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Theoretical and numerical analysis of a degenerate nonlinear cubic Schrödinger equation

MOHAMED ALAHYANE¹, ABDERRAZAK CHRIFI², YOUNES ECHARROUDI³

ABSTRACT. In this paper, we are interested in some theoretical and numerical studies of a special case of a degenerate nonlinear Schrödinger equation namely the so-called Gross-Pitaevskii Equation (GPE). More precisely, we will treat in a first time the well-posedness of GPE model with a degeneracy occurring in the interior of the space variable domain, i.e $\exists x_0 \in (0, L)$, s.t $k(x_0) = 0$, where k stands for the diffusion coefficient and L is a positive constant. Thereafter, we will focus ourselves on some numerical simulations showing the influence of a different parameters, especially the interior degeneracy, on the behavior of the wave solution corresponding to our model in a special case of the function k namely $k(x) = |x - x_0|^\alpha$, $\alpha \in (0, 1)$.

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¹ L2EP, Faculty of Sciences and Technologies, University of Lille, 59655 Villeneuve d'Ascq, France.

e-mail: mohamed.alahyane@univ-lille.fr

² Department of Mathematics, Faculty of Science and Technology, Cadi Ayyad University, B.P. 549, Av. Abdelkarim Elkhatabi, Guéliz, Marrakesh, 40000, Morocco.

e-mail: abderrazak.chrifi@gmail.com

³ Laboratory of mathematics and population dynamics, Faculty of sciences Semlalia Marrakesh, Morocco.

e-mail: yecharroudi@gmail.com (Corresponding Author) .

1. Introduction

Consider the system

$$\begin{aligned} i\frac{\partial u}{\partial t} + (k(x)u_x)_x + p(t, x)u + i\sigma u + \beta u|u|^2 &= \vartheta\chi_\omega & \text{in } Q, \\ u(t, L) = u(t, 0) &= 0 & \text{on } (0, T), \\ u(0, x) &= u_0(x) & \text{in } (0, L), \end{aligned} \quad (1.1)$$

where $Q = (0, T) \times (0, L)$, $T, L > 0$ and i is the complex number satisfying $i^2 = -1$. Herein, $\omega \subset\subset (0, 1)$ is the region where the control ϑ is acting. It is well-known that the wave-solution u of the Schrödinger equation (1.1) is a complex-valued function as well as the parameters k , p , ϑ and the initial data u_0 . Nevertheless, we will take in this paper k and p as real-valued functions and k satisfying some suitable conditions. We highlight that the parameters k , p ; $\sigma > 0$ and $\beta \in \mathbb{R}$ will be interpreted in the context of boson later on.

The paradigm (1.1) which belongs to the equations modeling the quantum mechanics phenomena, is a kind of a nonlinear Schrödinger equation called Gross-Pitaevskii equation (GPE). Actually, the main equation of (1.1) must be written as follows:

$$i\hbar\frac{\partial u}{\partial t} + \frac{\hbar^2}{2m}(k(x)u_x)_x + p(t, x)u + i\sigma u + \beta u|u|^2 = \vartheta\chi_\omega \quad \text{in } Q, \quad (1.2)$$

where $\hbar$ is the reduced Planck's constant and m is a positive constant interpreted as the mass of a particle. Nevertheless, as we are not concerned by the physical experiments of GPE then we will restrict ourselves to study (1.1) and the results obtained in this item can be extended to the model (1.2).

In our best knowledge, the GPE or in general the nonlinear Schrödinger equation belongs to the family of systems called boson systems. In quantum mechanics, a boson is a particle that follows the Bose-Einstein statistics and whose spin quantum number is zero or an integer number. Notice that there is two classes of elementary particles the other being fermions. An examples of bosons are fundamental particles such as photons, gluons, W and Z bosons, the Higgs boson and the hypothetical graviton of quantum gravity. An other interesting examples of bosons are mesons and stable nuclei of even mass number such as deuterium, helium-4 denoted by ${}^4\text{He}$.

Briefly, a boson system is the simplest system in nature which demonstrates a wide variety of physical phenomena such as superfluidity, magnetism, crystals, etc. We can also use boson system to study many different phases transition. In the light of last definition, the GPE is derived from the framework of second quantization and models the Bose-Einstein condensates in a macroscopic gaseous super-fluid wave-matter state in ultra-cold temperature. Afterwards, we see that the Bose-Einstein is tightly related to the notion of boson and hence the wave solution $u(t, x)$ of (1.1) is interpreted as the boson wave function or boson orbital function at time t and position x , $k(x)$ represents the dispersion or the quantum diffusion coefficient at the position x , $p(t, x)$ is the external trapping potential experienced by the boson at time t and position x , σ is the boson wave damping parameter, β is the nonlinearity coefficient called the Landau coefficient which is interpreted as the strength of effective contact interaction or the parameter that measures the bosons interactions. On the other hand, in the light of the

