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Capacitary characterization of variable exponent Sobolev trace spaces

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Abstract. Let $\Omega \subset \mathbb{R}^n$ be an open set. We give a new characterization of zero trace functions $f \in \mathscr{C}(\overline{\Omega}) \cap \mathcal{W}_0^{1,p(.)}(\Omega)$. If in addition Ω is bounded, then we give a sufficient condition for which the mapping $f \mapsto \mathcal{L}^{\Omega}_{p(.),f}$ from a set of real extended functions $f : \partial\Omega \longrightarrow \mathbb{R}$ to the nonlinear harmonic space $(\Omega, \mathcal{H}_{\mathcal{L}_{p(.)}})$ is injective, where $\mathcal{L}^{\Omega}_{p(.),f}$ denotes the Perron-Wiener-Brelot solution for the Dirichlet problem:

$$\begin{cases} \mathcal{L}_{\mathbf{p}(.)}u := -\Delta_{p(.)}u + \mathcal{B}(.,u) = 0 & \text{in } \Omega; \\ u = f & \text{on } \partial\Omega, \end{cases}$$

where $\mathcal{B}$ is a given Carathéodory function satisfies some structural conditions.

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1. Introduction

This paper has dual goals. First, it aims at using potential analysis to give a characterization of zero trace functions $f \in \mathscr{C}(\overline{\Omega}) \cap \mathcal{W}_0^{1,p(.)}(\Omega)$. A second goal is to discuss the axiomatic nonlinear potential theory associated with the perturbed $p(.)$-Laplacian operator

$$\mathcal{L}_{\mathbf{p}(.)}u := -\Delta_{p(.)}u + \mathcal{B}(.,u),$$

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where p is a measurable function, $\Delta_{p(.)} := \operatorname{div}\left(|\nabla u|^{p(.)-2}\nabla\right)$ is the $p(.)$-Laplacian operator and $\mathscr{B}$ is a given Carathéodory function satisfies some structural conditions. In the present introduction we motivate the question that we address and we state the main results.

Function spaces with variable exponents have been the subject of intensive investigation in recent years. Examples of such spaces include Lebesgue and Sobolev spaces with variable exponents. Introduced by Orlicz [17] in 1931, their properties were further developed by Nakano [16] as special cases of the theory of modular spaces. In the ensuing decades they were primarily considered as important examples of modular spaces or the class of Musielak–Orlicz spaces. Initially of theoretical interest, by the end of the twentieth century these function spaces moved beyond theory and into the area of variational problems and studies of the $p(x)$-Laplacian operator, which in turn further fueled the development of this theory. For more details on these spaces, see the monograph [10].

We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space. The ball with radius r and center $x \in \mathbb{R}^n$ will be denoted by $B(x,r)$. We denote by $L_0(\Omega)$ the space of all $\mathbb{R}$-valued measurable functions on Ω.

Next, we introduce variable exponent Lebesgue and Sobolev spaces on open sets Ω of $\mathbb{R}^n$. Let $p : \Omega \longrightarrow [1,\infty)$ be a measurable function (called the variable exponent on Ω). The set of variable exponent on Ω is denoted by $\mathcal{P}(\Omega)$. We set $p^+ = \operatorname{esssup}_{x\in\Omega} p(x)$ and $p^- = \operatorname{essinf}_{x\in\Omega} p(x)$. The variable exponent Lebesgue space $L^{p(.)}(\Omega)$ is the family of the equivalence classes of functions defined by

$$L^{p(.)}(\Omega) := \left\{ u \in L^0(\Omega) : \rho_{p(.)}(\lambda u) = \int_\Omega |\lambda u(x)|^{p(x)}\ dx < \infty, \text{ for some } \lambda > 0 \right\}.$$

The function $\rho_{p(.)} : L^{p(.)}(\Omega) \longrightarrow [0,\infty)$ is called the modular of the space $L^{p(.)}(\Omega)$. We define a norm, the so-called Luxembourg norm, in this space by

$$\|u\|_{L^{p(.)}} = \inf\left\{ \lambda > 0 : \rho_{p(.)}\left(\frac{1}{\lambda}u\right) \le 1 \right\}.$$

We define the variable exponent Sobolev space $\mathcal{W}^{1,p(.)}(\Omega)$ as follows:

$$\mathcal{W}^{1,p(.)}(\Omega) := \left\{ u \in L^{p(.)}(\Omega), |\nabla u| \in L^{p(.)}(\Omega) \right\}.$$

The space $\mathcal{W}^{1,p(.)}(\Omega)$ is endowed with the norm

$$\|u\|_{\mathcal{W}^{1,p(.)}(\Omega)} = \|u\|_{L^{p(.)}(\Omega)} + \|\nabla u\|_{L^{p(.)}(\Omega)}.$$

We define a modular on $\mathcal{W}^{1,p(.)}(\Omega)$ by

$$\rho^\Omega_{1,p(.)}(u) := \int_\Omega |u(x)|^{p(x)}\ dx + \int_\Omega |\nabla u(x)|^{p(x)}\ dx,$$

when $\Omega = \mathbb{R}^n$, we denote it by $\rho_{1,p(.)}$. The definition of the space $L^{p(.)}(\mathbb{R}^n)$ and $\mathcal{W}^{1,p(.)}(\mathbb{R}^n)$ is analogous to $L^{p(.)}(\Omega)$ and $\mathcal{W}^{1,p(.)}(\Omega)$; one just changes every occurrence of Ω by $\mathbb{R}^n$. By $\mathcal{W}^{1,p(.)}_0(\Omega)$, we represent the closure in $\mathcal{W}^{1,p(.)}(\Omega)$ of the set of all those functions in $\mathcal{W}^{1,p(.)}(\Omega)$ with compact support in Ω, clearly $\mathscr{C}^\infty_0(\Omega) \subset \mathcal{W}^{1,p(.)}_0(\Omega)$. In [10] it is proved that the space

$\mathcal{W}_0^{1,p(.)}(\Omega)$ is a closed Banach subspace of $\mathcal{W}^{1,p(.)}(\Omega)$. We design by $\mathcal{C}_c(\overline{\Omega})$ the space of continuous functions on $\overline{\Omega}$ with compact support in $\overline{\Omega}$.

