \in [t_1; T]\,;\, t_1 > 0\}$ be a diffusion process with values in $(0; \infty)$, where $X(t)$ is a stochastic Lundqvist-Korf diffusion process with parameters α, β and σ^2 (see, Nafidi and El Azri [6]) satisfies the following Itô's

SDE:

$$dX(t) = \frac{\alpha\beta}{t^{\alpha+1}} X(t)\, dt + \sigma X(t)\, dw(t); \quad X(t_1) = x_1 \tag{2.1}$$

with the initial condition $P(X(t_1) = x_1) = 1$ where x_1 is a positive real value and $w(t)$ is a standard Wiener process. The parameters α, β and σ^2 are all positive. The SDE (2.1) has a unique solution (see Kloeden and Platen [5]) is given by

$$X(t) = X(t_1) \exp\left[-\beta\left(\frac{1}{t^\alpha} - \frac{1}{t_1^\alpha}\right) - \frac{\sigma^2}{2}(t - t_1) + \sigma(w(t) - w(t_1))\right].$$

From this, we deduce that the process $(X(t) \mid X(s) = x_s)$ is distributed as the following one-dimensional lognormal distribution $\Lambda_1\left[\mu(s,t,x_s); \sigma^2(t-s)\right]$ with

$$\mu(s,t,x_s) = \log(x_s) - \beta\left(\frac{1}{t^\alpha} - \frac{1}{s^\alpha}\right) - \frac{\sigma^2}{2}(t-s).$$

2.2. The γ-SLKDP and its analytic expression. Let $\{X(t);\, t \in [t_1; T];\, t_1 > 0\}$ be a stochastic Lundqvist-Korf diffusion process. Then, the γ-SLKDP is defined by

$$X_\gamma(t) = X^\gamma(t); \qquad \gamma \in \mathbb{R}^* \tag{2.2}$$

The process $\{X_\gamma(t);\, t \in [t_1; T];\, t_1 > 0\}$ is also a diffusion process with values in $(0; \infty)$, and has the drift and diffusion coefficients are shown below.

By applying Itô's formula to the transform given in Eq. (2.2), we have

$$dX_\gamma(t) = \left(\frac{\gamma\alpha\beta}{t^{\alpha+1}} + \frac{\gamma(\gamma-1)\sigma^2}{2}\right) X_\gamma(t)\, dt + \gamma\sigma X_\gamma(t)\, dw(t); \tag{2.3}$$

The last SDE (2.3) has a unique solution (see, Kloeden and Platen [5]) can be obtained using the formula $X_\gamma(t) = X^\gamma(t)$ and Eq. (2.3), from which we have

$$X_\gamma(t) = X_\gamma(t_1) \exp\left[-\gamma\beta\left(\frac{1}{t^\alpha} - \frac{1}{t_1^\alpha}\right) - \frac{\gamma\sigma^2}{2}(t - t_1) + \gamma\sigma(w(t) - w(t_1))\right]. \tag{2.4}$$

According to this, we derive that the process $(X_\gamma(t) \mid X_\gamma(s) = x_s)$ is distributed as the one-dimensional lognormal distribution $\Lambda_1\left[m(s,t,x_s); \gamma^2\sigma^2(t-s)\right]$ with

$$m(s,t,x_s) = \log(x_s) - \gamma\beta\left(\frac{1}{t^\alpha} - \frac{1}{s^\alpha}\right) - \frac{\gamma\sigma^2}{2}(t-s).$$

2.3. The ptdf of the γ-SLKDP. When $(X_\gamma(t) \mid X_\gamma(s) = x_s)$ has the log-normal distribution $\Lambda_1\left[m(s,t,x_s); \gamma^2\sigma^2(t-s)\right]$, then the ptdf of the γ-SLKDP is

$$f(x,t \mid y,s) = \frac{1}{x\sqrt{2\pi\gamma^2\sigma^2(t-s)}} \exp\left(-\frac{[\log(x) - m(s,t,y)]^2}{2\gamma^2\sigma^2(t-s)}\right). \tag{2.5}$$