Born's statistical interpretation, the square of wave modulus $|u(t, x)|^2$ can be viewed as the probability density of finding the boson at position x and time t , $\vartheta\chi_\omega$ is an external force applied on an internal space control region $\omega \subset\subset (0, L)$ which can be related for example to the presence of an external magnetic or electric fields; the initial wave $u_0(x)$ is the equilibrium macroscopic boson wave function at position x . By the way, the quantity $\int_0^L |u_0(x)|^2 dx$ is exactly the total number of bosons throughout the medium of length L . Finally, if one applies GPE to a boson bound to a one dimensional box, then the Dirichlet boundary conditions $u(t, L) = u(t, 0) = 0$, $t \in (0, T)$ can be considered in the situation where the walls are rigid and impenetrable and this amounts to say that the studied boson is never found beyond the walls of the box.

The numerical study of the model (1.1) took and still take an intensive interest and there is a wide literature treating a various aspects of the Schrödinger equation either in its radial case or in the case when the orbital solution depends on the time variable and since in this item we are just concerned with the nonlinear Schrödinger equation (1.1) we will not discuss the first case, the interested reader is referred to [3, 4, 13, 16, 19, 21, 22, 24, 25, 28] for more details.

Among the pertinent papers wherein we can see different numerical methods applied for different cases of time dependent Schrödinger equation (TDSE), we find [7, 11, 23]. In fact, in the item [7] Belaouar and al. injected a stochastic term in a one dimensional nonlinear Schrödinger equation with the nonlinearity is cubic and which models the propagation of the light waves in optical fibers. To be more accurate, the stochastic case arises in the coefficient of velocity dispersion which is multiplied by a real Gaussian white noise in the sense of Stratonovich (see [29] for a numerical study of a similar equations). To obtain a numerical simulations related to this model, the authors used a semi-discretized Crank-Nicolson scheme of the noise together with a centered finite difference discretization with respect to the space. As an outcome of this, when the studied stochastic equation possesses a unique global solution, one can show that the probability to get a strong order of convergence is equal to 1. In a second step, some numerical investigations were done to test the influence of the nonlinearity orders on the behavior of the solution for the critical, subcritical and supercritical kinds. The Crank-Nicolson scheme plays again a crucial role but this time for a deterministic form of a non controlled Schrödinger equation treated in [11]. More exactly, the authors consider a TDSE which models the dynamics of an atom to an external field and the goal is to provide numerically the smooth and the strong oscillating solutions of the target equation. To this end, two strategies were formulated based on the finite-difference method and the finite-elements method. More precisely, one of them will be elaborated with the help of a generalization of Crank-Nicolson scheme of fourth order in time step while the second one uses a second order approximation in time step. To show the efficiency of such an algorithms, the authors try to propel their results on two models known as the Coulomb-like atoms in the electric field of a short laser pulse and the Pöschl-Teller potential (see [8, 26] for a related works to this type of potential). In [23], Sørøvik et al. consider a Schrödinger equation whose the right term is the Hamiltonian operator which is wrote under the sum of a kinetic energy term represented by the Laplacian operator and the classical potential energy denoted in literature by V .

This strategy called split-operator technique together with a spectral approximation with respect to space variable constitute the core of the numerical analysis done in this item. Concisely, the operator which generates the equation was expressed in the hyper-spherical coordinates, a choice justified to lead a more efficient results and required a specific discretization basis. Thereby, as the Fourier basis can not be incorporated with the current system of coordinates, the authors overcome this difficulty by the alternative of the Chebyshev polynomials and a suitable transformation (see the same reference for more details). Another problem which was remarked is that the splitting-operator technique involves a splitting error which creates a high frequency modes in the numerical solution. To tackle this difficult situation, the suggested remedy is to combine between the Coulomb potential and the Laplacian operator whose the spectral approximation is given in the Chebyshev basis. The paper is closed by showing the efficiency of this mixed technique as well as its limitations via a 1-D and 3-D numerical examples respectively illustrated by a TDSE with the null potential energy modeling a free wave packet propagation and the ground state energy of H_+^2 molecule. For a more extensive knowledge of numerical studies of TDSE and in a non exhaustive way, one can take a glance on [5, 6, 12, 14, 15].

As we said in the beginning, the Schrödinger operator that we will deal with is from the TDSE type and has an interior degeneracy, for example where the coefficient is defined by

$$k(x) = |x - x_0|^\alpha, \quad (1.3)$$

with $x_0 \in (0, L)$ and $\alpha \in (0, 1)$ or more generally when the conditions (2.1) beneath are satisfied. This non classical Laplacian operator is the main difference with the previous works and also the principal snag in the numerical simulations. Thus, the expression of k given by (1.3) satisfies

$$k(x_0) = 0. \quad (1.4)$$

Mathematically speaking, if (1.3) holds we say that (1.1) is an interior degenerate Gross-Pitaevskii equation with weak degeneracy since we impose the Dirichlet boundary condition on left hand side of the space domain, i.e in the point $x = 0$. Let us stress that if instead, one has the Neumann condition like $ku_x(t, 0) = 0$, $t \in (0, T)$, we talk about a strong degenerate problem which is equivalent in the case of the definition (1.3) that $\alpha \in [1, 2)$.