One of the most important motivations for the theory of Sobolev spaces with variable exponent comes from nonlinear potential theory. A typical task of nonlinear potential theory in variable exponent Sobolev spaces is to find a $p(.)$-harmonic function on a bounded open set $\Omega \subset \mathbb{R}^n$ which continuously extends the given boundary data $f \in \mathcal{C}(\partial\Omega)$ so that the $p(.)$-Laplace equation $\Delta_{p(.)}u = 0$ is satisfied in Ω and $u = f$ on $\partial\Omega$. This is the $p(.)$-Dirichlet problem. Its reformulation in terms of Sobolev spaces is to extend $f \in \mathcal{C}(\partial\Omega)$ to $u \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{W}^{1,p(.)}(\Omega)$ with $u - f \in \mathcal{W}_0^{1,p(.)}(\Omega)$. Moreover, a partial differential equation is in many cases solvable if and only if the boundary values are in the corresponding trace space. Therefore useful characterizations of trace space are desirable.

In 2007, L. Diening and Hästö present a simple approach to trace spaces; see [8], their philosophy is to move away as little as possible from the definition of trace space as consisting of those functions which can be extended to the whole space. To that end they defined an intrinsic norm for the trace space in terms of a an sharp operator of Hardy-Littlewood type. Under the assumption that the variable exponent p is globally log-Hölder continuous, they showed that the trace space depends only on the values of p on the boundary. Moreover, they characterized $\mathcal{W}_0^{1,p(.)}(\Omega)$ in terms of traces; see also[9, 11, 12].

Our main concern in this paper is to determine for which open subsets Ω of $\mathbb{R}^n$ that every $f \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{W}_0^{1,p(.)}(\Omega)$ is zero everywhere on $\partial\Omega$?

Our approach is in terms of a $p(.)$-relative capacity. For a measurable function $p : \Omega \longrightarrow [1, \infty)$, we let

$$\tilde{\mathcal{W}}^{1,p(.)}(\Omega) := \overline{\mathcal{W}^{1,p(.)}(\Omega) \cap \mathcal{C}_c(\overline{\Omega})}^{\mathcal{W}^{1,p(.)}(\Omega)}.$$

We define the $p(.)$-relative capacity with respect to Ω for an arbitrary subset E of $\overline{\Omega}$ by

$$C_{p(.)}^{\overline{\Omega}}(E) := \inf\left\{\rho_{1,p(.)}^{\Omega}(u) : u \in \tilde{\mathcal{W}}^{1,p(.)}(\Omega), \exists O \text{ open}, E \subset O \text{ and } u \geq 1 \text{ a.e on } \Omega \cap O\right\}.$$

For example in the case of the space $\tilde{\mathcal{W}}^{1,2}(\Omega) := \overline{\mathcal{W}^{1,2}(\Omega) \cap \mathcal{C}_c(\overline{\Omega})}^{\mathcal{W}^{1,2}(\Omega)}$, it is the capacity induced by the usual regular Dirichlet form on $L^2(\Omega, m)$ with domain $\tilde{\mathcal{W}}^{1,2}(\Omega)$, where m is the measure defined on $(\overline{\Omega}, \mathcal{B}(\overline{\Omega}))$ by $m(E) := \lambda_n(E \cap \Omega)$; see [13].

Notice that our $p(.)$-relative capacity $C_{p(.)}^{\overline{\Omega}}$ defined with the space $\tilde{\mathcal{W}}^{1,p(.)}(\Omega)$ slightly differs from the one introduced in [10, 14].

The main results of this paper are summarized in the following theorem:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Then every function in $\mathcal{W}_0^{1,p(.)}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is zero everywhere on the boundary $\partial\Omega$ if and only if $C_{p(.)}^{\overline{\Omega}}(B(x,r) \cap \partial\Omega) > 0$ for all $x \in \partial\Omega$.*

To get a corresponding result for nonlinear potential theory, let us suppose that Ω is bounded and assume that the variable exponent p satisfies the logarithmic Hölder continuity condition

with $1 < p^- \leq p(x) \leq p^+ < n$. Let $f \in \mathscr{C}(\partial\Omega)$, we consider the Dirichlet problem:

$$\begin{cases} \mathscr{L}_{\mathbf{p}(.)}u := -\Delta_{p(.)}u + \mathscr{B}(.,u) = 0 & \text{in } \Omega; \\ u = f & \text{on } \partial\Omega, \end{cases}$$

and we define the $\mathcal{H}_{\mathscr{L}_{p(.)}}$-harmonic sheaf as follows:

$$\mathcal{H}_{\mathscr{L}_{p(.)}}(\Omega) = \left\{ u \in \mathscr{C}(\Omega) : \mathscr{L}_{p(.)}u = 0 \right\}.$$

Element in the set $\mathcal{H}_{\mathscr{L}_{p(.)}}(\Omega)$ are called $\mathscr{L}_{p(.)}$-harmonic in Ω. We call $f \mapsto \mathscr{L}^{\Omega}_{p(.),f}$ the Perron-Wiener-Brelot operator from a set of real extended functions $f : \partial\Omega \longrightarrow \mathbb{R}$ to the nonlinear harmonic space $(\Omega, \mathcal{H}_{\mathscr{L}_{p(.)}})$ of the $\mathscr{L}_{p(.)}$-harmonic functions in Ω. As a consequence of Theorem 1.1 , we get the following result:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Assume that $C^{\overline{\Omega}}_{p(.)}(B(x,r) \cap \partial\Omega) > 0$ for all $x \in \partial\Omega$. Then the mapping $f \mapsto \mathscr{L}^{\Omega}_{p(.),f}$ from $\mathscr{C}(\partial\Omega)$ into $(\Omega, \mathcal{H}_{\mathscr{L}_{p(.)}})$ is injective.*

We now give the structure of the current paper. More detailed descriptions appear at the beginning of each section.

We start in Section 2 by study fine properties of Sobolev functions in $\tilde{\mathcal{W}}^{1,p(.)}(\Omega)$ as they are used to obtain our main results. In Section 3, we give a characterization of zero trace functions in $\mathcal{W}^{1,p(.)}_0(\Omega) \cap \mathscr{C}(\overline{\Omega})$. In Section 4, we apply our results to Dirichlet problem in nonlinear harmonic space.