2.4. The MFs of the γ-SLKDP. From the properties of the log-normal distribution, the rth conditional moment of the γ-SLKDP is

$$\begin{aligned} E\left(X_\gamma^r(t) \mid X_\gamma(s)=x_s\right) &= \exp\left(rm(s,t,x_s)+\frac{r^2\gamma^2\sigma^2}{2}(t-s)\right) \\ &= x_s^r \exp\left(-r\gamma\beta\left(\frac{1}{t^\alpha}-\frac{1}{s^\alpha}\right)+\frac{r\gamma(\gamma r-1)\sigma^2}{2}(t-s)\right). \end{aligned}$$

Then, for $r=1$, the conditional mean function (CMF) of the process is

$$E\left(X_\gamma(t) \mid X_\gamma(s)=x_s\right) = x_s \exp\left(-\gamma\beta\left(\frac{1}{t^\alpha}-\frac{1}{s^\alpha}\right)+\frac{\gamma(\gamma-1)\sigma^2}{2}(t-s)\right).$$

In addition, the MF (mean function) of the γ-SLKDP is given by the initial condition $P\left(X_\gamma(t_1)=x_1\right)=1$ and has the form

$$E\left(X_\gamma(t)\right) = x_1 \exp\left(-\gamma\beta\left(\frac{1}{t^\alpha}-\frac{1}{t_1^\alpha}\right)+\frac{\gamma(\gamma-1)\sigma^2}{2}(t-t_1)\right).$$

Remark. If $\gamma=1$, note that $E\left(X_\gamma(t) \mid X_\gamma(s)=x_s\right) = x_s \exp\left(-\beta\left(\frac{1}{t^\alpha}-\frac{1}{s^\alpha}\right)\right)$ is the conditional mean function of the stochastic Lundqvist-Korf diffusion process.

3. Statistical inference on the model

3.1. ML parameter estimation. Let us then consider in this section the ML estimation of the parameters α, β and σ^2 of the model from which we can obtain, by virtue of Zehna's theorem [14], we obtain the estimated mean functions of the process.

We consider a discrete sampling of the process, that is, for fixed times $t_1, t_2, ..., t_n$, where $n>2$, with the initial condition $P\left[X_\gamma(t_1)=x_1\right]=1$, we observe the variables $x(t_1)$, $x(t_2)$, $\cdots$, $x(t_n)$ whose values provide the basic sample for the inference process. In addition, we assume $t_i - t_{i-1} = h$, for $i=2,\cdots,n$. Let x_1, x_2, $\cdots$, x_n be the observed values of the sampling. The associated likelihood function is thus

$$\mathbf{L} = L_{x_1,x_2,\ldots,x_n}\left(\alpha,\beta,\sigma^2\right) = \prod_{j=2}^{n} f\left(x_j,t_j \mid x_{j-1},t_{j-1}\right),$$

which is of the form

$$\mathbf{L} = \prod_{j=2}^{n} \frac{1}{x_j\sqrt{2\pi\gamma^2\sigma^2 h}} \exp\left(-\frac{\left[\log\left(\frac{x_j}{x_{j-1}}\right)+\gamma\beta\left(\frac{1}{t_j^\alpha}-\frac{1}{t_{j-1}^\alpha}\right)+\frac{\gamma\sigma^2}{2}h\right]^2}{2\gamma^2\sigma^2 h}\right). \tag{3.1}$$

From Eq. (3.1), the log-likelihood function of the sample is

$$\log(\mathbf{L}) = -\frac{(n-1)}{2}\log\left(2\pi\gamma^2 h\right) - \frac{(n-1)}{2}\log\left(\sigma^2\right) - \sum_{i=2}^{n}\log(x_j) - \frac{1}{2\gamma^2\sigma^2 h}\sum_{j=2}^{n}\left\{\log\left(\frac{x_j}{x_{j-1}}\right) + \gamma\beta\left(\frac{1}{t_j^{\alpha}} - \frac{1}{t_{j-1}^{\alpha}}\right) + \frac{\gamma\sigma^2}{2}h\right\}^2. \tag{3.2}$$

However, estimating α, β and σ^2 poses some difficulties. In fact, the resulting system of equations is exceedingly complex and does not have an explicit solution, and numerical procedures must be employed.