To well understand the interpretation of (1.4), we must firstly recall the Hamiltonian energy (see [27] for instance) matched with our system (1.1) and which is given by

$$\mathcal{H}(u) = \mathcal{K}(u) + \mathcal{P}(u) + \mathcal{I}(u), \quad (1.5)$$

where

$$\mathcal{K}(u) := \int_0^L |(k(x)u_x)_x|^2 dx \quad (1.6)$$

is the kinetic energy of the boson,

$$\mathcal{P}(u) := \frac{1}{2} \int_0^L p(t, x) |u|^2 dx \quad (1.7)$$

represents the external trapping potential energy and

$$\mathcal{I}(u) := \frac{1}{4} \int_0^L \beta |u|^4 dx \quad (1.8)$$

is the bosons interaction potential energy.

Physically speaking, it is well-known that the kinetic energy of an object is the energy that it possesses due to its motion. It is defined as the work needed to accelerate a body of a given mass from rest to its stated velocity and naturally, the kinetic energy of an object at rest should be zero. On the other side, if one replace the condition (1.4) in the formula of the kinetic energy (1.6), we can see that $\mathcal{K}(u)(x_0, t) = 0$, $t \in (0, T)$ and this, following the last explanation our boson is somehow at rest at the position x_0 .

The remainder of this paper is organized as follows: The Section 2 is devoted to the evidence of the well-posedness result of (1.1) under a pertinent framework and with the diffusion coefficient k satisfies the interior degeneracy (2.1). Concerning the forthcoming sections, we will restrict ourselves to the case when k verifies (1.3) and we provide in Section 3 the proof of an adequate discretization scheme corresponding to (1.1). According to this, we build in the fourth section some numerical results to assess the impact of different model parameters on the behavior of solution and ultimately, we close this item by a global conclusion in Section 5.

2. Well-posedness result: existence and regularity properties

As we mentioned before, we study in this section the well-posedness of (1.1) under the suitable framework defined by (2.3). Such a framework was set recently in [10] to prove the well-posedness of (1.1) in the case where $\beta = 0$ and k satisfying (2.1) with $x_0 = 0$. We emphasize that for the non degenerate case, i.e $k(x) > 0$, $\forall x \in (0, 1)$ involving Dirac operator and under the so-called artificial boundary conditions, the well-posedness of Nonlinear Schrödinger equation can already be found in [1, 2] and in the recent paper [9].

On the other hand, we note that in this subsection, $|z|$, $\bar{z}$ and $\mathcal{R}e(z)$ will represent respectively the modulus, the adjoint and the real part of the complex z and we will assume also that the diffusion coefficient k verifies for an interior point $x_0 \in (0, L)$

$$\begin{cases} k \in C([0, L]) \cap C^1([0, L] \setminus \{x_0\}), \quad k > 0 \text{ in } [0, L] \setminus \{x_0\} \text{ and } k(x_0) = 0, \\ \exists \gamma \in [0, 1) : (x - x_0)k'(x) \leq \gamma k(x), \quad x \in [0, L] \setminus \{x_0\}. \end{cases} \quad (2.1)$$

The last hypothesis is satisfied in the case of $k(x) = |x - x_0|^\alpha$ that $0 \leq \alpha < 1$. Besides, we will require the following hypotheses

$$\begin{cases} \vartheta \in L^2(0, L), \\ p \in L^\infty(Q) \text{ and } \frac{\partial p}{\partial t} \in L^\infty(Q) \\ u_0 \in L^2(0, L). \end{cases} \quad (2.2)$$

to perform the proof of the well-posedness of (1.1), we need to introduce some special weighted

Sobolev space defined by:

$$H_k^1(0, L) := \{u \in L^2(0, L) : u \text{ is abs. cont. in } [0, L] : \sqrt{k}u_x \in L^2(0, L), u(L) = u(0) = 0\}, \quad (2.3)$$

endowed with the norm

$$\|u\|_{H_k^1(0, L)}^2 := \|u\|_{L^2(0, L)}^2 + \|\sqrt{k}u_x\|_{L^2(0, L)}^2, \quad u \in H_k^1(0, L), \quad (2.4)$$

We point out that the parameter γ given in (2.1) in the strong degenerate problem is taken on the interval $[1, 2)$ which means in the case of (1.3) that $\alpha \in [1, 2)$.

The main result of this section is stated as follows.