2. $p(.)$-relative quasi-continuous representative of equivalence class of Sobolev functions

Sobolev functions are defined only up to Lebesgue measure zero and thus it is not always clear how to use their pointwise properties. But one can also think of some representative of this equivalence class, perhaps defined at all points outside a set of measure zero.

In this section, we show that the equivalent class of Sobolev functions in $\tilde{\mathcal{W}}^{1,p(.)}(\Omega)$ is $p(.)$-relatively quasi-continuous. It turns out that Sobolev functions are defined up to a set of $p(.)$-relative capacity zero.

We follow the idea of the proof of the corresponding result in $\mathcal{W}^{1,p(.)}(\Omega)$; see [10]. The set function $E \mapsto C^{\overline{\Omega}}_{p(.)}$ has the following properties:

- (C_1) $C^{\overline{\Omega}}_{p(.)}(\varnothing) = 0$;
- (C_2) If $E_1 \subset E_2 \subset \Omega_2 \subset \Omega_1$, then

$$C^{\overline{\Omega}_1}_{p(.)}(E_1) \leqslant C^{\overline{\Omega}_1}_{p(.)}(E_2);$$

- (C_3) If $K_1 \supset K_2 \supset K_3 \ldots$ are compact subsets of Ω, then

$$C^{\overline{\Omega}}_{p(.)}(\cap_{i=1}^{\infty} K_i) = \lim_{i\to\infty} C^{\overline{\Omega}}_{p(.)}(K_i);$$

- (C_4) If K_1 and K_2 are compact subsets of Ω, then
$$C^{\overline{\Omega}}_{p(.)}(K_1 \cup K_2) + C^{\overline{\Omega}}_{p(.)}(K_1 \cap K_2) \leqslant C^{\overline{\Omega}}_{p(.)}(K_1) + C^{\overline{\Omega}}_{p(.)}(K_2);$$
- (C_5) If $E_1 \subset E_2 \ldots$ are subsets of Ω, then
$$C^{\overline{\Omega}}_{p(.)}(\cup_{i=1}^{\infty} E_i) = \lim_{i\to\infty} C^{\overline{\Omega}}_{p(.)}(E_i);$$
- (C_6) For $E_i \subset \Omega$, $i \in \mathbb{N}$, we have
$$C^{\overline{\Omega}}_{p(.)}(\cup_{i=1}^{\infty} E_i) \leqslant \sum_{i=1}^{\infty} C^{\overline{\Omega}}_{p(.)}(E_i).$$

This means that the $p(.)$-relative capacity $C^{\overline{\Omega}}_{p(.)}$ is an outer measure and a Choquet capacity.

A set $P \subset \overline{\Omega}$ is called $p(.)$-relatively polar if $C^{\overline{\Omega}}_{p(.)}(P) = 0$. We say that a property holds $p(.)$-relatively quasi-everywhere ($p(.)$-r.q.e. , for short) on a set $A \subset \overline{\Omega}$ if there exists a $p(.)$-relatively polar set $P \subset A$ such that the property holds everywhere on $A \setminus P$. A function $u : \overline{\Omega} \longrightarrow \mathbb{R}$ is said to be $p(.)$-relatively quasi-continuous ($(p(.)$-r.q.c., for short) if for every $\varepsilon > 0$, there exists a relatively open set $O \subset \overline{\Omega}$ such that $C^{\overline{\Omega}}_{p(.)}(O) < \varepsilon$ and u is continuous on $\overline{\Omega} \setminus O$.

By definition, $\mathcal{W}^{1,p(.)}(\Omega) \cap \mathcal{C}_c(\overline{\Omega})$ is dense in $\tilde{\mathcal{W}}^{1,p(.)}(\Omega)$ which is complete Banach space. The next result gives a way to find a $p(.)$-relatively quasi-everywhere converging subsequence.

Theorem 2.1. *Let $p \in \mathcal{P}(\Omega)$ satisfy $1 < p^- \leqslant p^+ < \infty$. For each Cauchy sequence with respect to the $\mathcal{W}^{1,p(.)}(\Omega)$-norm of functions in $\tilde{\mathcal{W}}^{1,p(.)}(\Omega)$, there exists a subsequence which converges $p(.)$-r.q.e. in $\overline{\Omega}$. Moreover, the convergence is uniform outside a set of arbitrary small $p(.)$-relative capacity.*

Proof. Let (u_i) be a Cauchy sequence in $\tilde{\mathcal{W}}^{1,p(.)}(\Omega)$. Without loss of generality, we denote again by (u_i) the subsequence of (u_i) such that:
$$\|u_{i+1} - u_i\|_{W^{1,p(.)}(\Omega)} \leq \frac{1}{4^i},\ i \in \mathbb{N}.$$
Put
$$G_i := \left\{ x \in \overline{\Omega} :\ |u_{i+1}(x) - u_i(x)| > \frac{1}{2^i},\ i \in \mathbb{N} \right\}$$
and
$$G_k := \bigcup_{i=k}^{\infty} G_i.$$
Hence G_i is an open set in $\overline{\Omega}$ and $2^i\,|u_{i+1}(x) - u_i(x)| > 1$ on G_i. By ([10, Proposition 8.1.9]), we deduce that $2^i\,|u_{i+1} - u_i| \in \tilde{\mathcal{W}}^{1,p(.)}(\Omega)$ and by assumption
$$\left\| 2^i\,|u_{i+1} - u_i| \right\|_{W^{1,p(.)}(\Omega)} \leqslant \frac{1}{2^i} \leqslant 1.$$

By the unit ball property ([10, Lemma 2.1.14]) we get that:

$$\rho_{1,p(.)}(2^i\,|u_{i+1}-u_i|) \leqslant 2^i\,\|u_{i+1}-u_i\|_{W^{1,p(.)}(\Omega)} \leqslant \frac{1}{2^i}.$$

