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Existence and Regularity for Solution to a Degenerate Problem with Singular Gradient Lower Order Term

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Abstract. We study the existence and regularity results for non-linear elliptic equation with degenerate coercivity and a singular gradient lower order term. The model problems is

$$\begin{cases} -\text{div}\left(b(x)\frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\gamma}}\right)+\frac{|\nabla u|^{p}}{|u|^{\theta}}=f, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where Ω is a bounded open subset in $\mathbb{R}^N$, $1 \leq \theta < 2$, $p > 2$ and $\gamma > 0$. We will show that, even if the lower order term is singular, we obtain existence and regularity of positive solution, under various assumptions on the summability of the source f.

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1. Introduction

In this paper, we deal with the following problem

$$\begin{cases} -\text{div}\left(b(x)\frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\gamma}}\right)+B\frac{|\nabla u|^{p}}{|u|^{\theta}}=f, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is an open bounded set of $\mathbb{R}^N$ ($N \geq 3$), $B, \gamma > 0$, $f \in L^m(\Omega)$ with $m > 1$, and

$$\operatorname{ess\,inf}\{f(x) : x \in \omega\} > 0, \quad \text{for every } \omega \subset\subset \Omega, \tag{1.2}$$

$$1 \leq \theta < 2, \quad \text{and} \quad 2 < p < N. \tag{1.3}$$

We assume that $b : \Omega \to \mathbb{R}$ is a measurable function such that for some positive constants α and β

$$\alpha \leq b(x) \leq \beta, \quad \text{for } a.e.\ x \in \Omega. \tag{1.4}$$

Starting from the pioneering work [14] many authors studied singular semi-linear elliptic equations, see e.g [13, 9, 2, 15, 17, 22]. Let us also mention the contributions in [18, 20] where related problems involving singular nonlinearities.

When the singular lower-order term does not appear in (1.1) (i.e., $B \equiv 0$), the existence and regularity of solutions to problem (1.1) are proved in [8]. under the hypothesis $Au = \operatorname{div}(a(x,u)\nabla u)$, where $a : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the following condition:

$$\frac{\alpha}{(1+|s|)^\gamma} \leq a(x,s) \leq \beta, \quad \text{with } 0 \leq \gamma \leq 1.$$

The corresponding results in the case $p = 2$ are developed in [13], the authors showed the existence of positive solutions for $0 \leq \theta < 2$ and non-existence for $\theta \geq 2$. In the case where $\gamma = B = 0$, $f \in L^1(\Omega)$, the solution u of problem (1.1) belongs only to $W_0^{1,s}(\Omega)$ for every $s < \frac{N(p-1)}{N-1}$, see [6, 7]. In [1], under the assumptions $B \equiv 0$ and $\gamma = \theta(p-1)$, the authors proved only the existence of entropy solutions u of problem (1.1) belongs only to the Marcinkiewicz space $M^\tau(\Omega)$ for every $\tau = \frac{N(p-1)(1-\theta)}{N-p}$, with $|\nabla u| \in M^q(\Omega)$ for $q = \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)}$.

In the study of problem (1.1), there are two difficulties, the first one, , the differential operator

$$Au = \operatorname{div}\left(b(x)\frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^\gamma}\right),$$

though well defined between $W_0^{1,p}(\Omega)$ and its dual $W^{-1,p'}(\Omega)$, but it fails to be coercive on $W_0^{1,p}(\Omega)$ when u is unbounded. Due to the lack of coercivity, the classical theory for elliptic operators acting between spaces in duality (see [23]) can not be applied even if the data f are sufficiently regular (see [24]). The second difficulty comes from the lower-order term: dependence with respect to the gradient and the singular dependence with respect to u. We overcome these difficulties by proof the existence of positive constant c_ω such that $u_n \geq c_\omega$ almost everywhere in ω for every $n \in \mathbb{N}$. For this using the following theorem :

Theorem 1. [3, 19, 21] *Let $G : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that for every $\omega \subset\subset \Omega$ there exists $m_\omega > 0$ such that $G(x,s) \geq m_\omega l(s)$ for a.e. $x \in \Omega$, and for every $s \geq 0$. Assume that $l : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function such that $\frac{l(s)}{s^{p-1}}$ is increasing for s sufficiently large and for some $t_0 > 0$*

$$\int_{t_0}^{+\infty} \frac{dt}{\left(\int_0^t l(s)ds\right)^{\frac{1}{p}}} < +\infty. \tag{1.5}$$

Then for every $\omega \subset\subset \Omega$ there exists a constant $C_\omega > 0$ such that every sub-solution $v \in W^{1,p}_{loc}(\Omega)$ of

$$-div(b(x)|\nabla v|^{p-2}\nabla v) + G(x,v) = 0,$$

such that $v^+ \in L^\infty_{loc}(\Omega)$ and $G(x,v^+) \in L^p_{loc}(\Omega)$ satisfies $v \leq C_\varepsilon$ in ω.

To prove our main results, we approximate the problem (1.1) by a sequence of non-degenerate and non-singular problems, procedure similarly to [13, 11, 3, 2, 15, 17, 16]. Then, we prove both a priori estimates and convergence results on the sequence of approximating solutions. Next, by the Theorem 1, we prove that the approximate solutions u_n is strictly positive in Ω. In the end, we pass to the limit in the approximate problems.

2. Statement of Main Results and Approximate Solutions

2.1. Statement of Main Results.

Definition 2. Let $f \in L^m(\Omega)$, with $m > 1$ a measurable function u is said to be solution in the sense of distributions to the problem (1.1), if $u \in W^{1,p}_0(\Omega)$ such that $u > 0$ in Ω, $\frac{|\nabla u|^p}{u^\theta} \in L^1(\Omega)$, and it holds

$$\int_\Omega b(x)\frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^\gamma}\nabla\varphi dx + B\int_\Omega \frac{|\nabla u|^p}{u^\theta}\varphi dx = \int_\Omega f\varphi dx, \tag{2.1}$$

for every $\varphi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$.

The first result deals with the case when the summability of f gives the existence of an infinite energy solution, belonging to $u \in W^{1,\sigma}_0(\Omega)$, with $1 < \sigma < p$.

Theorem 3. *Let $\gamma > \theta - 1$. Assume that (1.2)-(1.4) hold true and that $f \in L^m(\Omega)$, with*

$$1 < m < \frac{Np}{Np - \theta(N-p)}. \tag{2.2}$$

Then, the problem (1.1) has at least one distributional solution $u \in W^{1,\sigma}_0(\Omega)$, with

$$\sigma = \frac{mN(p-\theta)}{N - \theta m}. \tag{2.3}$$

Remark 4. *Notice that the condition (2.2) guarantees that $1 < \sigma < p$.*

The next result deals with a given f which yields unbounded solutions in energy space $W^{1,p}_0(\Omega)$.

Theorem 5. *Let $\gamma > \theta - 1$. Suppose that assumptions (1.2)-(1.4) hold, and furthermore suppose that $f \in L^m(\Omega)$, with*

$$m \geq \frac{Np}{Np - \theta(N-p)}. \tag{2.4}$$

Then, the problem (1.1) has at least one distributional solution $u \in W^{1,p}_0(\Omega)$.

