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On $(r;t;s)$-nuclear polynomials

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Abstract. In this work we extend the concept of $(r;t;s)$-nuclear operators presented by Lapresté in (Studia math., T. LVII. 1976, 47 – 83) to n-homogeneous polynomials. Factorization and inclusion properties are described. Under some conditions, we also characterize the topological dual of the studied space.

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1. Introduction

Since the appearance of the paper of Lapresté [14] and of the work of Pietsch [19], the theory of nuclear operators and also its connections with another classes have been intensively studied [5, 11, 20].

The study of nuclear homogeneous polynomials goes back, at least, to Dineen [12] and Gupta [13], the recent development of nonlinear theory of nuclear operators transported this question to polynomials and multilinear mappings by, for example, Matos [15, 16, 17] and Cerna [9]. Motivated by these ideas and developments, in this work, we will be mainly concerned with polynomials, this means that this paper is devoted to generalize the concept of $(r;t;s)$-nuclear

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operators to polynomials and we will prove a relation between the topological dual of $(r;t;s)$-nuclear polynomials and the absolutely $(r';t;s)$-summing polynomials.

In the first work [1], the authors generalize the work presented by Cerna [9], which in turn generalizes the work presented by Lapresté [14], in the linear case, to the multilinear framework. The difference between the two works, both are subject to the generalization technique and in our current work, we generalize Lapresté's work [14] to homogeneous polynomials situation.

In fact, Matos has studied the two generalizations in a single article [15], [16],... but in recent years the two generalizations have been separated in two different articles and the examples are numerous [3, 4]....

This work is organized as follows. In Section 1, we recall some notations and basic facts on sequence spaces and polynomials on Banach spaces. In Section 2, we study and characterize the ideal of $(r;t;s)$-nuclear polynomials. In Section 3, we prove that: the topological dual of $\mathcal{P}_{\mathcal{N},(r;t;s)}\left({}^{n}E;F\right)$ is isomorphic isometrically to $\mathcal{P}^{n}_{as(r';t;s)}\left({}^{n}E^{*};F^{*}\right)$ if E^{*} has the λ-bounded approximation property(λ-BAP, for short see [17, Proposition 4.8]).

2. Preliminaries

The definitions and notations used in this work are, in general, standard. Let E, F, G, H be Banach spaces over $\mathbb{R}$. As usual, $\mathcal{L}(E;F)$ is the space of bounded linear operators from E to F.

Given $1 \leq p < \infty$ we will write $\ell_p^n(E)$ for the space of all sequences $(x_i)_{i=1}^n$ in E with the norm

$$\|(x_i)_{i=1}^n\|_p = \left(\sum_{i=1}^n \|x_i\|^p\right)^{\frac{1}{p}},$$

and $\ell_p^{n,w}(E)$ for the space of all sequences $(x_i)_{i=1}^n$ in E with the norm

$$\|(x_i)_{i=1}^n\|_{p,w} = \sup_{\|\phi\|_{E^*}\leq 1}\left(\sum_{i=1}^n |\phi(x_i)|^p\right)^{\frac{1}{p}},$$

where E^* denotes the topological dual of E. The closed unit ball of E will be denoted by B_E. Let $\ell_p(E)$ be the Banach space of all absolutely p-summable sequences $(x_i)_{i=1}^\infty$ in E with the norm

$$\|(x_i)_{i=1}^\infty\|_p = \left(\sum_{i=1}^\infty \|x_i\|^p\right)^{\frac{1}{p}}.$$

We denote by $\ell_p^w(E)$ the Banach space of all weakly p-summable sequences $(x_i)_{i=1}^\infty$ in E with the norm

$$\|(x_i)_{i=1}^\infty\|_{p,w} = \sup_{\|\phi\|_{E^*}\leq 1}\left(\sum_{i=1}^\infty |\phi(x_i)|^p\right)^{\frac{1}{p}},$$

Note that $\ell_p^w(E) = \ell_p(E)$ for some $1 \leq p < \infty$ if, and only if, E is finite dimensional [17] for details.