Consequently

$$C^{\overline{\Omega}}_{p(.)}(G_i) \leqslant \frac{1}{2^i}.$$

The sub-additivity of the $p(.)$-relative capacity implies that:

$$\begin{aligned} C^{\overline{\Omega}}_{p(.)}(G_k) &= C^{\overline{\Omega}}_{p(.)}\left(\bigcup_{i=k}^{\infty} G_i\right) \\ &\leqslant \sum_{i=k}^{\infty} C^{\overline{\Omega}}_{p(.)}(G_i) \\ &\leqslant \sum_{i=k}^{\infty} \frac{1}{2^i} = \frac{1}{2^{k-1}}. \end{aligned}$$

Hence

$$\begin{aligned} C^{\overline{\Omega}}_{p(.)}\left(\bigcap_{k=1}^{\infty} G_k\right) &= C^{\overline{\Omega}}_{p(.)}\left(\bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty} G_i\right) \\ &\leqslant \lim_{k\to\infty} C^{\overline{\Omega}}_{p(.)}\left(\bigcup_{i=k}^{\infty} G_i\right) \\ &\leqslant \lim_{k\to\infty} \frac{1}{2^{k-1}} = 0. \end{aligned}$$

Thus

$$C^{\overline{\Omega}}_{p(.)}\left(\bigcap_{k=1}^{\infty} G_k\right) = 0.$$

Consequently $\bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty} G_i$ is a $p(.)$-relatively polar set. Moreover, (u_i) converges pointwise in $\overline{\Omega}\setminus\bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty} G_i$. Since $|u_{i+1}(x)-u_i(x)| \leqslant \frac{1}{2^i}$ in $\overline{\Omega}\setminus\cap_{k=1}^{\infty}\cup_{i=k}^{\infty} G_i$ for all $i \geq k$, we have that (u_i) is a sequence of continuous functions on $\overline{\Omega}$ which converges uniformly in $\overline{\Omega}\setminus G_k$.

Theorem 2.2. *Let $p \in \mathcal{P}(\Omega)$ with $1 < p^- \leqslant p^+ < \infty$. Then for every $u \in \tilde{\mathcal{W}}^{1,p(.)}(\Omega)$, there exists a unique (up to a $p(.)$-relative polar set) $p(.)$-r.q.c. function $\tilde{u} : \overline{\Omega} \longrightarrow \mathbb{R}$ such that $\tilde{u} = u$ a.e. in Ω.*

Proof. Let $u \in \tilde{\mathcal{W}}^{1,p(.)}(\Omega)$. There exists a sequence (u_i) in $\mathcal{W}^{1,p(.)}(\Omega)\cap\mathscr{C}_c(\overline{\Omega})$ such that $u_i \longrightarrow u$ in $\mathcal{W}^{1,p(.)}(\Omega)$. By Theorem 2.1, there exists a subsequence, we denote it again by (u_i), which converges $p(.)$-r.q.e. in $\overline{\Omega}$ and uniformly outside a $p(.)$-relative polar set. Let $\tilde{u}$ be the pointwise limit of (u_i). By the uniform convergence, we get that $\tilde{u} : \overline{\Omega} \longrightarrow \mathbb{R}$ is $p(.)$-r.q.c. and $\tilde{u} = u$ a.e. in Ω. For the uniqueness, we assume that there exists another $\tilde{v}$ $p(.)$-r.q.c. in $\overline{\Omega}$ such that $\tilde{v} = u$ a.e. in Ω. Hence $\tilde{u} - \tilde{v} = 0$ a.e. in Ω and $\tilde{u} - \tilde{v} = 0$ is $p(.)$-r.q.c. in $\overline{\Omega}$.

Corollary 2.1. *Let $p \in \mathcal{P}(\Omega)$ satisfy $1 < p^- \leqslant p^+ < \infty$. Let (u_i) be a sequence of $p(.)$-r.q.c. functions in $\tilde{\mathcal{W}}^{1,p(.)}(\Omega)$ which converges to a $p(.)$-r.q.c. function $u \in \tilde{\mathcal{W}}^{1,p(.)}(\Omega)$. Then there exists a subsequence which converges $p(.)$-r.q.e. to u in $\overline{\Omega}$.*

Proof. Let (u_{i_k}) be a subsequence of (u_i) such that:

$$\sum_{k=1}^{\infty} 2^{i_k} \left\| u_{i_k} - u \right\|_{W^{1,p(.)}(\Omega)} \leq 1.$$

Put

$$P := \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} G_k,$$

where

$$G_k := \left\{ x \in \overline{\Omega} : \ \left| u_{i_k}(x) - u(x) \right| > \frac{1}{2^{i_k}} \right\}.$$

There exists $j_0 \in \mathbb{N}$ such that:

$$\left| u_{i_k}(x) - u(x) \right| \leqslant \frac{1}{2^{i_k}}, \ \forall \ k \geq j_0.$$

Hence (u_{i_k}) converges uniformly in $\overline{\Omega} \setminus \bigcup_{k=j_0}^{\infty} G_k$ and everywhere in $\overline{\Omega} \setminus P$. By the same way in the proof of Theorem 2.1, we get that:

$$C_{p(.)}^{\overline{\Omega}}(P) = 0.$$

Hence P is $p(.)$-relatively polar set and the proof is finished.

3. Characterization of zero trace spaces

We now turn our attention to the question of traces of Sobolev functions on the boundary of the set of definition. According to Theorem 2.2, every function u in $\tilde{\mathcal{W}}^{1,p(.)}(\Omega)$ is defined $p(.)$-relatively quasi-everywhere. Moreover, if $E \subset \mathbb{R}^n$ with $C_{p(.)}^{\overline{\Omega}}(E) > 0$, then the trace of u to E is the restriction to E of any $p(.)$-relatively quasi-continuous representative of u.

The purpose of this section is to give a new characterization of variable exponent Sobolev zero trace space $\mathcal{W}_0^{1,p(.)}(\Omega)$. We begin by two useful lemmas.