3.2. Computational aspects. In this section we propose the use of the SA algorithm to maximize the likelihood function or, equivalently, its logarithm. Hereafter is the description of the method.

SA algorithm is a metaheuristic algorithm to approximating the solution of optimization problems of the type $\min_{x\in\theta} g(x)$, where g is the objective function to be optimized and θ is the solution space. It was developed by Kirkpatrick et al. [4]. Under this algorithm, in each iteration, let x be a current solution, x' be a new value selected in a neighborhood of x in the next iteration, and the objective difference is $\delta = g(x') - g(x)$. If $\delta \leq 0$, then x' is selected as the new solution. Otherwise, it could be accepted with probability $p = \exp(-\delta/T)$, where T is a scale factor called temperature.

In our case, the problem becomes maximizing the log-likelihood function $\log(\mathbf{L})$ Since the algorithm aforementioned is usually formulated for minimization problems, then, from Eq. (3.2), the objective function is a function of the parameters α, β and σ^2 and has the form:

$$g\left(\alpha,\beta,\sigma^2\right) = \frac{(n-1)}{2}\log\left(\sigma^2\right) + \frac{1}{2\gamma^2\sigma^2 h}\sum_{j=2}^{n}\left\{\log\left(\frac{x_j}{x_{j-1}}\right) + \gamma\beta\left(\frac{1}{t_j^{\alpha}} - \frac{1}{t_{j-1}^{\alpha}}\right) + \frac{\gamma\sigma^2}{2}h\right\}^2. \tag{3.3}$$

The parametric space θ related to the objective function (3.3), on which the selected algorithm must operate, is continuous and unbounded. Therefore,

$$\theta = \{(\alpha,\beta,\sigma) : \alpha > 0,\ \beta > 0,\ \sigma > 0\}.$$

The disadvantage is that the solution space may not be explored in sufficient depth. This forces us to find some arguments to bound this space. For this we use the strategies developed by Nafidi and El Azri [6]. Thus, for a known γ, the parametric space, which is obtained numerically for $a = \frac{1}{\alpha}$, $b = \exp(-\beta)$ and σ is bounded and takes the form

$$\theta' = \left(0,\ \frac{\kappa}{\gamma}\right) \times \left(0,\ \exp\left(-t_1^{\frac{\gamma}{\kappa}}\right)\right) \times (0,\ 0.1),$$

where $\kappa = \log\left(\frac{k_{\max}}{x_1^{\gamma}}\right) - 1$ and k_{max} is the maximum value of the sample path.

4. Simulation study

4.1. γ-SLKDP simulated sample paths. This section presents some examples of simulated trajectories of the γ-SLKDP model. We obtain the trajectory of the model by simulating the exact solution of the DHS (2.4). We get the simulated trajectories of the model by considering the time discretization of the interval $[t_1, T]$, with time points $t_i = t_{i-1} + (i-1)h$, $i = 2, .., N$ and step size $h = \dfrac{T - t_1}{N}$ for the sample size N.

The random variable $\gamma\sigma\left(w(t) - w(t_1)\right)$ in the Eq. (2.4) is distributed as one-dimensional normal distribution $\mathrm{N}_1\left(0, \gamma^2\sigma^2(t - t_1)\right)$. This simulation for $x_\gamma(t)$ ($\gamma = 1$; $\gamma = 1.5$ and $\gamma = 2$) is based on 20 sample paths with $t_1 = 1$, $T = 40$, $x_1 = 2$, and $N = 50$.

Figures 1, 2 and 3 shows some simulated sample paths of the γ-SLKDP for ($\gamma = 1$; $\gamma = 1.5$ and $\gamma = 2$) and its mean function for several values of σ with $\alpha = 0.5$ and $\beta = 4$.