Given a continuous n-linear mapping $T \in \mathcal{L}(^nE;F)$, the map $P : E \longrightarrow F : P(x) = T(x;x;...;x)$ for every $x \in E$ is called a continuous n-homogeneous polynomial. The space of all continuous n-homogeneous polynomials from E to F will be denoted by $\mathcal{P}(^nE;F)$, and it becomes a Banach space under the norm

$$\| P \| = \sup\{\|P(x)\| : \|x\| \leq 1\} = \inf\left\{C : \|P(x)\| \leq C \|x\|^n, x \in E\right\}.$$

The (finite) linear combinations of the n-homogeneous polynomials $x \to \varphi(x)^n y$, where $\varphi \in E^*$ and $y \in F$, are called polynomials of finite type. We use $\mathcal{P}_f(^nE;F)$ to denote the space of finite type n-homogeneous polynomials.

An ideal of homogeneous polynomials (or polynomial ideal) $\mathcal{Q}$ is a subclass of the class of all continuous homogeneous polynomials between Banach spaces such that for all $n \in \mathbb{N}$ and Banach spaces E and F, the components $\mathcal{Q}(^nE,F) = \mathcal{P}(^nE,F) \cap \mathcal{Q}$ satisfy:
(i) $\mathcal{Q}(^nE,F)$ is a linear subspace of $\mathcal{P}(^nE,F)$ which contains the n-homogeneous polynomials of finite type.
(ii) The ideal property: if $u \in \mathcal{L}(G,E), P \in \mathcal{Q}(^nE,F)$ and $v \in \mathcal{L}(F,H)$, Then the composition $v \circ P \circ u$ is in $\mathcal{Q}(^nG,H)$.
Moreover, assume that $0 < \beta \leq 1$ and the function $\|.\|_{\mathcal{Q}} : \mathcal{Q} \to \mathbb{R}^+$ satisfies the following three conditions:
(i') $(\mathcal{Q}(^nE;F);\|.\|_{\mathcal{Q}})$ is a β-Banach (Banach) space for all E, F and n,
(ii') $\|P^n : \mathbb{K} \to \mathbb{K} : P^n(x) = x^n\|_{\mathcal{Q}} = 1$ for all n,
(iii') If $u \in \mathcal{L}(G,E), P \in \mathcal{Q}(^nE,F)$ and $v \in \mathcal{L}(F,H)$, then

$$\|v \circ P \circ u\|_{\mathcal{Q}} \leq \|v\| \|P\|_{\mathcal{Q}} \|u\|^n.$$

Then $(\mathcal{Q}, \|.\|_{\mathcal{Q}})$ is called a β-Banach (Banach) polynomial ideal, see for example [6, 7, 8, 10].

The polynomial ideal $\mathcal{Q}$ is said to be closed if each component $\mathcal{Q}(^nE;F)$ is a closed subspace of $\mathcal{P}(^nE;F)$.
Let us recall the definition of absolutely $(r;t;s)$-summing polynomials.

Definition 2.1. [2] *Let $0 < r,t,s \leq \infty$ with $\frac{1}{r} \leq \frac{n}{t} + \frac{1}{s}$. An n-homogeneous polynomial $P : E \longrightarrow F$ is absolutely $(r;t;s)$-summing if there is a constant $C > 0$ such that for any $x_1,...,x_k \in E$, k is an arbitrary natural number, and $\varphi_1,...,\varphi_k \in F^*$,*

$$\left\|\left(\varphi_i\left(P\left(x_i\right)\right)\right)_{i=1}^k\right\|_r \leq C \left\|\left(x_i\right)_{i=1}^k\right\|_{t,w}^n \left\|\left(\varphi_i\right)_{i=1}^k\right\|_{s,w}. \tag{2.1}$$

The class of such polynomials is denoted by $\mathcal{P}^n_{as,(r;t;s)}(^nE;F)$. The infimum of the C such that inequality (2.1) holds defines a complete norm $\|\cdot\|_{\mathcal{P}^n_{as,(r;t;s)}}$ if $r \geq 1$ (quasi-norm if $0 < r < 1$) on $\mathcal{P}^n_{as,(r;t;s)}(^nE;F)$.