Lemma 3.1. *Let $u \in \mathcal{W}^{1,p(.)}(\Omega)$. Assume that u has compact support in Ω. Then $u \in \mathcal{W}_0^{1,p(.)}(\Omega)$.*

Proof. Let $u \in \mathcal{W}^{1,p(.)}(\Omega)$ and let $\psi \in \mathscr{C}_0^{\infty}(\Omega)$ be such that $\psi = 1$ on the support of u. If a sequence $(\psi_i)_i$ converges to u in $u \in \mathcal{W}^{1,p(.)}(\Omega)$, then $(\psi\psi_i)_i$ converges to $\psi u = u$ in $\mathcal{W}^{1,p(.)}(\Omega)$. Thus $\psi u \in \mathcal{W}_0^{1,p(.)}(\Omega)$ and hence $u \in \mathcal{W}_0^{1,p(.)}(\Omega)$.

A subspace I of $\mathcal{W}^{1,p(.)}(\Omega)$ is called an ideal if for $u \in I$, $v \in \mathcal{W}^{1,p(.)}(\Omega)$, $|v| \leqslant |u|$ a.e. implies that $v \in I$.

Lemma 3.2. *The space $\mathcal{W}_0^{1,p(.)}(\Omega)$ is a closed ideal in $\mathcal{W}^{1,p(.)}(\Omega)$.*

Proof. $\mathcal{W}_0^{1,p(.)}(\Omega)$ is a closed Banach subspace of $\mathcal{W}^{1,p(.)}(\Omega)$. Let $u \in \mathcal{W}_0^{1,p(.)}(\Omega)$, $v \in \mathcal{W}^{1,p(.)}(\Omega)$ with $|v| \leqslant |u|$ a.e. Let $(\varphi_i)_i \subset \mathscr{C}_0^\infty(\Omega)$ such that $(\varphi_i)_i$ converges to u in $\mathcal{W}_0^{1,p(.)}(\Omega)$. Then $v_i = \min\{v, \varphi_i\}$ has compact support and belongs to $\mathcal{W}^{1,p(.)}(\Omega)$. Hence by Lemma 3.1, we have that $v_i \in \mathcal{W}_0^{1,p(.)}(\Omega)$. Moreover, since $(v_i)_i$ converges to $\min\{v, u\} = v$ in $\mathcal{W}^{1,p(.)}(\Omega)$, as $i \longrightarrow \infty$, we have that $v \in \mathcal{W}_0^{1,p(.)}(\Omega)$.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Then*

$$\mathcal{W}_0^{1,p(.)}(\Omega) = \left\{ u \in \tilde{\mathcal{W}}^{1,p(.)}(\Omega) : \ \tilde{u} = 0 \ p(.)\text{-r.q.e. on } \partial\Omega \right\},$$

where $\tilde{u}$ denotes the $p(.)$-relative quasi-continuous representative of u.

Proof. From Lemma 3.2, we deduce that $\mathcal{W}_0^{1,p(.)}(\Omega)$ is also a closed ideal of $\tilde{\mathcal{W}}^{1,p(.)}(\Omega)$. We follow the idea of the proof giving in [18, Theorem 3], there exist a Borel set $\mathcal{B} \subset \overline{\Omega}$ such that

$$\mathcal{W}_0^{1,p(.)}(\Omega) = \left\{ u \in \tilde{\mathcal{W}}^{1,p(.)}(\Omega) : \ \tilde{u} = 0 \ p(.)\text{-r.q.e on } \mathcal{B} \right\}.$$

Let $(K_i)_i$ be a sequence of compact subset of Ω such that $K_i \subset K_{i+1}$ and $\Omega = \bigcup_{i\in\mathbb{N}} K_i$. For each $i \in \mathbb{N}$ there exist a cut-off function $\varphi_i \in \mathscr{C}_0^\infty(\Omega)$ such that $0 \leqslant \varphi_i \leqslant 1$ and $\varphi_i = 1$ on K_i. Since $\mathscr{C}_0^\infty(\Omega) \subset \mathcal{W}_0^{1,p(.)}(\Omega)$, it follows that $\varphi_i \in \mathcal{W}_0^{1,p(.)}(\Omega)$ and $\varphi_i = 0$ $p(.)$-r.q.e on $\mathcal{B}$. Hence $C_{p(.)}^{\overline{\Omega}}(K_i \cap \mathcal{B}) = 0$. Consequently:

$$\begin{aligned} C_{p(.)}^{\overline{\Omega}}(\mathcal{B} \cap \Omega) &= C_{p(.)}^{\overline{\Omega}}\left(\mathcal{B} \bigcap \bigcup_{i\in\mathbb{N}} K_i\right) \\ &= \lim_{i\longrightarrow\infty} C_{p(.)}^{\overline{\Omega}}\left(\mathcal{B} \bigcap K_i\right) \\ &= 0. \end{aligned}$$

Hence $\mathcal{B} \cap \Omega$ is $p(.)$-relatively polar set.

Let $u \in \tilde{\mathcal{W}}^{1,p(.)}(\Omega)$ such that $\tilde{u} = 0$ $p(.)$-r.q.e on $\partial\Omega$. Then $\tilde{u} = 0$ $p(.)$-r.q.e on $\mathcal{B}$ and hence $\tilde{u} \in \mathcal{W}_0^{1,p(.)}(\Omega)$. Consequently:

$$\left\{ u \in \tilde{\mathcal{W}}^{1,p(.)}(\Omega) : \ \tilde{u} = 0 \ p(.)\text{-r.q.e. on } \partial\Omega \right\} \subset \mathcal{W}_0^{1,p(.)}(\Omega).$$

To prove the converse, observe that the set:

$$\left\{ u \in \tilde{\mathcal{W}}^{1,p(.)}(\Omega) : \ \tilde{u} = 0 \ p(.)\text{-r.q.e. on } \partial\Omega \right\}$$

is a closed ideal of $\tilde{\mathcal{W}}^{1,p(.)}(\Omega)$ containing $\mathscr{C}_0^\infty(\Omega)$. Hence it also contains $\mathcal{W}_0^{1,p(.)}(\Omega)$.

The following corollary gives a necessary and sufficient condition in term of the $p(.)$-relative capacity for the equality $\mathcal{W}_0^{1,p(.)}(\Omega) = \tilde{\mathcal{W}}^{1,p(.)}(\Omega)$.