In the case where $0 < \gamma \leq \theta - 1$, we are able to prove the existence and regularity of a solution to problem (1.1).

Theorem 6. *Let $0 < \gamma \leq \theta - 1 < p - 1$. Suppose that assumptions (1.2)-(1.4) hold, and $f \in L^m(\Omega)$ be a positive function, with*

$$\frac{N}{Np-(N-1)(\gamma+1)} < m < \frac{Np}{Np-(N-p)(\gamma+1)}. \tag{2.5}$$

Then, the problem (1.1) has at least one distributional solution $u \in W_0^{1,q}(\Omega)$, with

$$q = \frac{Nm(p-1-\gamma)}{N-m(1+\gamma)}. \tag{2.6}$$

Theorem 7. *Let $0 < \gamma \leq \theta - 1 < p - 1$. If hypotheses (1.2)-(1.4) hold, and $f \in L^m(\Omega)$, with*

$$\frac{Np}{Np-(N-p)(\gamma+1)} \leq m < \frac{N}{p}. \tag{2.7}$$

Then, the problem (1.1) has at least one distributional solution $u \in W_0^{1,p}(\Omega) \cap L^r(\Omega)$, with

$$r = \frac{Nm(p-1-\gamma)}{N-pm}. \tag{2.8}$$

Remark 8. *The assumption (2.2) and the fact that*

$$\frac{Np}{Np-(N-p)(\gamma+1)} > \frac{Np}{Np-N-\gamma N+p^2},$$

implies $2 < p < r$.

Remark 9. *If $\gamma = p - 1$, the assumptions (2.5) and (2.7) become empty.*

Theorem 10. *Let $0 < \gamma \leq \theta - 1 < p - 1$. If hypotheses (1.2)-(1.4), hold and $f \in L^m(\Omega)$, with*

$$m > \frac{N}{p}. \tag{2.9}$$

Then there exists a distributional solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to problem (1.1).

2.2. Approximate Solutions. Hereafter, we denote by T_k the truncation function at the level $k > 0$, defined by $T_k(s) = \max\{-k, \min\{s, k\}\}$ for every $s \in \mathbb{R}$.

Let (f_n) $(f_n > 0)$ be a sequence of bounded functions defined in Ω which converges to $f > 0$ in $L^1(\Omega)$, and which verifies the inequalities $|f_n| \leq n$ and $|f_n| \leq |f|$ for every $n \geq 1$ (for example $f_n = T_n(f)$), we approximate the problem (1.1) by the following non-degenerate and non-singular problem

$$\begin{cases} -\operatorname{div}\left(b(x)\frac{|\nabla u_n|^{p-2}\nabla u_n}{(1+|T_n(u_n)|)^\gamma}\right) + B\frac{u_n|\nabla u_n|^p}{\left(|u_n|+\frac{1}{n}\right)^{\theta+1}} = T_n(f), & \text{in } \Omega, \\ u_n = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

The problem (2.10) admits at least one solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ (See [10], Thoerem 1., Remark 2.1..)

Proposition 11. *$u_n \geq 0$ almost everywhere in Ω.*

Proof. The proof uses the same technique as in [15] and is omited here.

Therefore, Proposition 11 implies that u_n satisfies

$$\begin{cases} -\text{div}\left(b(x)\frac{|\nabla u_n|^{p-2}\nabla u_n}{(1+T_n(u_n))^{\gamma}}\right) + B\frac{u_n|\nabla u_n|^p}{\left(u_n+\frac{1}{n}\right)^{\theta+1}} = T_n(f), & \text{in } \Omega, \\ u_n = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.11}$$

3. A priori estimates

Throughout this section, we will denote by C the positive constants depending only on the data of the problem, but not on n.
We are now going to prove some a priori estimates. The next lemma gives a control of the lower order term.

Lemma 12. [13, 12, 11] *Let u_n be the solutions to problems (2.11). Then it results*

$$B\int_\Omega \frac{u_n|\nabla u_n|^p}{\left(u_n+\frac{1}{n}\right)^{\theta+1}}dx \le \int_\Omega f dx. \tag{3.1}$$

Lemma 13. *Suppose that the hypotheses of Theorem 3 are satisfied. Then the sequence u_n is bounded in $W_0^{1,\sigma}(\Omega)$, where σ given by (2.3).*

Proof. Take $\phi = (u_n+1)^{\theta+ps} - 1$, with

$$s = \frac{p^* - \theta m'}{pm' - p^*}, \tag{3.2}$$

as a test function in problems (2.11), (note that $s < 0$ and $\theta + ps > 0$). Using (3.1),$T_n(f) \le f$, and dropping the nonegative first term, we obtain

$$B\int_\Omega \frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}}(u_n+1)^{\theta+ps}dx \le \int_\Omega f(u_n+1)^{\theta+ps}dx + C.$$

By working in $\{u_n \ge 1\}$, using the fact that

$$2u_n(u_n+1)^{\theta} \ge (u_n+\frac{1}{n})^{\theta+1}, \tag{3.3}$$

the previous inequality gives

$$\begin{aligned} \frac{B}{2(s+1)^p}\int_{\{u_n\ge 1\}}\left|\nabla\left((u_n+1)^{s+1}-2^{s+1}\right)\right|^p dx &= \frac{B}{2}\int_{\{u_n\ge1\}}|\nabla u_n|^p(u_n+1)^{ps}dx \\ &\le \int_{\{u_n\ge1\}} f(u_n+1)^{\theta+ps}dx + C \\ &\le C\left(\int_{\{u_n\ge1\}}(u_n+1)^{(ps+\theta)m'}dx\right)^{\frac{1}{m'}} + C. \end{aligned} \tag{3.4}$$

Using (3.4) and the Sobolev inequality on the left one imply

$$\begin{aligned}\mathcal{S}\frac{B}{2(s+1)^p}\left(\int_{\{u_n\geq 1\}}|(u_n+1)^{s+1}-2^{s+1}|^{p^*}dx\right)^{\frac{p}{p^*}}&\\ \leq \frac{B}{2(s+1)^p}\int_{\{u_n\geq 1\}}|\nabla\left((u_n+1)^{s+1}-2^{s+1}\right)|^p dx&\\ \leq C\left(\int_{\{u_n\geq 1\}}(u_n+1)^{(ps+\theta)m'}dx\right)^{\frac{1}{m'}}+C.&\end{aligned}\tag{3.5}$$