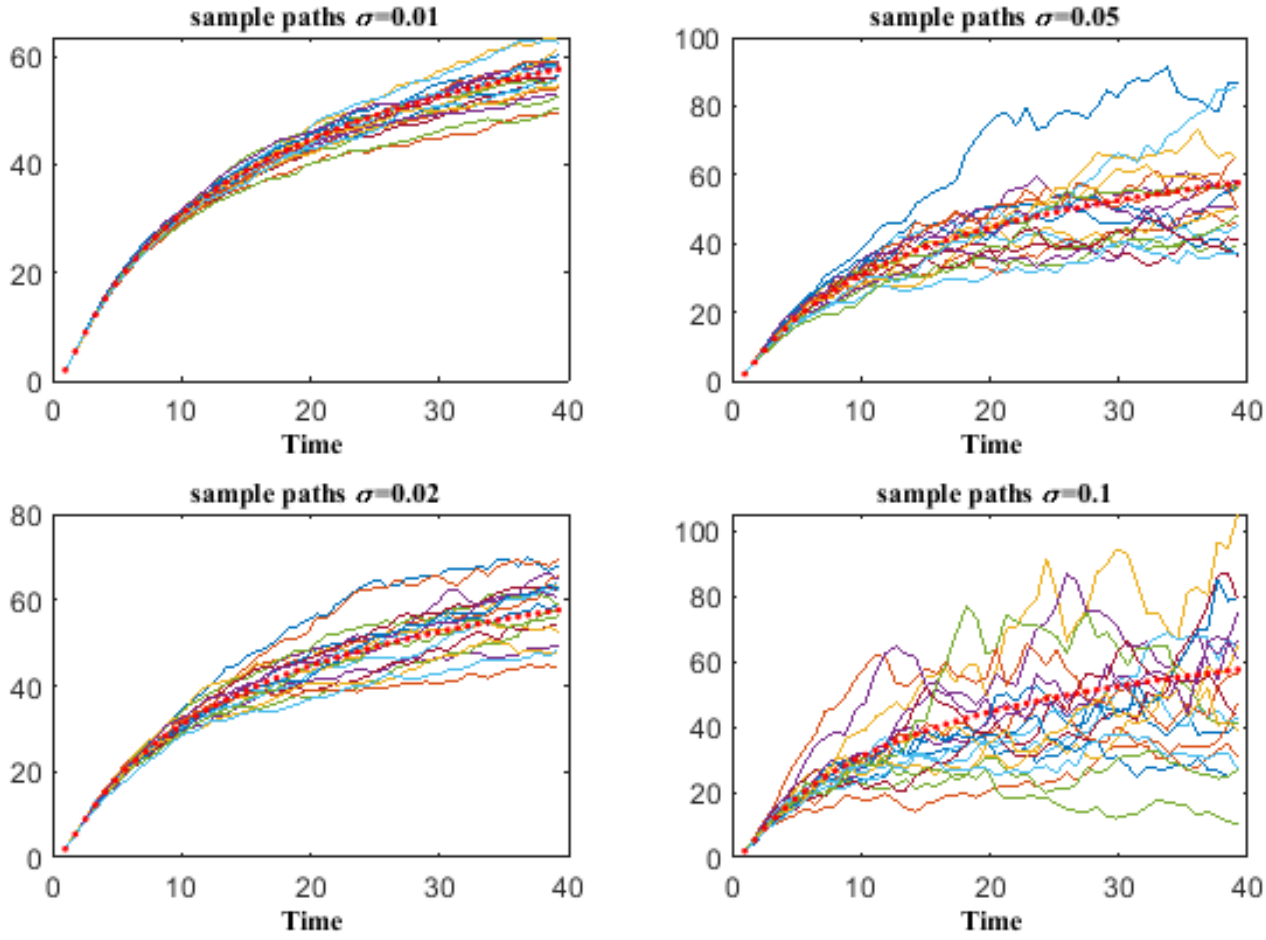


FIGURE 1. Simulated trajectories of the γ-SLKDP and its MF for several values of σ with $\alpha = 0.5$, $\beta = 4$ and $\gamma = 1$.