Theorem 2.1. *Let $u_0 \in H_k^1(0, L)$ be an initial data. Then, there exists a unique function $u \in C^0([0, +\infty[; H_k^1(0, L)) \cap C^1([0, +\infty[; [H_k^1(0, L)]')$ solution to equation (1.1), where $C^n(I; E)$ represents the space of n times continuously differentiable functions on $\mathbf{I}$ in $\mathbf{E}$, $[H_k^1(0, L)]'$ is the dual of $H_k^1(0, L)$.*

Proof. The main dilemma to bring out that the solution u of equation (1.1) lives in $C^0([0, +\infty[; H_k^1(0, L)) \cap C^1([0, +\infty[; [H_k^1(0, L)]')$ is the Galerkin method which is split into three main steps as detailed below and afterwards, our solution is shown to be only one in the aforementioned space.

First step: approximate problem:

Let $(\varphi_k)_k$ be an orthonormal basis of functions in $H_k^1(0, L)$. For $m \geq 1$, we set $\mathcal{V}_m = \text{Span}(\varphi_1, \dots, \varphi_m)$ and we define the orthogonal projection operator P_m by

$$\begin{aligned} P_m : H_k^1(0, L) &\rightarrow \mathcal{V}_m \\ u &\mapsto P_m(u) = \sum_{k=1}^m \langle u, \varphi_k \rangle_{H_k^1(0, L)} \varphi_k \end{aligned} \quad (2.5)$$

where $\langle \cdot, \cdot \rangle_{H_k^1(0, L)}$ is the scalar product in $H_k^1(0, L)$. For $m \geq 1$, we shall approximate u in (1.1) by

$$u_m(t) = \sum_{k=1}^m h_{km}(t) \varphi_k, \quad (2.6)$$

which satisfies, for all $k \in \{1, 2, \dots, m\}$

$$\begin{cases} \frac{d}{dt} \langle iu_m, \varphi_k \rangle_{L^2(0, L), L^2(0, L)} - \langle k(x)u_{mx}, \varphi_{kx} \rangle_{L^2(0, L), L^2(0, L)} + \langle p(t, x)u_m, \varphi_k \rangle_{L^2(0, L), L^2(0, L)} \\ + \beta \langle |u_m|^2 u_m, \varphi_k \rangle_{L^2(0, L), L^2(0, L)} + i\sigma \langle u_m, \varphi_k \rangle_{L^2(0, L), L^2(0, L)} = \langle \vartheta \chi_\omega, \varphi_k \rangle_{L^2(0, L), L^2(0, L)}, \\ u_m(0) = P_m(u_0). \end{cases} \quad (2.7)$$

We get the system

$$H'_m(t) = \mathbf{P}(H_m(t)), \quad (2.8)$$

where $\mathbf{P}$ is a polynomial function and

$$H_m(t) = \begin{pmatrix} h_{1m}(t) \\ \vdots \\ h_{mm}(t) \end{pmatrix} \in \mathbb{C}^m,$$

such that at $t = 0$ the components of $H_m(0)$ are

$$h_{mk}(0) = k^{th} \text{ element of } u_{m0}. \quad (2.9)$$

Hence, using Cauchy-Lipschitz theorem, we get a unique function $H_m = (h_{1m}, \dots, h_{mm})$ in $[0, T_m]$ solution of (2.8) with the initial condition (2.9).

Consequently, the approximate problem (2.7) has a unique solution u_m such that $u_m : [0, T_m] \rightarrow \mathcal{V}_m$. The existence of a maximal solution u_m (defined on $[0, T_{max}[$) is obtained by iterating m , where T_{max} is the maximum time of existence such that $u_m : [0, T_{max}[\rightarrow \mathcal{V}_m$. Then, we have $T_{max} < +\infty$ and $\lim_{t \rightarrow T_{max}} |u_m| = +\infty$, or $T_{max} = +\infty$.

Second step: A priori estimates:

Estimate in $L^2(0, L)$:

Multiplying (1.1) by $-i\bar{u}$, integrating over the space domain $]0, L[$, an integration by parts of the second term leads to

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2(0,L)}^2 + \sigma \|u(t)\|_{L^2(0,L)}^2 = \operatorname{Re} \left(i \int_0^L \vartheta \chi_\omega(x) \overline{u(x,t)} dx \right), \quad (2.10)$$

where we have just considered the real part.

Thus gathering Cauchy-Schwartz, Young inequalities, and Gronwall lemma, one can deduce that

$$\|u(t)\|_{L^2(0,L)}^2 \leq e^{-\sigma t} \|u_0\|_{L^2(0,L)}^2 + \frac{1 - e^{-\sigma t}}{\sigma^2} \|\vartheta \chi_\omega\|_{L^2(0,L)}^2. \quad (2.11)$$

Then, $\forall t \geq 0$, we have

$$\|u(t)\|_{L^2(0,L)}^2 \leq K_0^2 = \|u_0\|_{L^2(0,L)}^2 + \frac{1}{\sigma^2} \|\vartheta \chi_\omega\|_{L^2(0,L)}^2. \quad (2.12)$$

Estimate in $H_k^1(0, L)$:

Now, in order to get an upper bound of $\|u(t)\|_{H_k^1(0,L)}$, we need firstly to estimate $\|\sqrt{k(x)}u_x\|_{L^2(0,L)}^2$ in terms of $\Psi(u(t))$. In the sequel of this part of step 2, we denote by K any positive constant depending only on σ , $\|\vartheta \chi_\omega\|_{L^2(0,L)}$, $\|u_0\|_{H_k^1(0,L)}$, $\|p\|_{L^\infty(Q)}$ and $\|\frac{\partial p}{\partial t}\|_{L^\infty(Q)}$ and which can be change from an inequality to another.