We remark that (3.2) is equivalent to require $(s+1)p^*=(ps+\theta)m'$ moreover $\frac{p}{p^*}>\frac{1}{m'}$, due to the hypotheses on m and θ. Hence

$$\left(\int_{\{u_n\geq 1\}}|(u_n+1)^{s+1}-2^{s+1}|^{p^*}dx\right)^{\frac{p}{p^*}}\leq C\left(\int_{\{u_n\geq 1\}}(u_n+1)^{(s+1)p^*}dx\right)^{\frac{1}{m'}}+C,$$

so that

$$\int_{\{u_n\geq 1\}}(u_n+1)^{(s+1)p^*}dx\leq C.\tag{3.6}$$

Now, by Hölder inequality and that $\sigma<p$, (2.3),(3.2),(3.6), and that $-s\frac{\sigma p}{p-\sigma}=(s+1)p^*$, we obtain

$$\begin{aligned}\int_{\{u_n\geq 1\}}|\nabla u_n|^\sigma dx&=\int_{\{u_n\geq 1\}}\frac{|\nabla u_n|^\sigma}{(u_n+1)^{-s\sigma}}(u_n+1)^{-s\sigma}dx\\ &\leq\left(\int_{\{u_n\geq 1\}}\frac{|\nabla u_n|^p}{(u_n+1)^{-ps}}dx\right)^{\frac{\sigma}{p}}\times\left(\int_{\{u_n\geq 1\}}(u_n+1)^{-s\frac{\sigma p}{p-\sigma}}dx\right)^{\frac{p-\sigma}{p}}\\ &\leq\left(\int_{\{u_n\geq 1\}}|\nabla u_n|^p(u_n+1)^{ps}dx\right)^{\frac{\sigma}{p}}\times\left(\int_{\{u_n\geq 1\}}(u_n+1)^{(s+1)p^*}dx\right)^{\frac{p-\sigma}{p}}\\ &\leq C.\end{aligned}\tag{3.7}$$

It remains to analyse the behaviour of ∇u_n on $\{u_n\leq 1\}$. Taking $T_1(u_n)$ as a test function in (2.11), using the fact that $T_n(f)\leq f$ and dropping the non-negative the lower order term, we get

$$\int_{\{u_n\leq 1\}}|\nabla T_1(u_n)|^p dx\leq C.\tag{3.8}$$

As a consequence of estimates (3.7) and (3.8), $(u_n)_n$ is bounded in $W_0^{1,\sigma}(\Omega)$.

Lemma 14. *Suppose that the hypotheses of Theorem 5 are satisfied. Then the sequence u_n is bounded in $W_0^{1,p}(\Omega)$.*

Proof. If we take $\phi=(u_n+1)^\theta-1$ as a test function in problem (2.11), we obtain

$$B\int_\Omega\frac{|\nabla u_n|^p}{\left(u_n+\frac{1}{n}\right)^{\theta+1}}u_n(u_n+1)^\theta dx\leq B\int_\Omega\frac{u_n|\nabla u_n|^p}{\left(u_n+\frac{1}{n}\right)^{\theta+1}}dx+C\int_\Omega f u_n^\theta dx+C.$$

We can estimate the right hand side using (3.1) in order to get

$$B\int_\Omega \frac{|\nabla u_n|^p}{\left(u_n+\frac{1}{n}\right)^{\theta+1}}u_n(u_n+1)^\theta dx \le C\int_\Omega f u_n^\theta dx + C. \tag{3.9}$$

By working in $\{u_n \ge 1\}$, using (3.3) and the fact that

$$C\int_\Omega f u_n^\theta dx + C \le C\int_{\{u_n\ge 1\}} f u_n^\theta dx + C,$$

and (3.9), we get

$$\frac{B}{2}\int_{\{u_n\ge 1\}} |\nabla u_n|^p dx \le C\int_{\{u_n\ge 1\}} f(u_n-1)^\theta dx + C.$$

Using Sobolev's inequality (with exponent $\frac{p^*}{\theta}$ on the left-hand side), and Hölder's inequality (on the right-hand side), we obtain

$$\begin{aligned}\mathcal{S}\frac{B}{2}\left(\int_{\{u_n\ge 1\}}(u_n-1)^{p^*}\right)^{\frac{p}{p^*}} dx &\le \frac{B}{2}\int_{\{u_n\ge 1\}}|\nabla u_n|^p dx \\ &\le C\left(\int_{\{u_n\ge 1\}}(u_n-1)^{p^*}dx\right)^{\frac{\theta}{p^*}} + C. \end{aligned}\tag{3.10}$$

Since we are assuming $\theta < p$, we deduce that

$$\int_{\{u_n\ge 1\}}(u_n-1)^{p^*}dx \le C, \tag{3.11}$$

inequality (3.10) and (3.11) imply

$$\int_{\{u_n\ge 1\}}|\nabla u_n|^p dx \le C. \tag{3.12}$$

Let us search for the same kind of estimate in $\{u_n < 1\}$. Taking $T_1(u_n)$ as a test function in problem (2.11), and dropping the non-negative lower order term, we get

$$\int_{\{u_n<1\}}|\nabla T_1(u_n)|^p dx \le C. \tag{3.13}$$

As a consequence of estimates (3.12) and (3.13), u_n is bounded in $W_0^{1,p}(\Omega)$.

Lemma 15. *Suppose that the hypotheses of Theorem 6 are satisfied. Then the sequence u_n is bounded in $W_0^{1,q}(\Omega)$, where q given by (2.6).*

Lemma 16. *Suppose that the hypotheses of Theorem 7 are satisfied. Then the sequence u_n is bounded in $W_0^{1,p}(\Omega)\cap L^r(\Omega)$, where r given by (2.8).*

Proof of Lemma 15 and Lemma 16. In problems (2.11) consider as a test function the same test functions as in [4]. With this choice, the lower order term is non-negative and we can take into account only the term given by the operator. Therefore one can follow the same proofs as in [4] to get the above estimates.

To prove the $L^\infty(\Omega)$ a priori estimate, we will need the following result

Lemma 17 ([6]). *Let w be a function in $W_0^{1,\delta}(\Omega)$ such that, for k greater than some k_0,*

$$\int_{A_k} |\nabla w|^\delta dx \le Ck^{\frac{\gamma\delta}{p-1}} |A_k|^{\frac{\delta}{\delta^*}+\varepsilon_1}, \tag{3.14}$$

where $\varepsilon_1 > 0$, $0 \le \gamma < p-1$, $\delta^* = \frac{N\delta}{N-\delta}$ and

$$A_k = \{x \in \Omega : \quad w(x) > k\}.$$

Then the norm of w in $L^\infty(\Omega)$ is bounded by a constant which depends on $C, \gamma, \delta, N, \varepsilon_1, k_0$ and $|\Omega|$.

Lemma 18. *Suppose that the hypotheses of Theorem 10 are satisfied. Then the sequence u_n is bounded in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.*

Proof. Let us start with the estimate in $L^\infty(\Omega)$. Define, for x in $\mathbb{R}$ and for $k > 0$,

$$G_k(x) = (|x| - k)_+ sign(x) = x - T_k(x).$$

For $k > 0$, if we take $G_k(u_n)$ as test function in (2.11), by (1.4), Hölder inequality and dropping the nonegative lower order term, we obtain

$$\alpha \int_{A_k} \frac{|\nabla u_n|^p}{(1+u_n)^\gamma} dx \le C \left(\int_{A_k} |G_k(u_n)|^{m'} dx \right)^{\frac{1}{m'}}. \tag{3.15}$$

Therefore, if $\delta < p$, we can write, by the Hölder inequality and (3.15), we get

$$\begin{aligned} \int_{A_k} |\nabla u_n|^\delta dx &= \int_{A_k} \frac{|\nabla u_n|^\delta}{(1+u_n)^{\frac{\gamma\delta}{p}}} (1+u_n)^{\frac{\gamma\delta}{p}} dx \\ &\le C \left(\int_{A_k} |G_k(u_n)|^{m'} dx \right)^{\frac{\delta}{pm'}} \left(\int_{A_k} (1+u_n)^{\frac{\gamma\delta}{p-\delta}} dx \right)^{\frac{p-\delta}{p}}. \end{aligned} \tag{3.16}$$