3. $(r;t;s)$-nuclear n-Homogeneous Polynomials

We give in this section a natural generalization of $(r;t;s)$-nuclear linear operators to the category of polynomial mappings.

Definition 3.1. *Let $r \in]0,+\infty]$, $1 \leq t,s \leq \infty$ be such that*

$$\frac{1}{\beta} = \frac{1}{r} + \frac{n}{t} + \frac{1}{s} \geq 1.$$

An n-homogeneous polynomial $P : E \longrightarrow F$ is $(r;t;s)$-nuclear if it has a representation of the form:

$$P(x) = \sum_{j=1}^{\infty} \lambda_j \varphi_j^n(x)\, y_j \text{ for all } x \in E. \tag{3.1}$$

with $(\lambda_j)_j \in \ell_r$, if $r < \infty$ or $(\lambda_j)_j \in c_0$, if $r = +\infty$, $(\varphi_j)_j \in \ell_t^w(E^*)$ and $(y_j)_j \in \ell_s^w(F)$.

The vector space of all such operators is denoted by $\mathcal{P}_{\mathcal{N},(r;t;s)}({}^nE;F)$. We define on it the function

$$\mu_{p,(r;t;s)}(P) = \inf\left\{ \left\|(\lambda_j)_j\right\|_r \left\|(\varphi_j)_j\right\|_{t,w}^n \left\|(y_j)_j\right\|_{s,w} \right\}$$

where the infimum is taken over all representation of P as in (3.1).

Theorem 1. *Let $r \in]0,+\infty]$, $1 \leq t,s \leq \infty$ be such that*

$$\tfrac{1}{\beta} = \tfrac{1}{r} + \tfrac{n}{t} + \tfrac{1}{s} \geq 1.$$

We have, the function $\mu_{p,(r;t;s)}(.)$ is a β-norm on $\left(\underbrace{(E^*) \otimes_s \cdots \otimes_s (E^*)}_{n}\right) \otimes F$.

Proof. Let us P_1, P_2 two elements of $E \otimes F$, we can write:

$$P_i(x) = \sum_{j=1}^{\infty} \lambda_{i,j} \varphi_{i,j}^n(x)\, y_{i,j};\ i = 1,2.$$

It should, firstly, be shown that:

$$\mu_{p,(r;t;s)}(P_1 + P_2) \leq \left(\mu_{p,(r;t;s)}^{\beta}(P_1) + \mu_{p,(r;t;s)}^{\beta}(P_2)\right)^{\frac{1}{\beta}}.$$

For all $\epsilon > 0$. We have for all i in $\{1,2\}$

$$\begin{aligned}
\left\|\left(\varphi_{i,j}^k\right)_{j<n}\right\|_{t,w} &\leq \left(\mu_{p,(r;t;s)}^{\beta}(P_i) + \epsilon\right)^{\frac{1}{t}};\\
\left\|(y_{i,j})_{j<n}\right\|_{s,w} &\leq \left(\mu_{p,(r;t;s)}^{\beta}(P_i) + \epsilon\right)^{\frac{1}{s}},\\
\left\|(\lambda_{i,j})_{j<n}\right\|_{r} &\leq \left(\mu_{p,(r;t;s)}^{\beta}(P_i) + \epsilon\right)^{\frac{1}{r}},
\end{aligned}$$

and consequently:

$$\begin{aligned}\left\|\left(\varphi_{i,j}^{k}\right)_{i,j}\right\|_{t,w} &\leq \left(\mu_{p,(r;t;s)}^{\beta}(P_1)+\mu_{p,(r;t;s)}^{\beta}(P_2)+\epsilon\right)^{\frac{1}{t}};\\ \left\|\left(y_{i,j}\right)_{i,j}\right\|_{s,w} &\leq \left(\mu_{p,(r;t;s)}^{\beta}(P_1)+\mu_{p,(r;t;s)}^{\beta}(P_2)+\epsilon\right)^{\frac{1}{s}},\\ \left\|\left(\lambda_{i,j}\right)_{i,j}\right\|_{r} &\leq \left(\mu_{p,(r;t;s)}^{\beta}(P_1)+\mu_{p,(r;t;s)}^{\beta}(P_2)+\epsilon\right)^{\frac{1}{r}}.\end{aligned}$$