To this end, we multiply the first equality of (1.1) by $\overline{u_t + \sigma u}$, integrating over $(0, L)$ and applying again the real part, we have

$$\frac{d}{dt} \Psi(u(t)) + 2\sigma \Psi(u(t)) = \sigma \beta \|u\|_{L^4(0,L)}^4 - \int_0^L \frac{\partial p}{\partial t}(t, x) |u|^2 dx + 2\sigma \operatorname{Re} \left(\int_0^L \vartheta \chi_\omega \bar{u} dx \right), \quad (2.13)$$

with,

$$\Psi(u(t)) = \|\sqrt{k(x)}u_x\|_{L^2(0,L)}^2 - \int_0^L p(t, x) |u|^2 dx - \frac{\beta}{2} \|u\|_{L^4(0,L)}^4 + 2\operatorname{Re} \left(\int_0^L \vartheta \chi_\omega \bar{u} dx \right). \quad (2.14)$$

The last equality is equivalent to

$$\|\sqrt{k(x)}u_x\|_{L^2(0,L)}^2 = \Psi(u(t)) + \int_0^L p(t, x) |u|^2 dx + \frac{\beta}{2} \|u\|_{L^4(0,L)}^4 - 2\operatorname{Re} \left(\int_0^L \vartheta \chi_\omega \bar{u} dx \right). \quad (2.15)$$

In the light of Gagliardo-Nirenberg and Young inequalities, we obtain

$$\begin{aligned} \frac{\beta}{2} \|u\|_{L^4(0,L)}^4 &\leq \frac{1}{4} \|\sqrt{k(x)}u_x\|_{L^2(\Omega)}^2 + K \\ &\leq \frac{1}{2} \|\sqrt{k(x)}u_x\|_{L^2(\Omega)}^2 + K. \end{aligned} \quad (2.16)$$

On the other hand, (2.12) and Cauchy-Schwarz imply that

$$-2\operatorname{Re} \left(\int_0^L \vartheta \chi_\omega \bar{u} dx \right) \leq 2 \|\vartheta \chi_\omega\|_{L^2(\Omega)} K_0 \leq K. \quad (2.17)$$

Keeping in the mind the assumptions (2.2) and again (2.12), we get

$$\int_0^L p(t, x) |u|^2 dx \leq \|p\|_{L^\infty(Q)} K_0^2 \leq K. \quad (2.18)$$

and

$$\int_0^L \frac{\partial p}{\partial t}(t, x) |u|^2 dx \leq \left\| \frac{\partial p}{\partial t} \right\|_{L^\infty(Q)} K_0^2 \leq K. \quad (2.19)$$

Subsequently, with the aid of (2.15) as well as (2.17), (2.18) and (2.16) we infer that

$$\|\sqrt{k(x)}u_x\|_{L^2(0,L)}^2 \leq 2\Psi(u(t)) + K. \quad (2.20)$$

Combining the inequalities (2.19), (2.17), (2.16) and (2.20), we infer that

$$\frac{d}{dt} \Psi(u(t)) + \sigma \Psi(u(t)) \leq K$$

Consequently, the Gronwall lemma, (2.20) and (2.12) $\forall t \geq 0$ lead to

$$\|u(t)\|_{H_k^1(0,L)} \leq K. \quad (2.21)$$

Finally, the sequence $(u_m)_m$ remains bounded in $\mathcal{C}_b([0, +\infty[; H_k^1(0, L)])$ and $(u'_m)_m$ remains bounded in $\mathcal{C}_b([0, +\infty[; [H_k^1(0, L)]')$.

Third step: from approximated to continuous model:

Since the sequence $(u_m)_m$ remains bounded in $\mathcal{C}_b([0, +\infty[; H_k^1(0, L)])$.

Then, Banach-Alaoglu Theorem involves that $(u_m)_m$ admits a subsequence still denoted $(u_m)_m$ such that

$$u_m \rightharpoonup u \text{ weakly } \star \text{ in } L^\infty(0, T; H_k^1(0, L)). \quad (2.22)$$

On the other hand, the sequence $(\frac{du_m}{dt})_m$ is bounded in $L^\infty(0, T; [H_k^1(0, L)]')$.

Thus, it admits a subsequence which converges weakly $\star$ to $h \in L^\infty(0, T; [H_k^1(0, L)]')$.