Let us choose δ such that its Sobolev conjugate exponent $\delta^* = \frac{\delta N}{N-\delta}$ is equal to m, that is,

$$\delta = \frac{Nm}{Nm + m - N}.$$

It is easy to check that the hypotheses on m imply $\delta < \frac{N}{N-p+1} < p$. From (3.16) and Sobolev's embedding theorem we obtain

$$\int_{A_k} |\nabla u_n|^\delta dx \le C \left(\int_{A_k} |\nabla u_n|^\delta dx \right)^{\frac{1}{p}} \left(\int_{A_k} (1+u_n)^{\frac{\gamma\delta}{p-\delta}} dx \right)^{\frac{p-\delta}{p}},$$

and therefore

$$\int_{A_k} |\nabla u_n|^\delta dx \le C \left(\int_{A_k} (1+u_n)^{\frac{\gamma\delta}{p-\delta}} dx \right)^{\frac{p-\delta}{p-1}}.$$

Since if $k \ge 1$, one has on A_k that $1 + u_n \le 2(k + G_k(u_n))$, we can write

$$\int_{A_k} |\nabla u_n|^\delta dx \le C \left[k^{\frac{\gamma\delta}{p-1}} |A_k|^{\frac{p-\delta}{p-1}} + \left(\int_{A_k} G_k(u_n)^{\frac{\gamma\delta}{p-\delta}} dx \right)^{\frac{p-\delta}{p-1}} \right].$$

Since $\gamma < p-1$ and $m > \frac{N}{p}$, with our choice of γ one has

$$\frac{\gamma\delta}{p-\delta} < \delta^*,$$

and therefore, using Hôlder's, Sobolev's and Young's inequalities, one obtains

$$\begin{aligned}
\int_{A_k} |\nabla u_n|^\delta dx &\leq Ck^{\frac{\gamma\delta}{p-1}}|A_k|^{\frac{p-\delta}{p-1}} + C\left(\int_{A_k} G_k(u_n)^{\delta^*} dx\right)^{\frac{\gamma\delta}{(p-1)\delta^*}} |A_k|^{\frac{p-\delta}{p-1}-\frac{\gamma\delta}{(p-1)\delta^*}} \\
&\leq C\left[k^{\frac{\gamma\delta}{p-1}}|A_k|^{\frac{p-\delta}{p-1}} + \left(\int_{A_k} |\nabla u_n|^\delta dx\right)^{\frac{\gamma}{p-1}} |A_k|^{\frac{p-\delta}{p-1}-\frac{\gamma\delta}{(p-1)\delta^*}}\right] \\
&\leq C\left[k^{\frac{\gamma\delta}{p-1}}|A_k|^{\frac{p-\delta}{p-1}} + \varepsilon\int_{A_k} |\nabla u_n|^\delta dx + \varepsilon(p,\gamma)|A_k|^{\frac{(p-\delta)\delta^*-\gamma\delta}{\delta^*(p-1-\gamma)}}\right].
\end{aligned}$$

If we choose $\varepsilon = \frac{1}{2C}$, we can take the term containing the gradient to the right hand side, obtaining

$$\int_{A_k} |\nabla u_n|^\delta dx \leq C\left[k^{\frac{\gamma\delta}{p-1}}|A_k|^{\frac{p-\delta}{p-1}} + |A_k|^{\frac{(p-\delta)\delta^*-\gamma\delta}{\delta^*(p-1-\gamma)}}\right].$$

Since $\delta < \frac{N}{N-p+1}$ we have

$$\frac{(p-\delta)}{p-1} < \frac{(p-\delta)\delta^*-\gamma\delta}{\delta^*(p-1-\gamma)}.$$

So that, for $k > 1$, we can write, observing that $|A_k| \leq |\Omega|$,

$$\int_{A_k} |\nabla u_n|^\delta dx \leq C_{22}k^{\frac{\gamma\delta}{p-1}}|A_k|^{\frac{p-\delta}{p-1}}.$$

Let now ε_1 be such that $\frac{p-\delta}{p-1} = \frac{\delta}{\delta^*} + \varepsilon_1$; then it is easy to see that $\varepsilon_1 > 0$.

Therefore we can apply Lemma 17, with $w = u_n$, and obtain a bound for u_n in $L^\infty(\Omega)$. The estimate in $W_0^{1,p}(\Omega)$ is now very easy. Taking u_n as test function in (2.11), using hypothesis (1.4) one obtains,

$$\frac{\alpha}{(1+c)^\gamma}\int_\Omega |\nabla u_n|^p dx \leq \int_\Omega f u_n dx,$$

and the right hand side is trivially bounded since f belongs to $L^1(\Omega)$.

Remark 19. *By Lemmata 13-16 and 18, the sequence $(u_n)_n$ is bounded in $W_0^{1,\eta}(\Omega)$ where η equals p or σ or q. Therefore, there exists a function u belongs to $W_0^{1,\eta}(\Omega)$ such that*

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_0^{1,\eta}(\Omega), \\ u_n \to u & \text{a.e. in } \Omega. \end{cases} \tag{3.17}$$

Lemma 20. *Let $1 \leq \theta < 2$. Let u_n and u be as in (3.17). Then the sequence $\nabla u_n(x)$ converges a.e. to $\nabla u(x)$.*

Proof. The proof is in the spirit of ([13], Lemma 2.6) and also ([5], Lemma 2.3), we fix $h,k>0$. Plugging $T_h(u_n - T_k(u))$ as a test function in (2.11), By estimate (3.1) and hypothesis (1.4), we get

$$\int_\Omega \frac{\alpha}{(1+T_n(u_n))^\gamma}|\nabla u_n|^{p-2}\nabla u_n \nabla T_h(u_n - T_k(u))dx \le Ch. \tag{3.18}$$

We recall the following well-known inequalities, that hold for any two real vectors ξ,η and a real $p>2$

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi-\eta) \ge 2^{2-p}|\xi-\eta|^p. \tag{3.19}$$

Using (3.18) and (3.19), we have

$$\begin{aligned}\int_{A_{k,h,n}} \frac{|\nabla(u_n - T_k(u))|^p}{(1+u_n)^\gamma}dx &\le \int_{A_{k,h,n}} C\frac{(|\nabla u_n|^{p-2}\nabla u_n - |\nabla T_k(u)|^{p-2}\nabla T_k(u))\nabla T_h(u_n - T_k(u))}{(1+T_n(u_n))^\gamma}dx \\ &\le Ch - C\int_{A_{k,h,n}} \frac{|\nabla T_k(u)|^{p-2}\nabla T_k(u)\nabla T_h(u_n - T_k(u))}{(1+T_n(u_n))^\gamma}dx, \end{aligned}\tag{3.20}$$

where $A_{k,h,n} = \{|u_n - T_k(u)| \le h, |u| \le k\}$. We pass to the limit as $n \to +\infty$ in (3.20), using the fact that $\frac{1}{(1+T_n(u_n))^\gamma} \le 1$ and $\nabla T_h(u_n - u) \rightharpoonup 0$ in $L^p(\Omega)$, we get

$$\limsup_{n\to+\infty} \int_{A_{k,h,n}} \frac{|\nabla(u_n - T_k(u))|^p}{(1+u_n)^\gamma}dx \le Ch.$$