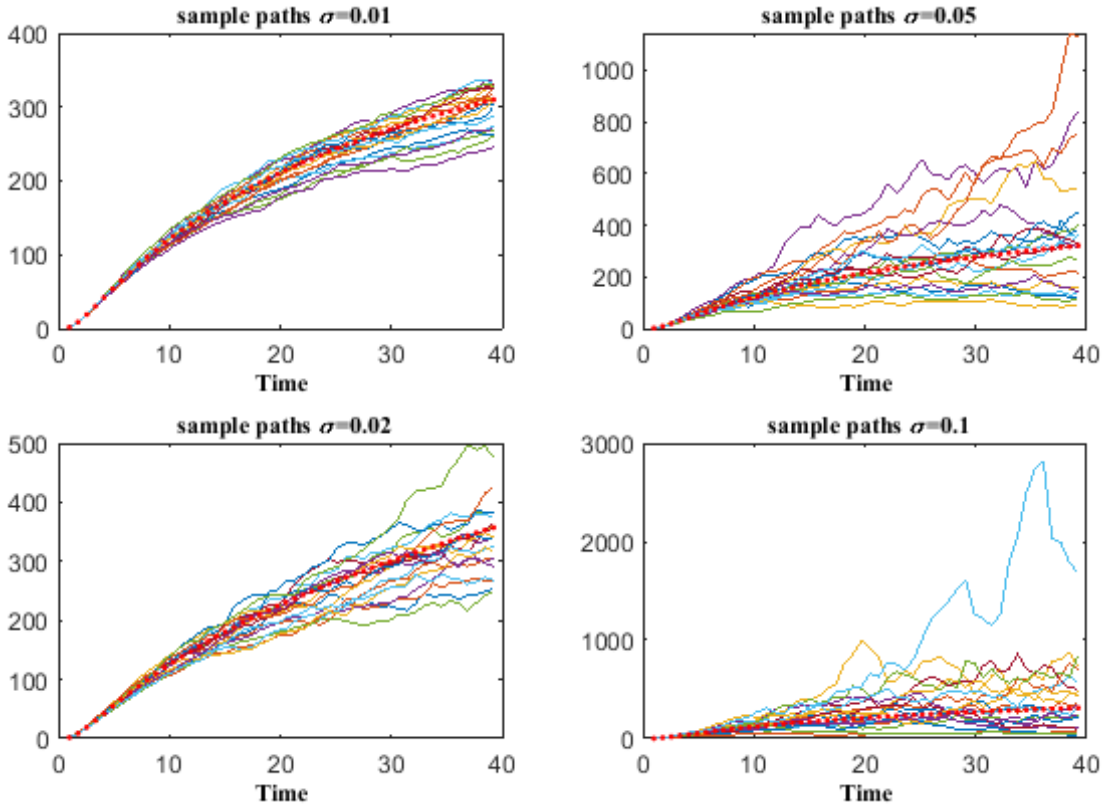


FIGURE 2. Simulated trajectories of the γ-SLKDP and its MF for several values of σ with $\alpha = 0.5$, $\beta = 4$ and $\gamma = 1.5$.