Corollary 3.1. *Let $\Omega \subset \mathbb{R}^n$ be open. Then the following assertions are equivalent.*

(1) $C^{\overline{\Omega}}_{p(.)}(\partial\Omega) = 0$;
(2) $\mathcal{W}^{1,p(.)}_0(\Omega) = \tilde{\mathcal{W}}^{1,p(.)}(\Omega)$.

In the sequel, we prove that every function $u \in \mathcal{W}^{1,p(.)}_0(\Omega) \cap \mathscr{C}(\overline{\Omega})$ is zero everywhere on the boundary $\partial\Omega$ if and only if $C^{\overline{\Omega}}_{p(.)}(B(x,r) \cap \partial\Omega) > 0$ for all $x \in \partial\Omega$. Let us begin by recalling the definition of the $p(.)$-Sobolev capacity, we refer to [10]. Let $p \in \mathcal{P}(\Omega)$. For a set $E \subset \mathbb{R}^n$,we denote

$$\mathcal{A}_{p(.)}(E) := \left\{ u \in \mathcal{W}^{1,p(.)}(\mathbb{R}^n) : u \geq 1 \text{ a.e on a neighbourhood of } E \right\}.$$

The $p(.)$- Sobolev capacity of the set E is the number defined by

$$C_{p(.)}(E) := \inf_{u \in \mathcal{A}_{p(.)}(E)} \rho_{1,p(.)}(u).$$

In case $\mathcal{A}_{p(.)}(E) = \varnothing$, we set $C_{p(.)}(E) = \infty$.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Then the following assertions are equivalent.*
(1) $u \in \mathscr{C}(\partial\Omega)$ and $u = 0$ $p(.)$-q.e on $\partial\Omega$ implies that $u(x) = 0$ for every $x \in \partial\Omega$;
(2) Every function $u \in \mathcal{W}^{1,p(.)}_0(\Omega) \cap \mathscr{C}(\overline{\Omega})$ is zero everywhere on the boundary $\partial\Omega$;
(3) $C_{p(.)}(B(x,r) \setminus \Omega) > 0$ for all $x \in \partial\Omega$;
(4) $C_{p(.)}(B(x,r) \cap \partial\Omega) > 0$ for all $x \in \partial\Omega$;
(5) $C^{\overline{\Omega}}_{p(.)}(B(x,r) \cap \partial\Omega) > 0$ for all $x \in \partial\Omega$;
(6) $u \in \mathscr{C}(\partial\Omega)$ and $u = 0$ $p(.)$-r.q.e. on $\partial\Omega$ implies that $u(x) = 0$ for every $x \in \partial\Omega$.

Proof. (1) $\Longrightarrow$ (2): This follows from the fact that every function $u \in \mathcal{W}^{1,p(.)}_0(\Omega) \cap \mathscr{C}(\overline{\Omega})$ is continuous and is zero everywhere on the boundary $\partial\Omega$.
(2) $\Longrightarrow$ (3): Assume that there exist $x_0 \in \partial\Omega$ such that $C_{p(.)}(B(x,r) \cap \partial\Omega) = 0$. Since

$$C^{\overline{\Omega}}_{p(.)}(B(x_0,r) \setminus \Omega) \leqslant C_{p(.)}(B(x_0,r) \setminus \Omega),$$

we get that:

$$C^{\overline{\Omega}}_{p(.)}(B(x_0,r) \cap \partial\Omega) = 0.$$

Let $\varphi \in \mathscr{C}^{\infty}_0(B(x_0,r))$ be such that $\varphi = 1$ on $B(x_0, \frac{r}{2})$. Hence, by Theorem 3.1, we have that $\varphi_{|\Omega} \in \mathcal{W}^{1,p(.)}_0(\Omega) \cap \mathscr{C}(\overline{\Omega})$, but $\varphi(x_0) \neq 0$.
(3) $\Longrightarrow$ (4): Let $x_0 \in \partial\Omega$ such that $C_{p(.)}(B(x,r) \cap \partial\Omega) = 0$. We have that:

$$B(x_0,r) \setminus \Omega = (B(x_0,r) \setminus \overline{\Omega}) \cup ((B(x_0,r) \cap \partial\Omega)).$$

By the monotonicity of the $p(.)$-Sobolev capacity, we get that:

$$\begin{aligned} C_{p(.)}(B(x_0,r) \setminus \Omega) &\leqslant C_{p(.)}(B(x_0,r) \setminus \overline{\Omega}) + C_{p(.)}(B(x_0,r) \cap \partial\Omega) \\ &\leqslant C_{p(.)}(B(x_0,r) \setminus \overline{\Omega}). \end{aligned}$$

By assumption $C_{p(.)}(B(x_0,r) \setminus \Omega) > 0$, it follows that $C_{p(.)}(B(x_0,r) \setminus \overline{\Omega}) > 0$. Hence, the open set $B(x_0,r) \setminus \overline{\Omega}$ is not empty. Consequently, there exist x_1 and $\varepsilon_1 > 0$ such that $B(x_1,\varepsilon_1) \subset$

$B(x_0,r)\setminus\overline{\Omega}$. Moreover, there exist x_2 and $\varepsilon_2>0$ such that $B(x_2,\varepsilon_2)\subset B(x_0,r)\cap\Omega$. Let $\varepsilon\in(0,r)$ be such that $B(x_1,\varepsilon_1)\cup B(x_2,\varepsilon_2)\subset B(x_0,\varepsilon)$. Let $u\in\mathscr{C}_0^\infty(B(x_0,r))$ be such that $u=1$ on $B(x_0,\varepsilon)$. Define the function v by

$$v:=\begin{cases} u & \text{on } B(x_0,r)\cap\Omega;\\ 0 & \text{on } \Omega^c.\end{cases}$$

According to Theorem 3.1, we have $v\in\mathcal{W}_0^{1,p(.)}(\Omega)\subset\mathcal{W}^{1,p(.)}(\Omega)$. On the other hand, $\nabla v=0$ a.e. on $B(x_0,\varepsilon)$, hence $v=C$ a.e. on $B(x_0,\varepsilon)$. This contradicts the fact that $v=1$ on $B(x_0,\varepsilon_2)$ and $v=0$ on $B(x_0,\varepsilon_1)$.