Then,

$$\left\|\left(\lambda_{i,j}\right)_{i,j}\right\|_{r}\left\|\left(\varphi_{i,j}^{k}\right)_{i,j}\right\|_{t,w}^{n}\left\|\left(y_{i,j}\right)_{i,j}\right\|_{s,w}\leq\left(\mu_{(r;t;s)}^{\beta}(P_1)+\mu_{(r;t;s)}^{\beta}(P_2)+\epsilon\right)^{\frac{1}{\beta}}.$$

Thus,

$$\mu_{p,(r;t;s)}(P_1+P_2)\leq\left(\mu_{p,(r;t;s)}^{\beta}(P_1)+\mu_{p,(r;t;s)}^{\beta}(P_2)\right)^{\frac{1}{\beta}}.$$

It is easy to note, that $\mu_{p,(r;t;s)}$ is positively homogeneous.

We have for $P=\sum_{i=1}^{\infty}\lambda_i\varphi_i^n(x)\,y_i$ and $P'=\sum_{j=1}^{\infty}\lambda_j\varphi_j'^n(x)\otimes y_j'$,

$$\begin{aligned}\left|\langle P,P'\rangle\right| &= \left|\sum_{i,j}\lambda_i\lambda_j\left(\left\langle x_i^k,x_j'^k\right\rangle\right)^n\left\langle y_i,y_j'\right\rangle\right|\\ 0<\beta\leq 1 &\leq \left|\sum_{i,j}\left|\lambda_i\right|^{\beta}\left|\lambda_j\right|^{\beta}\left(\left|\left\langle\varphi_i^k,\varphi_j'^k\right\rangle\right|^{\beta}\right)^n\left|\left\langle y_i,y_j'\right\rangle\right|^{\beta}\right|^{\frac{1}{\beta}};\end{aligned}$$

and, consequently, by the inequality of Hölder generalized:

$$\left|\langle P,P'\rangle\right|\leq\left\|\left(\left\langle\varphi_i^k,\varphi_j'^k\right\rangle\right)_{i,j}\right\|_{t,w}^{n}\left\|\left(\left\langle y_i,y_j'\right\rangle\right)_{i,j}\right\|_{s,w}\left\|\left(\lambda_i\lambda_j'\right)_{i,j}\right\|_{r}$$

i.e.

$$\left|\langle P,P'\rangle\right|\leq\mu_{p,(r;t;s)}(P)\cdot\mu_{p,(r;t;s)}(P').$$

Thus, $\mu_{p,(r;t;s)}(P)=0$ imply $P=0$.

We note that; *if* $\beta=1$, then $\mu_{p,(r;t;s)}$ *is a norm.*

Example 1. 1) By definition, continuous n-homogeneous polynomials of finite type P are in $\mathcal{P}_{\mathcal{N},(r;t;s)}\left({}^{n}E;F\right)$; for $P=\sum_{j=1}^{l}\lambda_j\varphi_j^n\otimes y_j$, we define

$$\mu_{f,p,(r;t;s)}\left(P\right)=\inf\left\{\left\|\left(\lambda_j\right)_{j=1}^{l}\right\|_r\left\|\left(\varphi_j\right)_{j=1}^{l}\right\|_{t,w}^{n}\left\|\left(y_j\right)_{j=1}^{l}\right\|_{s,w}\right\}.$$