We conclude via the uniqueness of the limit in $\mathcal{D}'((0, T) \times (0, L))$ that $h = \frac{du}{dt}$ and this implies that

$$\frac{du_m}{dt} \rightharpoonup \frac{du}{dt} \text{ weakly } \star \text{ in } L^\infty(0, T; [H_k^1(0, L)]'). \quad (2.23)$$

On the other side, consider $Y \in \mathcal{D}([0, T])$ such that $Y(T) = 0$.

Passing to the limit on the equation

$$\int_0^T \langle iu'_m, Y(t) \varphi_k \rangle_{L^2(0,L), L^2(0,L)} dt - \int_0^T \langle k(x) u_{mx}, Y(t) \varphi_{kx} \rangle_{L^2(0,L), L^2(0,L)} dt$$

$$\begin{aligned}
& + \int_0^T \langle p(t, x) u_m, Y(t) \varphi_k \rangle_{L^2(0, L), L^2(0, L)} dt + \beta \int_0^T \langle |u_m|^2 u_m, Y(t) \varphi_k \rangle_{L^2(0, L), L^2(0, L)} dt \\
& + i\sigma \int_0^T \langle u_m, Y(t) \varphi_k \rangle_{L^2(0, L), L^2(0, L)} dt = \int_0^T \langle \vartheta \chi_\omega, Y(t) \varphi_k \rangle_{L^2(0, L), L^2(0, L)} dt,
\end{aligned} \tag{2.24}$$

and using (2.22) and (2.23), we conclude therefore that $u_m \rightarrow u$ strongly in $\mathcal{C}^0([0, T]; L^2(0, L))$.

Now, Let K be a compact of $(0, 1)$. Since $u_m |u_m|^2$ belongs to a bounded set of $L^\infty(0, T, L^2(K))$, we then extract a subsequence of $(u_m)_m$ (denoted again $(u_m)_m$) such that $u_m |u_m|^2 \rightharpoonup w$ weakly $\star$ in $L^\infty(0, T, L^2(K))$.

Afterwards, the compactness lemma mentioned in [17] allows us to say that

$$|u_m|^2 u_m \rightarrow |u|^2 u \text{ strongly in } \mathcal{C}^0([0, T]; L^2(0, L)).$$

Hence

$$\lim_{m \rightarrow +\infty} \int_0^T \langle |u_m|^2 u_m, Y(t) \varphi_k \rangle_{L^2(0, L), L^2(0, L)} dt = \int_0^T \langle |u|^2 u, Y(t) \varphi_k \rangle_{L^2(0, L), L^2(0, L)} dt. \tag{2.25}$$

On the other hand, we will check out that $\forall \phi \in H_k^1(0, L)$ we have

$$\left\langle i \frac{\partial u}{\partial t} + (k(x) u_x)_x + p(t, x) u + \beta |u|^2 u + i\sigma u - \vartheta \chi_\omega, \phi \right\rangle_{[H_k^1(0, L)]', H_k^1(0, L)} = 0. \tag{2.26}$$

In fact, one has $\forall \phi \in H_k^1(0, L)$

$$\begin{aligned}
& - \langle i u_0, Y(0) \phi \rangle_{L^2(0, L), L^2(0, L)} - \int_0^T \langle i u, Y'(t) \phi \rangle_{H_k^1(0, L), [H_k^1(0, L)]'} dt \\
& + \int_0^T \langle (k(x) u_x)_x, Y(t) \phi \rangle_{H_k^1(0, L), [H_k^1(0, L)]'} dt \\
& + \int_0^T \langle p(t, x) u, Y(t) \phi \rangle_{H_k^1(0, L), [H_k^1(0, L)]'} dt + \beta \int_0^T \langle |u|^2 u, Y(t) \varphi_k \rangle_{L^2(0, L), L^2(0, L)} dt \\
& + i\sigma \int_0^T \langle u, Y(t) \phi \rangle_{H_k^1(0, L), [H_k^1(0, L)]'} dt = \int_0^T \langle \vartheta \chi_\omega, Y(t) \phi \rangle_{H_k^1(0, L), [H_k^1(0, L)]'} dt,
\end{aligned} \tag{2.27}$$

For $Y \in \mathcal{D}(0, T)$, we deduce that u verifies

$$\int_0^T \left\langle i \frac{\partial u}{\partial t} + (k(x) u_x)_x + p(t, x) u + \beta |u|^2 u + i\sigma u - \vartheta \chi_\omega, \phi \right\rangle_{[H_k^1(0, L)]', H_k^1(0, L)} Y(t) dt = 0,$$

and this achieves the task of (2.26).

Now, it remains to verify that u satisfies $u(0) = u_0$. To this end, let $Y \in \mathcal{D}([0, T])$ be such that $Y(T) = 0$.