This, imply

$$\limsup_{n\to+\infty} \int_{A_{k,h,n}} |\nabla(u_n - T_k(u))|^p dx \le Ch(1+k+h)^p. \tag{3.21}$$

We recall that u_n is uniformly bounded in $W_0^{1,\eta}(\Omega)$, where η equals p or σ or q, according to the statements of Lemmata 13-16 and 18. Let $\tau \in (1,\eta)$. We can write

$$\begin{aligned}\int_\Omega |\nabla(u_n - u)|^\tau dx \le &\int_{\{|u_n-u|\le h, |u|\le k\}} |\nabla(u_n-u)|^\tau dx + \int_{\{|u_n-u|\le h, |u|>k\}} |\nabla(u_n-u)|^\tau dx \\ &+ \int_{\{|u_n-u|>h\}} |\nabla(u_n-u)|^\tau dx.\end{aligned}$$

Using the Hölder inequality, and that u_n is uniformly bounded in $W_0^{1,\eta}(\Omega)$, we have

$$\begin{aligned}\int_\Omega |\nabla(u_n-u)|^\tau dx \le C\left(\int_{\{|u_n-u|\le h,|u|\le k\}} |\nabla(u_n-u)|^p dx\right)^{\frac{\tau}{p}} + C\mu(\{|u|>k\}))^{1-\frac{\tau}{\eta}} \\ + C\mu(\{|u_n-u|>h\}))^{1-\frac{\tau}{\eta}}.\end{aligned}\tag{3.22}$$

By (3.21) and (3.22), we obtain

$$\limsup_{h\to 0}\limsup_{n\to\infty}\int_\Omega |\nabla(u_n-u)|^\tau dx \le C(\mu(\{|u|>k\}))^{1-\frac{\tau}{\eta}}, \quad \forall k>0.$$

At the limit as $k\to+\infty$, $\mu(\{|u|>k\})$ converges to 0. Therefore $\nabla u_n\to\nabla u$ in $L^\tau(\Omega)$. Up to a subsequence, $\nabla u_n\to\nabla u$ a.e. in Ω.

Remark 21. *The technique used for the proof of Lemma 20 under the hypotheses of theorem 3, is the same compared to Theorems 5-10 .*

4. Proof of Main Results

4.1. Positive Solution of (2.11). As in the above case, we need to prove that the function u is not 0 in Ω. To this aim, we are going to prove that for every $\omega\subset\subset\Omega$ there exists a positive constant c_ω such that the solutions u_n to problems (2.11) satisfy $u_n\ge c_\omega$ in ω for every $n\in\mathbb{N}$.

Proposition 22. *Let $1\le\theta<2$. Let u_n be a weak solution of problem (2.11). Then for every $w\subset\subset\Omega$, there exists a strictly positive constant c_w, such that*

$$u_n\ge c_w, \quad \text{in } w,\ \forall n\in\mathbb{N}. \tag{4.1}$$

Proof. **Step 1.** Let u_n be a $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ solution to (2.11). We perform a change of variable in order to get a sub-solution of an elliptic semi-linear problem, as in Theorem 1. We set $h_n(s)=\frac{1}{(1+T_n(s))^\gamma}$. Then u_n satisfies, distributionally,

$$-\mathrm{div}(b(x)h_n(u_n)|\nabla u_n|^{p-2}\nabla u_n)+\frac{B}{u_n^\theta}|\nabla u_n|^p\ge T_1(f),$$

that is,

$$-\mathrm{div}(b(x)|\nabla u_n|^{p-2}\nabla u_n)h_n(u_n)-h_n'(u_n)b(x)|\nabla u_n|^p+\frac{B}{u_n^\theta}|\nabla u_n|^p\ge T_1(f). \tag{4.2}$$

We define, for $s,t\ge 0$

$$k_n(t)=\int_1^t\frac{B}{\alpha r^\theta h_n(r)}dr, \quad \psi_n(s)=\int_s^1 e^{-\frac{1}{p-1}k_n(t)}(h_n(t))^{\frac{\beta}{\alpha(p-1)}}dt. \tag{4.3}$$

We remark that (4.3) implies

$$\begin{cases}\psi_n'(s)=-e^{-\frac{1}{p-1}k_n(s)}(h_n(s))^{\frac{\beta}{\alpha(p-1)}},\\ \frac{\psi_n''(s)}{\psi_n'(s)}=\frac{1}{p-1}\left(-\frac{B}{\alpha s^\theta h_n(s)}+\beta\frac{h_n'(s)}{\alpha h_n(s)}\right).\end{cases} \tag{4.4}$$

We define $v_n=\psi_n(u_n)$. Then

$$\begin{aligned}\mathrm{div}\left(b(x)|\nabla v_n|^{p-2}\nabla v_n\right)&=|\psi_n'(u_n)|^{p-2}\psi_n'(u_n)\mathrm{div}\left(b(x)|\nabla u_n|^{p-2}\nabla u_n\right)\\&\quad+(p-1)b(x)|\nabla u_n|^p\psi_n''(u_n)|\psi_n'(u_n)|^{p-2},\end{aligned}$$

and therefore

$$-h_n(u_n)\mathrm{div}\left(b(x)|\nabla u_n|^{p-2}\nabla u_n\right) = -h_n(u_n)\frac{\mathrm{div}\left(b(x)|\nabla v_n|^{p-2}\nabla v_n\right)}{|\psi_n'(u_n)|^{p-2}\psi_n'(u_n)} + (p-1)h_n(u_n)b(x)\frac{\psi_n''(u_n)}{\psi_n'(u_n)}|\nabla u_n|^p. \tag{4.5}$$

By inequality (4.2) and (4.5), we have

$$T_1(f) \le -h_n(u_n)\frac{\mathrm{div}\left(b(x)|\nabla v_n|^{p-2}\nabla v_n\right)}{|\psi_n'(u_n)|^{p-2}\psi_n'(u_n)} + (p-1)h_n(u_n)b(x)\frac{\psi_n''(u_n)}{\psi_n'(u_n)}|\nabla u_n|^p - h_n'(u_n)b(x)|\nabla u_n|^p + \frac{B}{u_n^\theta}|\nabla u_n|^p. \tag{4.6}$$

Using that $h_n'(s) \le 0$, $\frac{\psi_n''(s)}{\psi_n'(s)} \le 0$, hypothesis (1.4) and (4.4),(4.6), we obtain

$$T_1(f) \le -h_n(u_n)\frac{\mathrm{div}\left(b(x)|\nabla v_n|^{p-2}\nabla v_n\right)}{|\psi_n'(u_n)|^{p-2}\psi_n'(u_n)}, \tag{4.7}$$

using the fact that

$$|\psi_n'(s)|^{p-2}\psi_n'(s) = -e^{-k_n(s)}(h_n(s))^{\frac{\beta}{\alpha}} \le 0. \tag{4.8}$$