(4) $\Longrightarrow$ (5): Suppose that there exists $x_0\in\partial\Omega$ such that:

$$C_{p(.)}^{\overline{\Omega}}(B(x_0,r)\cap\partial\Omega)=0.$$

Let $u\in\mathscr{C}_0^\infty(B(x_0,r))$ be such that $u=1$ on $B(x_0,\frac{r}{2})$. By Theorem 3.1, we have $u_{|_\Omega}\in\mathcal{W}_0^{1,p(.)}(\Omega)$. We define the function v by

$$v:=\begin{cases} u & \text{on } \Omega;\\ 0 & \text{on } \Omega^c.\end{cases}$$

Then it follows that $v\in\mathcal{W}_0^{1,p(.)}(\Omega)\subset\mathcal{W}^{1,p(.)}(\Omega)$ and $\nabla v=0$ a.e. on $B(x_0,\frac{r}{2})$. Consequently, we obtain $v=1$ a.e. on $B(x_0,\frac{r}{2})$. This implies that $\tilde{v}=1$ $p(.)$-q.e. on $B(x_0,\frac{r}{2})$, where $\tilde{v}$ denotes the $p(.)$-quasi-continuous representative of v. In particular, $\tilde{v}=1$ $p(.)$-q.e. on $B(x_0,\frac{r}{2})\cap\partial\Omega$ and $\tilde{v}=0$ $p(.)$-q.e. on $\mathbb{R}^n\setminus\Omega$. It follows that $C_{p(.)}(B(x_0,\frac{r}{2})\cap\partial\Omega)=0$.

(5) $\Longrightarrow$ (6): We suppose that there exists $x_0\in\partial\Omega$ such that $u(x_0)\neq 0$. Since $u\in C(\partial\Omega)$, we deduce that there exists an open ball $B(x_0,r)$ such that $u(x)\neq 0$ for all $x\in B(x_0,r)\cap\partial\Omega$. Using the fact that $u=0$ $p(.)$-q.e. on $\partial\Omega$, it follows that $C_{p(.)}^{\overline{\Omega}}(B(x_0,r)\cap\partial\Omega)=0$.

(6) $\Longrightarrow$ (1): Let $u\in\mathscr{C}(\partial\Omega)$ be such that $u=0$ $p(.)$-q.e. on $\partial\Omega$. By assertion (6), we deduce that $u(x)=0$ for every $x\in\partial\Omega$.

4. Dirichlet problem in nonlinear harmonic space

Let $\Omega\subset\mathbb{R}^n$ be a bounded open set and $p\in\mathcal{P}(\Omega)$ with $1<p^-\leqslant p(x)\leqslant p^+<n$. We assume that the variable exponent p satisfies the logarithmic Hölder continuity condition introduced by Zhikov in [19], namely we suppose that:

$$|p(x)-p(y)|\leq\frac{C}{-\log(|x-y|)}\text{ for }(x,y)\in\Omega\times\Omega\text{ with }|p(x)-p(y)|<\frac{1}{2},$$

where $C>0$ is independent of (x,y). Under this assumption and as it was observed in [19], the space of smooth functions is dense in the variable exponent Sobolev space. We refer also to the monograph [10] for the details in this direction.

Let $f \in \mathscr{C}(\partial\Omega)$, we consider the Dirichlet problem:

$$\begin{cases} \mathscr{L}_{\mathbf{p}(.)}u := -\Delta_{p(.)}u + \mathscr{B}(.,u) = 0 & \text{in } \Omega; \\ u = f & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_{p(.)} := \operatorname{div}\left(|\nabla u|^{p(.)-1}\nabla\right)$ is the $p(.)$-Laplacian operator and $\mathscr{B} : \Omega \times \mathbb{R} \to \mathbb{R}$ is a given Carathéodory function satisfies the following structural condition for a positive constant C:

- $|\mathscr{B}(.,\zeta)| \leqslant a(.) + C\,|\zeta|^{p(.)-1}$ a.e. in Ω and for all $\zeta \in \mathbb{R}^n$, where a is a positive measurable function lying in $L^{p'(.)}(\Omega)$.
- $\zeta \mapsto \mathscr{B}(.,\zeta)$ is increasing function on Ω.

A typical example when the above structural condition holds is $\mathscr{B}(.,u) = |u|^{p(.)-2}u$.

In the sequel, we discuss the nonlinear potential theory associated with the $\mathcal{H}_{\mathscr{L}_{p(.)}}$-harmonic sheaf. For that, we use Theorem 3.2 and adapt some definitions from [6].

We say that Ω is $\mathscr{L}_{p(.)}$-regular if the following conditions are satisfied:

- (C_1): for every continuous function $f : \partial\mathcal{V} \longrightarrow \mathbb{R}$, there exists a unique $\mathscr{L}_{p(.)}$-harmonic function in Ω, denoted by $\mathscr{L}^{\Omega}_{p(.),f}$, such that:

$$\lim_{x \longrightarrow y} \mathscr{L}^{\Omega}_{p(.),f}(x) = f(y) \quad \text{for every } y \in \partial\Omega;$$

- (C_2): For each pair $f, g \in \mathscr{C}(\partial\Omega)$ the condition $f \leqslant g$ implies $\mathscr{L}^{\Omega}_{p(.),f} \leqslant \mathscr{L}^{\Omega}_{p(.),g}$ in Ω.

The pair $(\Omega, \mathcal{H}_{\mathscr{L}_{p(.)}})$ is called a nonlinear harmonic space, with respect to the sheaf $\mathcal{H}_{\mathscr{L}_{p(.)}}$ if the following three axioms are satisfied:

- (A_1) The family of the $\mathscr{L}_{p(.)}$-regular open sets is a basis for the topology of Ω.
- (A_2) The sheaf $\mathcal{H}_{\mathscr{L}_{p(.)}}$ is non degenerate: For every $x \in \Omega$, there exists an open neighborhood V of x and a function $h \in \mathcal{H}_{\mathscr{L}_{p(.)}}$ such that $h(x) \neq 0$.
- (A_3) Convergence property: for every subset U of Ω and every monotone locally bounded sequence $(h_n)_n$ in $\mathcal{H}_{\mathscr{L}_{p(.)}}$, we have $h = \lim_{n \longrightarrow +\infty} h_n \in \mathcal{H}_{\mathscr{L}_{p(.)}}$.