2) If $\lambda=\left(\lambda_j\right)_{j\in\mathbb{N}}$ is in ℓ_r, if $1\leq r<\infty$, or in c_0, if $r=\infty$, then

$$\begin{array}{rcccl}\mathcal{D}_\lambda & : & \ell_t & \longrightarrow & \ell_{s'}\\ & & \mathcal{D}_\lambda\left(\left(\xi_j\right)_{j=1}^{\infty}\right) & = & \left(\lambda_j\xi_j^n\right)_{j\in\mathbb{N}},\end{array}$$

is in $\mathcal{P}_{\mathcal{N},(r;t;s)}\left({}^{n}\ell_t;\ell_{s'}\right)$ whenever $1\leq\frac{1}{r}+\frac{n}{t}+\frac{1}{s}$.

and

$$\mu_{p,(r;t;s)}\left(\mathcal{D}_\lambda\right)\leq\left\|\left(\lambda_j\right)_{j\in\mathbb{N}}\right\|_r.$$

To prove this, it is enough to take

$$\mathcal{D}_\lambda=\sum_{j=1}^{\infty}\lambda_j\pi_j^n\otimes e_j\ \text{where}\pi_j\left(\xi\right):=\xi_j\ for\ \xi=\left(\xi_j\right)_{j\in\mathbb{N}}\in\ell_t.$$

3) Let's take an example of polynomials of finite type $P=\varphi^n y$, where $\varphi\in E^*$ and $y\in F$, we have

$$\mu_{p,(r;t;s)}\left(\varphi^n y\right)=\left\|\varphi\right\|^n\left\|y\right\|.$$

In fact, by (3.1) it follows that

$$\mu_{p,(r;t;s)}\left(\varphi^n y\right)\geq\left\|\varphi\right\|^n\left\|y\right\|.$$

On the other hand it is clear that

$$\mu_{p,(r;t;s)}\left(\varphi^n y\right)\leq\left\|\varphi\right\|^n\left\|y\right\|.$$

Theorem 2. *(Factorization theorem). We consider $0<r\leq+\infty$ and $1\leq t,s\leq+\infty$ with $\frac{1}{r}+\frac{n}{t}+\frac{1}{s}\geq 1$. Let $P\in\mathcal{P}\left({}^{n}E;F\right)$. The following assertions are equivalent:*

(a) P is in $\mathcal{P}_{\mathcal{N},(r;t;s)}\left({}^{n}E;F\right)$.

(b) There are operators $u\in\mathcal{L}\left(E;\ell_t\right),v\in\mathcal{L}\left(\ell_{s'};F\right)$ and a polynomial $\mathcal{D}_\lambda\in\mathcal{P}\left({}^{n}\ell_t;\ell_{s'}\right)$ of the form $\mathcal{D}_\lambda(z)=\left(\lambda_j z_j^n\right)_{j=1}^{\infty}$ where $\lambda=\left(\lambda_j\right)_{j=1}^{\infty}\in\ell_r$ and $z=\left(z_j\right)_{j=1}^{\infty}\in\ell_t$, such that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{P} & F \\ \downarrow u & & \uparrow v \\ \ell_t & \xrightarrow{\mathcal{D}_\lambda} & \ell_{s'} \end{array}$$

Proof. $(a) \Longrightarrow (b)$. If P is in $\mathcal{P}_{\mathcal{N},(r;t;s)}\left({}^nE;F\right)$, there are bounded sequences $(\varphi_j)_j \subset E^*$ and $(y_j) \subset F$ such that (3.1) holds.

Define $u, \mathcal{D}_\lambda$ and v by $u(x) = \left(\varphi_j(x)\right)_{j=1}^{\infty}, v\left(\left(\xi_j\right)_j\right) = \sum_{j=1}^{\infty} y_j\xi_j$ and $\mathcal{D}_\lambda(z) = \left(\lambda_j z_j^n\right)_{j=1}^{\infty}$.