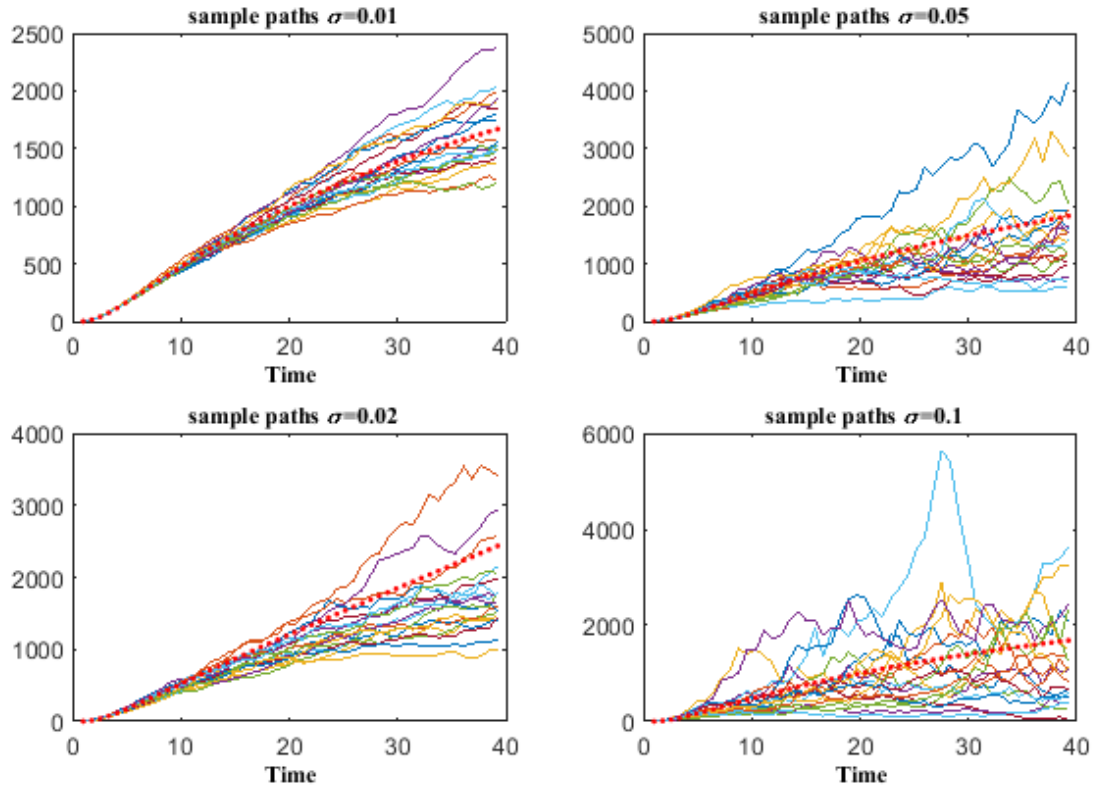


FIGURE 3. Simulated trajectories of the γ-SLKDP and its MF for several values of σ with $\alpha = 0.5$, $\beta = 4$ and $\gamma = 2$.

4.2. Parameters estimation. This section shows various examples in order to validate the estimation procedure developed above. For this purpose, the equation (2.4) was simulated 25 times under the following hypotheses, for each one, we consider the equidistant time discretization of $[t_1, T]$ with time points $t_i = t_{i-1} + (i-1)h$; for $i = 2, ..., N$ where $T = 50$ and step size $h = \dfrac{T - t_1}{N}$, starting from instant $t_1 = 0.1$ and $x_1 = 2$. As for the sample size, values 100, 250 and 500 have been considered for N.

The SA algorithm has been applied for estimating the parameters of the process.

The results obtained are summarised in Tables 2 and 3 which shows for each of the above data sets the empirical mean, the standard deviation (std), and the coefficient of variation (CV) obtained for $a = \dfrac{1}{\alpha}$, $b = \exp(-\beta)$ and σ given by Table 1.

The results obtained show the performance as well as the usefulness and importance of the methodology studied.

TABLE 1. The empirical mean, the std and the CV for a, b and σ (M is the sample paths)

Empirical mean	std	CV
$\overline{a} = \frac{1}{M}\sum_{i=1}^{M} a_i$	$\mathrm{std}(a) = \left(\frac{1}{M-1}\sum_{i=1}^{M}(a_i - \overline{a})^2\right)^{\frac{1}{2}}$	$\mathrm{CV}(a) = \frac{\mathrm{std}(a)}{\overline{a}}$
$\overline{b} = \frac{1}{M}\sum_{i=1}^{M} b_i$	$\mathrm{std}(b) = \left(\frac{1}{M-1}\sum_{i=1}^{M}\left(b_i - \overline{b}\right)^2\right)^{\frac{1}{2}}$	$\mathrm{CV}(b) = \frac{\mathrm{std}(b)}{\overline{b}}$
$\overline{\sigma} = \frac{1}{M}\sum_{i=1}^{M} \sigma_i$	$\mathrm{std}(\sigma) = \left(\frac{1}{M-1}\sum_{i=1}^{M}(\sigma_i - \overline{\sigma})^2\right)^{\frac{1}{2}}$	$\mathrm{CV}(\sigma) = \frac{\mathrm{std}(\sigma)}{\overline{\sigma}}$

5. Conclusions

This paper introduces a study of the γ-stochastic Lundqvist-Korf diffusion process, including its probabilistic features and the related statistical inference.