By (4.7) and (4.8), we get

$$0 \ge -\mathrm{div}(b(x)|\nabla v_n|^{p-2}\nabla v_n) + T_1(f)e^{-k_n(\psi_n^{-1}(v_n))}(h_n(\psi_n^{-1}(v_n)))^{\frac{\beta}{\alpha}-1}.$$

Step 2. We now study, for $s \ge 0$

$$s \mapsto e^{-k_n(\psi_n^{-1}(s))}(h_n(\psi_n^{-1}(s)))^{\frac{\beta}{\alpha}-1}$$

We remark that $\psi_n^{-1}(s) \le 1$, since $s \ge 0 = \psi_n(1)$ and ψ_n is decreasing. Since h_n is decreasing, then

$$h_n^{\frac{\beta}{\alpha}-1}(\psi_n^{-1}(s)) \ge h_n^{\frac{\beta}{\alpha}-1}(1) = h_0. \tag{4.9}$$

New, by (4.3), we have $h_n(s) = h_1(s)$ if $s \le 1$ and $h_n(s) \le h_1(s)$ if $s > 1$, it is not difficult to prove that

$$\psi_n(s) \ge \psi_1(s), \tag{4.10}$$

Now, inequality (4.10) and the fact that ψ_n is decreasing imply that $\psi_n^{-1}(s) \le \psi_1^{-1}(s)$ for every $s \ge 0$. Recalling that $\psi_n^{-1}(s) \le 1$ and $h_n(s) = h_1(s) \ge 0$ for $s \ge 1$, we deduce easily that

$$e^{-k_n(\psi_n^{-1}(s))} \ge e^{-k_1(\psi_1^{-1}(s))}. \tag{4.11}$$

Due to (4.9) and (4.11), v_n satisfies

$$\begin{cases} -\mathrm{div}\left(b(x)|\nabla v_n|^{p-2}\nabla v_n\right) + G(x,v_n) \le 0, \\ G(x,s) = \begin{cases} T_1(f)h_0 l(s), & \text{if} \quad s \ge 0 \\ 0, & \text{if} \quad s \le 0 \end{cases}, \\ \text{where} \;\; l(s) = \frac{1}{p-1}(e^{-k_1(\psi_1^{-1}(s))} - 1). \end{cases} \tag{4.12}$$

Step 3. We are going to prove that l satisfies the hypotheses of Theorem 1. We observe that l is continuous and increasing, since ψ_1^{-1} is decreasing and k_1 is increasing. We claim that $\frac{l(s)}{s^{p-1}}$ is increasing for s sufficiently large. This is equivalent to prove that $Y(t) = \frac{l(\psi_1(t))}{(\psi_1(t))^{p-1}}$ is decreasing for small positive t. Now, $Y'(t) < 0$ if and only if

$$\frac{1}{p-1} l'(\psi_1(t))\psi_1(t) - \int_1^t l'(\psi_1(s))\psi_1'(s)ds > 0. \tag{4.13}$$

We remark that

$$l'(\psi_1(s)) = \frac{B}{\alpha s^\theta h_1^{\frac{\beta}{(p-1)\alpha}+1}(s)}.$$

Let $w_0 \in (0,1)$ be such that

$$g(t) = l'(\psi_1(t)) = \frac{1}{p-1}\frac{B}{\alpha t^\theta h_1^{\frac{\beta}{(p-1)\alpha}+1}(t)},$$

is decreasing in $(0, w_0]$. Therefore

$$\begin{aligned} l'(\psi_1(t))\psi_1(t) - \int_1^t l'(\psi_1(s))\psi_1'(s)ds &= \int_t^1 e^{-\frac{1}{p-1}k_1(s)} h_1^{\frac{\beta}{(p-1)\alpha}}(s)[\frac{1}{p-1}g(t) - g(s)]ds \\ &\geq \int_{w_0}^1 e^{-\frac{1}{p-1}k_1(s)} h_1^{\frac{\beta}{(p-1)\alpha}}(s)[\frac{1}{p-1}g(t) - g(s)]ds. \end{aligned}$$

Let to the choice of w_0. Let

$$M_1 = \int_{w_0}^1 e^{-\frac{1}{p-1}k_1(s)} h_1^{\frac{\beta}{(p-1)\alpha}}(s)ds, \quad M_1 = \int_{w_0}^1 e^{-\frac{1}{p-1}k_1(s)} h_1^{\frac{\beta}{(p-1)\alpha}}(s)g(s)ds.$$

We have proved that

$$l'(\psi_1(t))\psi_1(t) - \int_1^t l'(\psi_1(s))\psi_1'(s)ds \geq \frac{1}{p-1}M_1 g(t) - M_2.$$

It t is sufficiently small, the last quantity is positive, since h is decreasing for small positive t. Therefore (4.13) holds.

We are going to study the last condition on l, that is, the existence of a positive t_0 such that

$$\int_{t_0}^{+\infty} \frac{dt}{\left(\int_0^t l(s)ds\right)^{\frac{1}{p}}} < \infty. \tag{4.14}$$

Using the change of variable $\tau = \psi_1^{-1}(s)$ we get

$$\begin{aligned} \int_0^t l(s)ds &= \int_0^t (e^{k_1(\psi_1^{-1}(s))} - 1)ds \\ &= \int_{\psi_1^{-1}(t)}^1 [e^{-k_1(\tau)} - 1] h_1^{\frac{\beta}{(p-1)\alpha}}(\tau) e^{-\frac{1}{p-1}k_1(\tau)} d\tau. \end{aligned}$$

It is easy to see that $e^{-k_1(\tau)} - 1 \geq \frac{1}{2}e^{-k_1(\tau)}$ for $\tau \leq \tau_0$ sufficiently small. Moreover $h_1(\tau) \geq \frac{1}{2}$, for $\tau \leq 1$. Therefore it suffices to find t_0 sufficiently large ($t_0 > \psi_1(\tau_0)$) such that

$$\int_{t_0}^{+\infty} \frac{dt}{\left(\int_{\psi_1^{-1}(t)}^1 e^{-\frac{p}{p-1}k_1(\tau)}d\tau\right)^{\frac{1}{p}}} < \infty.$$

The last integral can be estimated, using the change $w = \psi_1^{-1}(t)$ and the fact that $h_1(s) \leq 1$, in the following way

$$\int_{t_0}^{+\infty} \frac{dt}{\left(\int_{\psi_1^{-1}(t)}^1 e^{-\frac{p}{p-1}k_1(\tau)}d\tau\right)^{\frac{1}{p}}} \leq \int_0^{\psi_1^{-1}(t_0)} \frac{dw}{\left(\int_w^{w_0} e^{\frac{p}{p-1}(k_1(w)-k_1(\tau))}d\tau\right)^{\frac{1}{p}}},$$

where w_0 is chosen in such a way that k_1' is decreasing in $(0, w_0]$. We observe that $\int_0^1 k_1'(t)^{\frac{1}{p}}dt < +\infty$ as $\theta < p$.
Hence it suffices to prove that there exists a strictly positive constant c such that

$$k_1'(w)\int_w^{w_0} e^{\frac{p}{p-1}(k_1(w)-k_1(\tau))}d\tau \geq c.$$

Now, since $k_1'(\tau)$ is decreasing in $(0, w_0]$,

$$\begin{aligned} k_1'(w)\int_w^{w_0} e^{\frac{p}{p-1}(k_1(w)-k_1(\tau))}d\tau &\geq \int_w^{w_0} k_1'(\tau)e^{\frac{p}{p-1}(k_1(w)-k_1(\tau))}d\tau \\ &= \frac{p-1}{p}\left(e^{\frac{p}{p-1}(k_1(w)-k_1(w_0))} - 1\right). \end{aligned}$$