$(b) \Longrightarrow (a)$. By Examples 4, $\mathcal{D}_\lambda \in \mathcal{P}_{\mathcal{N},(r;t;s)}\left({}^n\ell_t;\ell_{s'}\right)$. Theorem $5-b$ implies that $P = v \circ \mathcal{D}_\lambda \circ u \in \mathcal{P}_{\mathcal{N},(r;t;s)}\left({}^nE;F\right)$. ■

Proposition 3. *The class of $(r;t;s)$-nuclear n-homogeneous polynomials with the function $\mu_{p,(r;t;s)}(.)$ is a β-Banach polynomial ideal.*

Proof. The proof is standard (For example see [6, 7, 10] to understand the proof).

Theorem 4. *(Inclusion theorem). Let $r \leq r_1$, $t_1 \leq t$, $s_1 \leq s$. Furthermore, suppose that*

$$n\left(\frac{1}{t_1} - \frac{1}{t}\right) + \left(\frac{1}{s_1} - \frac{1}{s}\right) \leq \frac{1}{r} - \frac{1}{r_1}. \tag{3.2}$$

Then

$$\mathcal{P}_{\mathcal{N},(r;t;s)}\left({}^nE;F\right) \subset \mathcal{P}_{\mathcal{N},(r_1;t_1;s_1)}\left({}^nE;F\right).$$

Proof. Put $\frac{1}{c} := \frac{1}{s_1} - \frac{1}{s}; \frac{1}{b} := \left(\frac{n}{t_1} - \frac{n}{t}\right) := \frac{n}{b_1}$ and $\frac{1}{r} := \frac{1}{c} + \frac{1}{b} + \frac{1}{a}$; from (3.2), we deduce $a \leq r_1$.

Let $P \in \mathcal{P}_{\mathcal{N},(r;t;s)}\left({}^nE;F\right)$. Given $\varepsilon > 0$, we choose a representation

$$P = \sum_{j=1}^{\infty} \lambda_j \varphi_j^n \otimes y_j$$

such that

$$\left\|(\lambda_j)\right\|_r \left\|(\varphi_j)\right\|_{t,w}^n \left\|(y_j)\right\|_{s,w} \leq (1+\varepsilon)\,\mu_{p,(r;t;s)}(P)$$

and $\lambda_j \geq 0$.

We have

$$P = \sum_{j=1}^{\infty} \lambda_j^{\frac{r}{a}} \left(\lambda_j^{\frac{r}{b_1}} \varphi_j\right)^n \left(\lambda_j^{\frac{r}{c}} y_j\right),$$

and

$$\left\|(\lambda_j^{\frac{r}{a}})\right\|_{r_1} \leq \left\|(\lambda_j^{\frac{r}{a}})\right\|_a = \left\|(\lambda_j)\right\|_r^{\frac{r}{a}},$$

and

$$\left\|(\lambda_j^{\frac{r}{b_1}}\varphi_j)\right\|_{t_1,w} \leq \left\|(\lambda_j^{\frac{r}{b_1}})\right\|_{b_1} \left\|(\varphi_j)\right\|_{t,w} = \left\|(\lambda_j)\right\|_r^{\frac{r}{b_1}} \left\|(\varphi_j)\right\|_{t,w}$$

and

$$\left\|(\lambda_j^{\frac{r}{c}}y_j)\right\|_{s_1,w} \leq \left\|(\lambda_j^{\frac{r}{c}})\right\|_c \left\|(y_j)\right\|_{s,w} = \left\|(\lambda_j)\right\|_r^{\frac{r}{c}} \left\|(y_j)\right\|_{s,w}.$$

Finally,

$$\left\|(\lambda_j^{\frac{r}{a}})\right\|_{r_1} \left\|(\lambda_j^{\frac{r}{b_1}}\varphi_j)\right\|_{t_1,w}^n \left\|(\lambda_j^{\frac{r}{c}}y_j)\right\|_{s_1,w} \leq \left\|(\lambda_j)\right\|_r^{\frac{r}{a}} \left\|(\lambda_j)\right\|_r^{\frac{r}{c}} \left\|(y_j)\right\|_{s,w} \left\|(\lambda_j)\right\|_r^{\frac{r}{b}} \left\|(\varphi_j)\right\|_{t,w}^n.$$