Multiplying equation (2.26) by $Y(t)$ and integrating over $(0, T)$ we obtain

$$\begin{aligned}
& - \langle i u(0), Y(0) \phi \rangle_{L^2(0, L), L^2(0, L)} - \int_0^T \langle i u, Y'(t) \phi \rangle_{H_k^1(0, L), [H_k^1(0, L)]'} dt \\
& + \int_0^T \langle (k(x) u_x)_x, Y(t) \phi \rangle_{H_k^1(0, L), [H_k^1(0, L)]'} dt
\end{aligned}$$

$$\begin{aligned}
& + \int_0^T \langle p(t, x)u, Y(t)\phi \rangle_{H_k^1(0,L), [H_k^1(0,L)]'} dt + \beta \int_0^T \langle |u|^2 u, Y(t)\phi \rangle_{H_k^1(0,L), [H_k^1(0,L)]'} dt \\
& + i\sigma \int_0^T \langle u, Y(t)\phi \rangle_{H_k^1(0,L), [H_k^1(0,L)]'} dt = \int_0^T \langle \vartheta \chi_\omega, Y(t)\phi \rangle_{H_k^1(0,L), [H_k^1(0,L)]'} dt,
\end{aligned} \tag{2.28}$$

This together with (2.27) and the condition $Y(0) = 1$ lead to $u(0) = u_0$.

Abstractly, we proved that there exists $u \in \mathcal{C}^0([0, +\infty[; H_k^1(0, L)) \cap \mathcal{C}^1([0, +\infty[; [H_k^1(0, L)]')$ solution to equation 1.1. The forthcoming step is to perform the uniqueness of such a solution.

Uniqueness and continuous dependence of the solutions:

To reach this purpose and to check out the continuity with respect to the initial datum of the solution of (1.1), we consider $u(t)$ and $\tilde{u}(t)$ a solutions satisfying (1.1) which are related respectively to initial data u_0 and $\tilde{u}_0$. We set $w(t) = u(t) - \tilde{u}(t)$ with the initial condition $w(0) = u_0 - \tilde{u}_0$. Then, w satisfies the following system

$$\begin{aligned}
i \frac{\partial w}{\partial t} + (k(x)w_x)_x + p(t, x)w + i\sigma w + \beta (|u|^2 u - |\tilde{u}|^2 \tilde{u}) &= 0 & \text{in } Q, \\
w(t, 1) = w(t, 0) &= 0 & \text{on } (0, T), \\
w(0, x) &= u_0(x) - \tilde{u}_0(x) & \text{in } (0, L),
\end{aligned} \tag{2.29}$$

Multiplying (2.29) by $\bar{w}$ and taking the imaginary part, we arrive to

$$\frac{1}{2} \frac{d}{dt} \|w(t)\|_{L^2(0,L)}^2 + \sigma \|w(t)\|_{L^2(0,L)}^2 + \text{Im} \left(\int_0^L (|u|^2 u - |\tilde{u}|^2 \tilde{u}) \bar{w} dx \right) = 0.$$

Applying Cauchy-Schwarz's inequality and taking into account that injection of $H_k^1(0, L)$ in $L^\infty(0, L)$ is continuous, we deduce that

$$\frac{1}{2} \frac{d}{dt} \|w(t)\|_{L^2(0,L)}^2 + \sigma \|w(t)\|_{L^2(0,L)}^2 \leq K \|w(t)\|_{L^2(0,L)}^2.$$

Consequently, by mean again of Gronwall's lemma we arrive to

$$\|w(t)\|_{L^2(0,L)}^2 \leq e^{-2(\sigma-K)t} \|w(0)\|_{L^2(0,L)}^2 = e^{-2(\sigma-K)t} \|u_0 - \tilde{u}_0\|_{L^2(0,L)}^2,$$

and this prove the existence of a unique solution $u \in \mathcal{C}^0([0, +\infty[; H_k^1(0, L)) \cap \mathcal{C}^1([0, +\infty[; [H_k^1(0, L)]')$ to equation (1.1). Moreover, the map

$$\begin{aligned}
H_k^1(0, L) &\longrightarrow H_k^1(0, L) \\
u_0 &\longmapsto u(t),
\end{aligned}$$

is continuous on a bounded subset of $H_k^1(0, L)$ for the strong topology of $L^2(0, L)$.