Using the notion of traces of functions and uniform approximation in $\mathcal{W}^{1,p(.)}(\Omega)$, Baalal and Berghout [2] showed the following theorem:

Theorem 4.1. *Let $f \in \mathscr{C}(\partial\Omega)$. Then there exists a unique continuous $\mathscr{L}_{p(.)}$-harmonic extension of f in $\overline{\Omega}$.*

Note that the comparison principle given in [4] can be extended immediately to functions in $\mathscr{C}(\partial\Omega)$ in the following way:

Proposition 4.1. *(Comparison principal). Let $f, g \in \mathscr{C}(\partial\Omega)$. If $f \leqslant g$ on $\partial\Omega$, then $\mathscr{L}^{\Omega}_{p(.),f} \leqslant \mathscr{L}^{\Omega}_{p(.),g}$ in Ω.*

Combining Theorem 4.1 with Proposition 4.1, we obtain the following corollary:

Corollary 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with non empty boundary $\partial\Omega$, then Ω is $\mathcal{H}_{\mathscr{L}_{p(.)}}$-regular. Moreover, the family of the $\mathcal{H}_{\mathscr{L}_{p(.)}}$-regular open sets is a basis for the topology of Ω.*

Theorem 4.2. *Let $\mathcal{H}_{\mathcal{L}_{p(.)}}$ be the sheaf of $\mathcal{L}_{p(.)}$-harmonic functions. Then the pair $(\Omega, \mathcal{H}_{\mathcal{L}_{p(.)}})$ is a nonlinear harmonic space.*

Proof. By Corollary 4.1, the family of the $\mathcal{H}_{\mathcal{L}_{p(.)}}$-regular open sets is a basis for the topology of Ω. From [5], we deduce the following form of the Harnack inequality: for every non empty open set $\Omega \subset \mathbb{R}^n$, for every constant $M > 0$ and every compact $K \subset \Omega$, there exists a constant $C = C(K, M)$ such that:

$$\operatorname{esssup}_{x\in K} u(x) \leqslant C(\operatorname{essinf}_{x\in K} u(x) + R),$$

for every positive u in $\mathcal{H}_{\mathcal{L}_{p(.)}}(\Omega)$ with $u \leqslant M$, it follows that the sheaf $\mathcal{H}_{\mathcal{L}_{p(.)}}(\Omega)$ is non degenerate.
Let $(u_i)_i$ be a locally bounded increasing sequence in $\mathcal{H}_{\mathcal{L}_{p(.)}}(\Omega)$. Then by Harnack inequality the family

$$\mathcal{F} := \left\{u_i(x), x \in \overline{V}, i \in \mathbb{N}, \text{ for every } \ V \subset \overline{V} \subset \Omega\right\}$$

is uniformly equicontinuous on $\overline{V}$ and it is a bounded subset of $\mathcal{C}(\overline{V})$. By Ascoli–Arzelà Theorem, $\mathcal{F}$ has compact closure in $\mathcal{C}(\overline{V})$; consequently, the sequence $(u_i)_i$ converges locally and uniformly in Ω to a continuous function u. Hence, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that:

$$(\forall i \geq n_0); \sup_{x\in\overline{V}} |u_i(x) - u(x)| \leq \varepsilon.$$

By the comparison principal

$$\mathcal{L}_{p(.)} u - \varepsilon \leq u_i \leq \mathcal{L}_{p(.)} u + \varepsilon,$$

letting $i \longrightarrow \infty$, we obtain that:

$$\mathcal{L}_{p(.)} u - \varepsilon \leq u \leq \mathcal{L}_{p(.)} u + \varepsilon.$$

Now, letting $\varepsilon \longrightarrow 0$, we get that $\mathcal{L}_{p(.)} u = u$. Hence the space $(\Omega, \mathcal{H}_{\mathcal{L}_{p(.)}})$ satisfies the convergence property.

Notice that by the uniqueness of the solution, we deduce that the $\mathcal{L}_{p(.)}$-harmonic function and the solution given by Brelot-Perron-Wiener Method coincides. In this case the space of continuous functions on $\partial\Omega$ is resolutive.

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Assume that $C^{\overline{\Omega}}_{p(.)}(B(x,r) \cap \partial\Omega) > 0$ for all $x \in \partial\Omega$. Then the mapping $f \mapsto \mathcal{L}^{\Omega}_{p(.),f}$ from $\mathcal{C}(\partial\Omega)$ into $(\Omega, \mathcal{H}_{\mathcal{L}_{p(.)}})$ is injective.*

Proof. Let $f, g \in \mathcal{C}(\partial\Omega)$ and assume that $\mathcal{L}^{\Omega}_{p(.),f} = \mathcal{L}^{\Omega}_{p(.),g}$. By Theorem 4.1, $\mathcal{L}^{\Omega}_{p(.),f}$ and $\mathcal{L}^{\Omega}_{p(.),g}$ belong to $\mathcal{C}(\overline{\Omega})$. Hence

$$\mathcal{L}^{\Omega}_{p(.),f} = f \ p(.)\text{-r.q.e on } \partial\Omega$$

and

$$\mathcal{L}^{\Omega}_{p(.),g} = g \ p(.)\text{-r.q.e on } \partial\Omega.$$

Again by Theorem 4.1, we deduce that:

$$f - g = 0 \ p(.)\text{-r.q.e on } \partial\Omega.$$

Since $C^{\overline{\Omega}}_{p(.)}(B(x,r)\cap\partial\Omega) > 0$ for all $x \in \partial\Omega$, it follows from Theorem 3.2 that $f - g = 0$ everywhere on $\partial\Omega$, which implies that $f = g$ everywhere on $\partial\Omega$. Hence, the mapping $f \mapsto \mathcal{L}^{\Omega}_{p(.),f}$ from $\mathcal{C}(\partial\Omega)$ into $\mathcal{H}_{\mathcal{L}_{p(.)}}(\Omega)$ is injective.

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