Then

$$\left\|(\lambda_j^{\frac{r}{a}})\right\|_{r_1} \left\|(\lambda_j^{\frac{r}{b_1}}\varphi_j)\right\|_{t_1,w}^n \left\|(\lambda_j^{\frac{r}{c}}y_j)\right\|_{s_1,w} \leq \left\|(\lambda_j)\right\|_r \left\|(\varphi_j)\right\|_{t,w}^n \left\|(y_j)\right\|_{s,w}.$$

Hence P is in $\mathcal{P}_{\mathcal{N},(r_1;t_1;s_1)}\left({}^nE;F\right)$, with

$$\mu_{p,(r_1;t_1;s_1)}\left(P\right) \leq (1+\varepsilon)\,\mu_{p,(r;t;s)}\left(P\right), \text{ for all } \varepsilon > 0. \qquad \blacksquare$$

4. Duality

We present now our main result of this work.

Theorem 5. *We consider $0 < r \leq +\infty$ and $1 \leq t,s \leq +\infty$ with $\frac{1}{r} + \frac{n}{t} + \frac{1}{s} \geq 1$. Let $\lambda \geq 1$, if E^* has the λ-BAP (λ-bounded approximation property), the topological dual of $\mathcal{P}_{\mathcal{N},(r;t;s)}\left({}^nE;F\right)$ is isomorphic isometrically to $\mathcal{P}^n_{as,(r';t;s)}\left({}^nE^*;F^*\right)$ through the mapping $\mathcal{B}(\Phi)(\varphi)(y) = \Phi(\varphi^n y)$, for all $y \in F, \varphi \in E^*$, and Φ in the required dual.*

Proof. We assume that $\Phi \in \left(\mathcal{P}_{\mathcal{N},(r;t;s)}\left({}^nE;F\right)\right)^*$. We try to show that $\mathcal{B}(\Phi)$ belongs to $\mathcal{P}^n_{as,(r';t;s)}\left({}^nE^*;F^*\right)$. We consider $m \in \mathbb{N}$ and $\varphi_j \in E^*$ for $j = 1,...,m$. There is $(\lambda_j)_{j=1}^m \in \ell_r^m$ such that $\left\|(\lambda_j)_{j=1}^m\right\|_r = 1$, and

$$\left\|\left(\mathcal{B}\left(\Phi\right)\left(\varphi_j\right)\right)_{j=1}^m\right\|_{r'} = \sum_{j=1}^m \lambda_j \left\|\mathcal{B}\left(\Phi\right)\left(\varphi_j\right)\right\| \tag{4.1}$$

For each $\varepsilon > 0$, we can find $b_j \in F$, $\left\|b_j\right\| = 1, j = 1,...,m$, such that

$$(4.1) \leq \varepsilon + \sum_{j=1}^m \lambda_j \left|\mathcal{B}\left(\Phi\right)\left(\varphi_j\right)\left(b_j\right)\right|;$$

then

$$(4.1) \leq \varepsilon + \sum_{j=1}^{m} \lambda_j sgn\left(\mathcal{B}\left(\Phi\right)\left(\varphi_j\right)\left(b_j\right)\right)\mathcal{B}\left(\Phi\right)\left(\varphi_j\right)\left(b_j\right).$$

By the definition of $\mathcal{B}(\Phi)$, we can write

$$\begin{aligned}(4.1) &\leq \varepsilon + \sum_{j=1}^{m} \lambda_j sgn\left(\mathcal{B}\left(\Phi\right)\left(\varphi_j\right)\left(b_j\right)\right)\Phi(\varphi_j^n b_j)\\ &\leq \varepsilon + \Phi\left(\sum_{j=1}^{m} \lambda_j sgn\left(\mathcal{B}\left(\Phi\right)\left(\varphi_j\right)\left(b_j\right)\right)\varphi_j^n b_j\right).\end{aligned}$$