This paper presents a study of the γ-stochastic Lundqvist-Korf diffusion process, including all its probabilistic features and the corresponding statistical inference.

A simulation study was performed, analyzing different process trajectories.

The inferential study is performed on the basis of discrete sampling via ML method. Since a complex system of equation appeared, which cannot be solved via classical numerical procedures, we suggest the use of metaheuristic optimization algorithms such as simulated annealing algorithm.

As an interesting line of study for further research, it will be possible to apply this model to fit real data and to obtain goodness of fit results between the process and the data.

TABLE 2. Estimation values, the std and the CV of a, b and σ for several values of γ ($a = 2, b = 0.02, \sigma = 0.01$).

$\gamma = 1$									
N	$\overline{a}$	$\overline{b}$	$\overline{\sigma}$	std$(a).10^{-4}$	std$(b).10^{-4}$	std$(\sigma).10^{-4}$	CV$(a).10^{-4}$	CV$(b).10^{-4}$	CV$(\sigma).10^{-4}$
100	1.98886	0.02082	0.00987	72.79	5.194	5.757	36.60	249.4	583.2
250	1.99025	0.02070	0.01007	60.13	4.500	5.283	30.21	217.4	524.3
500	1.98826	0.02083	0.01001	28.36	1.964	2.479	14.27	94.29	247.7
$\gamma = 1.5$									
N	$\overline{a}$	$\overline{b}$	$\overline{\sigma}$	std$(a).10^{-4}$	std$(b).10^{-4}$	std$(\sigma).10^{-4}$	CV$(a).10^{-4}$	CV$(b).10^{-4}$	CV$(\sigma).10^{-4}$
100	2.01439	0.01964	0.01187	733.8	51.04	32.54	364.3	2599	2742
250	2.04887	0.01713	0.01141	592.7	34.11	24.50	289.3	1992	2147
500	2.00738	0.02031	0.01129	812.0	64.45	14.16	404.5	3173	1254
$\gamma = 2$									
N	$\overline{a}$	$\overline{b}$	$\overline{\sigma}$	std$(a).10^{-4}$	std$(b).10^{-4}$	std$(\sigma).10^{-4}$	CV$(a).10^{-4}$	CV$(b).10^{-4}$	CV$(\sigma).10^{-4}$
100	1.97972	0.02169	0.01108	503.4	37.79	17.57	254.3	1742	1586
250	1.99655	0.02082	0.01033	372.1	32.47	8.482	186.4	1559	821.0
500	1.99498	0.02059	0.01027	411.2	34.31	8.762	206.1	1665	853.0

TABLE 3. Estimation values, the std and the CV of a, b and σ for several values of γ ($a = 0.02, b = 0.7, \sigma = 0.02$).

N	$\overline{a}$	$\overline{b}$	$\overline{\sigma}$	std$(a).10^{-4}$	std$(b).10^{-4}$	std$(\sigma).10^{-4}$	CV$(a).10^{-4}$	CV$(b).10^{-4}$	CV$(\sigma).10^{-4}$
$\gamma = 1$									
100	2.00558	0.01961	0.01991	38.12	2.861	14.05	19.01	145.9	706.1
250	2.00529	0.01962	0.01994	24.94	2.413	9.925	12.44	122.9	497.8
500	2.00402	0.01969	0.01986	38.97	2.904	7.244	19.44	147.5	364.7
$\gamma = 1.5$									
N	$\overline{a}$	$\overline{b}$	$\overline{\sigma}$	std$(a).10^{-4}$	std$(b).10^{-4}$	std$(\sigma).10^{-4}$	CV$(a).10^{-4}$	CV$(b).10^{-4}$	CV$(\sigma).10^{-4}$
100	1.99800	0.02039	0.02005	466.1	38.64	16.65	233.3	1894	830.3
250	2.00177	0.01996	0.01978	308.2	22.88	10.02	153.9	1147	506.2
500	1.98623	0.02128	0.02001	429.1	35.48	8.020	216.1	1667	400.6
$\gamma = 2$									
N	$\overline{a}$	$\overline{b}$	$\overline{\sigma}$	std$(a).10^{-4}$	std$(b).10^{-4}$	std$(\sigma).10^{-4}$	CV$(a).10^{-4}$	CV$(b).10^{-4}$	CV$(\sigma).10^{-4}$
100	1.96747	0.02273	0.02071	619.0	50.83	18.01	314.6	2236	869.7
250	1.99415	0.02073	0.02023	462.6	39.00	14.69	231.9	1881	726.2
500	1.99418	0.02075	0.02019	487.2	43.67	9.208	244.3	2104	455.9