Observe that $e^{\frac{p}{p-1}k_1(w)} \to 0$ as $w \to 0$, since $k_1(w) = \int_1^w \frac{B}{\alpha t^\theta h_1(t)}dt \to -\infty$ as $w \to 0$, by hypothesis $\theta \geq 1$. Therefore (4.14) is proved.
Step 4. Theorem 1 applies and gives, for every $\omega \subset\subset \Omega$, the existence of a constant $C_\omega > 0$ such that $v_n \leq C_\omega$. Recalling that $\psi_n(s) \geq \psi_1(s)$ by (4.10), we have $C_\omega \geq v_n = \psi_n(u_n) \geq \psi_1(u_n)$. Since ψ_1 is decreasing, $u_n \geq \psi_1^{-1}(C_\omega) = c_\omega > 0$ in $\omega \subset\subset \Omega$.

4.2. Proof of Theorem 3-10. We are now going to prove of Theorem 3 and Theorem 5 in a unique proof. As we will see the only difference is the choice of the test functions φ. Theorems 6-10 can be proved with the same technique.

Corollary 23. *Let $1 \leq \theta < 2$. Let u_n and u be as in (3.17). Then $\frac{|\nabla u|^p}{u^\theta} \in L^1(\Omega)$.*

Proof. We pass to the limit in (3.1), using (3.17), the a.e. convergence of ∇u_n to ∇u (see Lemma 20), by Proposition 22 and Fatou's lemma, we have

$$B\int_\Omega \frac{|\nabla u|^p}{u^\theta}dx \leq \int_\Omega f dx. \tag{4.15}$$

.

Corollary 24. *For every $\omega \subset\subset \Omega$ there exists a positive constant $\widetilde{c}_\omega$ such that*

$$\frac{u_n}{(u_n+\frac{1}{n})^{1+\theta}} \leq \widetilde{c}_\omega \quad \forall x \in \omega. \tag{4.16}$$

Proof. It is sufficient to observe that in every subset $\omega \subset\subset \Omega$

$$\frac{u_n}{(u_n+\frac{1}{n})^{\theta+1}} \leq \frac{1}{u_n^\theta} \leq \frac{1}{c_\omega^\theta} = \widetilde{c}_\omega,$$

since $u_n \geq c_\omega > 0$ in ω by Proposition 22.

Lemma 25. *Let u_n be the solutions to problems (2.11) and u be the function found in (3.17). Then up to a subsequence, $T_k(u_n) \to T_k(u)$ in $W^{1,p}_{loc}(\Omega)$.*

Proof. We are going to prove that

$$\lim_{n\to\infty} \int_\Omega |\nabla(T_k(u_n) - T_k(u))|^p \phi dx = 0.$$

For all positive $\phi \in C_0^\infty(\Omega)$. Let $\varphi_\lambda(s) = se^{\lambda s^2}$, $\lambda > 0$. As in [9], we will consider as a test function $\varphi_\lambda(T_k(u_n) - T_k(u))\phi$, where λ will be chosen later. In the sequel $\varepsilon(n)$ will denote any quantity converging to 0, as $n \to \infty$. From (2.11) we get

$$\begin{aligned}
&\int_\Omega \frac{b(x)}{(1+T_n(u_n))^\gamma} |\nabla u_n|^{p-2} \nabla u_n \nabla (T_k(u_n) - T_k(u)) \varphi_\lambda'(T_k(u_n) - T_k(u)) \phi dx \\
&\quad + B \int_\Omega \frac{u_n |\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} \varphi_\lambda(T_k(u_n) - T_k(u)) \phi dx \\
(\mathbf{A}) \qquad &= -\int_\Omega \frac{b(x)}{(1+T_n(u_n))^\gamma} |\nabla u_n|^{p-2} \nabla u_n \nabla \phi \varphi_\lambda(T_k(u_n) - T_k(u)) dx \\
(\mathbf{B}) \qquad &+ \int_\Omega T_n(f) \varphi_\lambda(T_k(u_n) - T_k(u)) \phi dx.
\end{aligned} \tag{4.17}$$

Using the Lebesgue Theorem for $(\mathbf{B})$. For $(\mathbf{A})$ it is sufficient to observe that ∇u_n converges weakly in some Sobolev space given by the statements of Lemmata 13-16, 18 and that

$$\frac{b(x)}{(1+T_n(u_n))^\gamma} |\nabla u_n|^{p-2} \nabla u_n \nabla \phi \varphi_\lambda(T_k(u_n) - T_k(u)),$$

is uniformly bounded with respect to n. As $n \to +\infty$, we get

$$(\mathbf{A}) = \varepsilon(n), \quad \text{and} \quad (\mathbf{B}) = \varepsilon(n). \tag{4.18}$$

We choose $\omega_\phi \subset\subset \Omega$, with $\text{supp}(\phi) \subset \omega_\phi$. Using (4.16) (Corollary 24), we have

$$B \int_\Omega \frac{u_n |\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} \varphi_\lambda(T_k(u_n) - T_k(u)) \phi dx$$

$$\geq -B\widetilde{c}_{\omega_\phi}\int_\Omega |\nabla T_k(u_n)|^p|\varphi_\lambda(T_k(u_n)-T_k(u))|\phi dx. \tag{4.19}$$

We remark that

$$\int_{\{u_n\geq k\}} \frac{b(x)|\nabla u_n|^{p-2}\nabla u_n}{(1+T_n(u_n))^\gamma}\nabla(T_k(u_n)-T_k(u))\varphi'_\lambda(T_k(u_n)-T_k(u))\phi dx = \varepsilon(n). \tag{4.20}$$

By (4.17)-(4.20), we have

$$\begin{aligned}\int_{\{u_n\leq k\}} \frac{b(x)|\nabla u_n|^{p-2}\nabla u_n}{(1+T_n(u_n))^\gamma}\nabla(T_k(u_n)-T_k(u))\varphi'_\lambda(T_k(u_n)-T_k(u))\phi \\ - B\widetilde{c}_{\omega_\phi}\int_\Omega |\nabla T_k(u_n)|^p|\varphi_\lambda(T_k(u_n)-T_k(u))|\phi dx \leq \varepsilon(n).\end{aligned} \tag{4.21}$$

Remark that

$$\int_{\{u_n\leq k\}} \frac{b(x)|\nabla T_k(u)|^{p-2}\nabla T_k(u)}{(1+T_n(u_n))^\gamma}\nabla(T_k(u_n)-T_k(u))\varphi'_\lambda(T_k(u_n)-T_k(u))\phi \to 0.$$