And by the continuity of Φ, we have

$$\begin{aligned}(4.1) &\leq \varepsilon + \|\Phi\|\left\|\left(\lambda_j sgn\left(\mathcal{B}\left(\Phi\right)\left(\varphi_j\right)\left(b_j\right)\right)\right)_{j=1}^{m}\right\|_r \left\|\left(\varphi_j\right)_{j=1}^{m}\right\|_{t,w}^{n} \left\|\left(b_j\right)_{j=1}^{m}\right\|_{s,w}\\ &= \varepsilon + \|\Phi\|\left\|\left(\varphi_j\right)_{j=1}^{m}\right\|_{t,w}^{n} \left\|\left(b_j\right)_{j=1}^{m}\right\|_{s,w}.\end{aligned}$$

This shows that $\mathcal{B}\left(\Phi\right) \in \mathcal{P}^n_{as,(r';t;s)}\left({}^nE^*;F^*\right)$ and

$$\|\mathcal{B}\left(\Phi\right)\| \leq \|\Phi\|.$$

Now, we consider $P \in \mathcal{P}^n_{as,(r';t;s)}\left({}^nE^*;F^*\right)$ and define the linear functional Ψ_P on the space $\left(\mathcal{P}_f\left({}^nE;F\right);\mu_{f,p,(r;t;s)}\left(.\right)\right)$ by

$$\Psi_P\left(S\right) = \sum_{j=1}^{l} \lambda_j P(\varphi_j)\left(b_j\right)$$

for every $S \in \mathcal{P}_f\left({}^nE;F\right)$ of the form

$$S = \sum_{j=1}^{l} \lambda_j \varphi_j^n \otimes b_j$$

By Hölder's inequality, we get

$$\left|\Psi_P\left(S\right)\right| \leq \left\|\left(\lambda_j\right)_{j=1}^{l}\right\|_r \left\|\left(P(\varphi_j)\left(b_j\right)\right)_{j=1}^{l}\right\|_{r'}$$

Since P is belongs to $\mathcal{P}^n_{as,(r';t;s)}\left({}^nE^*;F^*\right)$, we find

$$\left|\Psi_P\left(S\right)\right| \leq \|P\|_{\mathcal{P}^n_{as,(r';t;s)}} \left\|\left(\lambda_j\right)_{j=1}^{l}\right\|_r \left\|\left(\varphi_j\right)_{j=1}^{l}\right\|_{t,w}^{n} \left\|\left(b_j\right)_{j=1}^{l}\right\|_{s,w}$$

This implies that

$$\left|\Psi_P\left(S\right)\right| \leq C\|P\|_{\mathcal{P}^n_{as,(r';t;s)}}.\mu_{f,p,(r;t;s)}\left(S\right).$$

By the λ-BAP property of E^*, $\mu_{f,p,(r;t;s)}$ and $\mu_{p,(r;t;s)}$ coincide on $\mathcal{P}_f\left({}^nE;F\right)$ (see the argument of [17, Proposition 7.2.14]). We conclude that Ψ_P is continuous on $\mathcal{P}_f\left({}^nE;F\right)$ for $\mu_{f,p,(r;t;s)}\left(.\right)$ and

$$\left\|\Psi_P\right\| \leq \left\|P\right\|_{\mathcal{P}^n_{as,(r';t;s)}}.$$

By the density of $\mathcal{P}_f\left({}^nE;F\right)$ in $\mathcal{P}_{\mathcal{N},(r;t;s)}\left({}^nE;F\right)$ (see analogous result in [17, Proposition 7.2.10]), we extend Ψ_P to a continuous functional $\widetilde{\Psi}_P$ on $\mathcal{P}_{\mathcal{N},(r;t;s)}\left({}^nE;F\right)$ in a unique way, with

$$\left\|\widetilde{\Psi}_P\right\| \leq \left\|P\right\|_{\mathcal{P}^n_{as,(r';t;s)}}.$$

Now, we note that $\mathcal{B}(\widetilde{\Psi}_P) = P$. ■

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