Acknowledgments

The authors thank the anonymous referee for their very careful reading and valuable comments.

Conflicts of Interest

The authors have no relevant financial or non-financial interests to disclose.

References

[1] A. Amaro, D. Reed, M. Tomé, I. Themido, Modelling dominant height growth: Eucalyptus plantations in Portugal, For. Sci., 44 (1998), 37-46.
[2] R. Gutiérrez, A. Nafidi, R. Gutiérrez-Sánchez, Forecasting total natural-gas consumption in Spain by using the stochastic Gompertz innovation diffusion model, Applied Energy, 80(2) (2005), 115-124.
[3] R. Gutiérrez, P. Román, F. Torres, Inference and first-passagetime for the lognormal diffusion process with exogenous factors: application to modelling in economics, Appl. Stoch. Models Bus. Ind., 15(4) (1999), 325-332.
[4] S. Kirkpatrick, C.D. Gelatt, M.P. Vecchi, Optimization by simulated annealing, Science, 220(4598) (1983), 671-680.
[5] P. Kloeden, E. Platen, The numerical solution of stochastic differential equations, Springer, Berlin, Germany, 1992.
[6] A. Nafidi, A. El Azri, A stochastic diffusion process based on the Lundqvist-Korf growth: Computational aspects and simulation, Mathematics and Computers in Simulation, 182 (2021), 25–38.
[7] A. Nafidi, A. El Azri, R. Gutiérrez-Sánchez, The Stochastic Modified Lundqvist-Korf Diffusion Process: Statistical and Computational Aspects and Application to Modeling of the CO_2 Emission in Morocco, Stochastic Environmental Research and Risk Assessment, 36 (2022), 1163–1176.
[8] Ramón Gutiérrez Jáimez, Ramón Gutiérrez-Sánchez, Ahmed Nafidi, E. Ramos-Ábalos, A τ-power stochastic gamma diffusion process: Computational statistical inference and simulation aspects. A real example, Appl. Math. Comput., 219 (2012), 1576–1588.
[9] E.M. Ramos-Ábalos, R. Gutirrez-Snchez, A. Nafidi, Powers of the Stochastic Gompertz and Lognormal Diffusion Processes, Statistical Inference and Simulation, Mathematics, 8(4) (2020), 588–588.
[10] P. Román-Román, D. Romero, M.A. Rubio, F. Torres-Ruiz, Estimating the parameters of a Gompertz-type diffusion process by means of Simulated Annealing, Applied Mathematics and Computation, 218 (2012), 5121–5131.
[11] P. Román-Román, F. Torres-Ruiz, Modelling logistic growth by a new diffusion process:application to biological systems, BioSystems, 110(1) (2012), 9-21.
[12] M. Sánchez-González, M. Tomé, G. Montero, Modelling height and diameter growth of dominant cork oak trees in Spain, Ann. For. Sci., 62 (2005), 633-643.
[13] T. Sghaier, M. Palahi, S. Garchi, B. José Antonio, Ammari Youssef, Pique Miriam, Modeling Dominant Height Growth in Planted Pinus pinea Stands in Northwest of Tunisia, Hindawi Publishing Corporation, International Journal of Forestry Research, (2012), 1–12.
[14] P.W. Zehna, Invariance of maximum likelihood estimators, Ann. Math. stat., 37 (1996), 744-744.