Adding the above quantity in both sides of (4.21) and by hypothesis (1.4) on b, we obtain

$$\begin{aligned}\int_{\{u_n\leq k\}} \frac{\alpha}{(1+k)^\gamma}|\nabla(T_k(u_n)-T_k(u))|^p\varphi'_\lambda(T_k(u_n)-T_k(u))\phi dx \\ - B\widetilde{c}_{\omega_\phi}\int_\Omega |\nabla T_k(u_n)|^p|\varphi_\lambda(T_k(u_n)-T_k(u))|\phi dx \leq \varepsilon(n).\end{aligned} \tag{4.22}$$

It is easy to prove that

$$\begin{aligned}\int_\Omega |\nabla T_k(u_n)|^p|\varphi_\lambda(T_k(u_n)-T_k(u))|\phi dx \\ \leq \int_{\{u_n\leq k\}} 2|\nabla(T_k(u_n)-T_k(u))|^p|\varphi_\lambda(T_k(u_n)-T_k(u))|\phi dx + \varepsilon(n).\end{aligned} \tag{4.23}$$

By (4.22) and (4.23), we get

$$\int_{\{u_n\leq k\}} \left[\frac{\alpha}{(1+k)^\gamma}\varphi'_\lambda(T_k(u_n)-T_k(u)) - 2B\widetilde{c}_{\omega_\phi}|\varphi_\lambda(T_k(u_n)-T_k(u))|\right]|\nabla(T_k(u_n)-T_k(u))|^p\phi \to 0. \tag{4.24}$$

Now, φ_λ has the following property, for $a,b>0$, $a\varphi'_\lambda(s)-b|\varphi_\lambda(s)|\geq \frac{a}{2}$ if $\lambda > \frac{b^2}{4a^2}$. Therefore there exists $\lambda>0$ such that

$$\frac{\alpha}{(1+k)^\gamma}\varphi'_k(s) - 2B\widetilde{c}_{\omega_\phi}|\varphi_\lambda(s)| \geq \frac{\alpha}{2(1+k)^\gamma} \quad \forall s\in\mathbb{R}.$$

Applying this inequality to the quantity (4.24), the statement of the theorem is proved.

By Remark 19 the solutions u_n to (2.11) are uniformly bounded in $W_0^{1,\sigma}(\Omega)$ and $W_0^{1,p}(\Omega)$ respectively; moreover ∇u_n converges to ∇u a.e. in Ω up to a subsequence, by Lemma 20. The solutions u_n satisfy

$$\int_\Omega \frac{b(x)}{(1+T_n(u_n))^\gamma}|\nabla u_n|^{p-2}\nabla u_n.\nabla\varphi + B\int_\Omega \frac{|\nabla u_n|^p u_n}{(u_n+\frac{1}{n})^{1+\theta}}\varphi = \int_\Omega T_n(f)\varphi.$$

For the proof of Theorem 3 we consider for φ a bounded $C_0^1(\Omega)$ function. For the proof of Theorem 5, φ is a $W_0^{1,p}(\Omega)$ function. To compute the limit of the first term in the case where u_n weakly converges to u in $W_0^{1,\sigma}(\Omega)$ (Theorem 3), with $\sigma < p$ it is sufficient to use that $\frac{b(x)}{(1+T_n(u_n))^\gamma}\nabla\varphi$ strongly converges to $\frac{b(x)}{(1+u)^\gamma}\nabla\varphi$ in $(L^r(\Omega))^N$ for every $r \geq 1$ and for every $\varphi \in C_0^1(\Omega)$, and $|\nabla u_n|^{p-2}\nabla u_n$ weakly converges to $|\nabla u|^{p-2}\nabla u$ in $L^{p'}(\Omega)$. In the case where u_n weakly converges to u in $W_0^{1,p}(\Omega)$, (Theorem 5), one uses that $\frac{b(x)}{(1+T_n(u_n))^\gamma}\nabla\varphi$ strongly converges to $\frac{b(x)}{(1+u)^\gamma}\nabla\varphi$ in $(L^p(\Omega))^N$ for every $\varphi \in L^\infty(\Omega)\cap W_0^{1,p}(\Omega)$.

We are going to prove that $\frac{|\nabla u_n|^p u_n}{(u_n+\frac{1}{n})^{1+\theta}}$ is equi-integrable. Let $E \subset\subset \omega \subset\subset \Omega$. Using (4.16), we have

$$\begin{aligned}\int_E \frac{|\nabla u_n|^p u_n}{(u_n+\frac{1}{n})^{1+\theta}}dx &\leq \int_{E\cap\{u_n\leq k\}} \frac{|\nabla u_n|^p u_n}{(u_n+\frac{1}{n})^{1+\theta}}dx + \int_{E\cap\{u_n\geq k\}} \frac{|\nabla u_n|^p u_n}{(u_n+\frac{1}{n})^{1+\theta}}dx \\ &\leq \widetilde{c}_\omega \int_{E\cap\{u_n\leq k\}} |\nabla T_k(u_n)|^p + \int_{\{u_n\geq k\}} \frac{|\nabla u_n|^p u_n}{(u_n+\frac{1}{n})^{1+\theta}}dx. \qquad (4.25)\end{aligned}$$

Now, if we choose $T_1(u_n - T_{k-1}(u_n))$ in problems (2.11) and dropping the non-negative operator term, we get

$$B\int_{\{u_n\geq k\}} \frac{|\nabla u_n|^p u_n}{(u_n+\frac{1}{n})^{1+\theta}}dx \leq \int_{\{u_n\geq k-1\}} f. \qquad (4.26)$$

Observe that there exists a constant $C > 0$ such that $\mu(\{u_n \geq k-1\}) \leq \frac{C}{k-1}$, as u_n are uniformly bounded in $L^1(\Omega)$. This implies that the right hand side of (4.26) converges to 0 as $k \to \infty$, uniformly with respect to n. We deduce that there exists $k_0 > 1$ such that

$$\int_{\{u_n\geq k\}} \frac{|\nabla u_n|^p u_n}{(u_n+\frac{1}{n})^{1+\theta}}dx \leq \frac{\varepsilon}{2}, \quad \forall k \geq k_0, \ \ \forall n \in \mathbb{N}. \qquad (4.27)$$

Moreover, since $T_k(u_n) \to T_k(u)$ in $W_{loc}^{1,p}(\Omega)$ by Lemma 25, there exist $n_\varepsilon, \delta_\varepsilon$ such that for every $E \subset\subset \Omega$ with $\mu(E) < \delta_\varepsilon$ we have

$$\int_{E\cap\{u_n\leq k\}} |\nabla T_k(u_n)|^p dx = \int_E |\nabla T_k(u_n)|^p dx \leq \frac{\varepsilon}{2\widetilde{c}_\omega}, \quad \forall n \geq n_\varepsilon. \qquad (4.28)$$

(4.25),(4.27) and (4.28) imply that $\frac{|\nabla u_n|^p u_n}{(u_n+\frac{1}{n})^{1+\theta}}$ is equi-integrable. Now, recall that $\frac{|\nabla u_n|^p u_n}{(u_n+\frac{1}{n})^{1+\theta}}$ converges a.e. to $\frac{|\nabla u|^p}{u^\theta} \in L^1(\Omega)$ by Corollary 23. By Vitali's Theorem we have the result.

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