3. Stability of discretization scheme

In this section, we deal with the stability of the semi-discretized scheme of (1.1) basing ourselves on the Crank-Nicolson method. In summuray, the Crank-Nicolson scheme is a second-order method in time that has the property to be unconditionally stable. To be more precise, let $T > 0$ be a maximal time. The interval $[0, T]$ is discretized into N intervals. $\Delta T = \frac{T}{N}$ represents

the time step and set $t_n = n\Delta t$. Furthermore, U^n is an approximation of $u(t_n)$. Under these assumptions, the associated Crank-Nicolson scheme for the problem (1.1) is given by

$$\begin{cases} \frac{2i}{\Delta t} V^{n+1} + \partial_x (k(x) \partial_x V^{n+1}) + p(t_{n+1}, x) V^{n+1} + i\sigma V^{n+1} \\ + \beta |V^{n+1}|^2 V^{n+1} = \frac{2i}{\Delta t} U^n + \vartheta \chi_\omega \text{ in } (0, L) \\ V^{n+1}(L) = V^{n+1}(0) = 0, \\ U^0 = u_0 \text{ in } (0, L) \end{cases} \quad (3.1)$$

where

$$V^{n+1} = \frac{U^{n+1} + U^n}{2}, \quad (3.2)$$

and for any $n = 0, 1, \dots, N-1$.

The key result is the following theorem.

Theorem 3.1. *Let $(U^n)_{n \in \mathbb{N}}$ be a solution of the system (3.1). The following inequality holds*

$$\forall n \in \{1, 2, \dots, N\} \quad \|U^n\|_{L^2(0,L)}^2 \leq \|U^0\|_{L^2(0,L)}^2 + \frac{T}{2\sigma} \|\vartheta \chi_\omega\|_{L^2(0,L)}^2. \quad (3.3)$$

Proof. Let $n \in \{1, 2, \dots, N\}$ and j an integer between 0 and $n-1$. At time t_{j+1} , the first equation of (3.1) reads as

$$i \frac{U^{j+1} - U^j}{\Delta t} + \partial_x (k(x) \partial_x V^{j+1}) + p(t_{j+1}, x) V^{j+1} + i\sigma V^{j+1} + \beta |V^{j+1}|^2 V^{j+1} = \vartheta \chi_\omega \quad (3.4)$$

Multiplying (3.4) by $-i\overline{V^{j+1}}$, a careful computations steer to

$$(U^{j+1} - U^j) \overline{V^{j+1}} = \frac{1}{2} (|U^{j+1}|^2 - |U^j|^2 + 2i\mathcal{I}m(U^{j+1}\overline{U^j})). \quad (3.5)$$

Then, integrating by parts over $(0, L)$, the Cauchy-Schwartz inequality implies

$$\frac{\|U^{j+1}\|_{L^2(0,L)}^2 - \|U^j\|_{L^2(0,L)}^2}{2\Delta t} + \sigma \|V^{j+1}\|_{L^2(0,L)}^2 \leq \|\vartheta \chi_\omega\|_{L^2(0,L)} \|U^{j+1}\|_{L^2(0,L)}, \quad (3.6)$$

Applying now Young inequality, we get

$$\frac{\|U^{j+1}\|_{L^2(0,L)}^2 - \|U^j\|_{L^2(0,L)}^2}{2\Delta t} \leq \frac{1}{4\sigma} \|\vartheta \chi_\omega\|_{L^2(0,L)}^2. \quad (3.7)$$

As an outcome of (3.7), the following inequality occurs

$$\|U^n\|_{L^2(0,L)}^2 - \|U^0\|_{L^2(0,L)}^2 \leq \frac{N\Delta t}{2\sigma} \|\vartheta \chi_\omega\|_{L^2(0,L)}^2, \quad (3.8)$$

summing the N inequalities for $j \in \{1, 2, \dots, n-1\}$ and afterwards the thesis follows.

4. Numerical results

In this section, we present the numerical results concerning the solution of the problem (1.1) for different cases in order to study the interior degenerate effects. The expression of the initial data is given by:

$$u_0(x) = \frac{\exp(i120x)}{\cosh(40(x - 0.2))}, \quad (4.1)$$

and the diffusion coefficient is defined by

$$k(x) = |x - x_0|^\alpha,$$

and we take the degenerate point at $x_0 = 0.4$. Recall that the external force $\vartheta\chi_\omega$ is given by

$$\vartheta\chi_\omega(x) = \begin{cases} \vartheta(x) & \text{if } x \in \omega \\ 0 & \text{otherwise} \end{cases} \quad (4.2)$$

where $\omega = [0.2, 0.6]$.

In the following, we present some numerical tests based on the numerical scheme (3.1) for time, where $T = 0.015$ and $\Delta t = 10^{-4}$. For the spatial discretization of $(0, L) = (0, 1)$, we use a finite element method with $\mathbb{P}_1$ shape functions and spatial step $\Delta x = 5 \cdot 10^{-4}$.

Test 1: For the first test, we consider a simple case of the problem. Where $p(t, x) = 0$, $\sigma = 0$, $\beta = 0$ and $\vartheta(x) = 0$. In figure 1, we plot the modulus of the approximate solution $|u(x, t)|$ at different instants for $\alpha = 0$, while figures 2 and 3 represent respectively the numerical results for $\alpha = 0.5$ and $\alpha = 1$. Then, figure 4 shows the plot of the modulus $|u(x, t)|$ in the plane (x, t) for the three cases ($\alpha = 0$, $\alpha = 0.5$ and $\alpha = 1$).

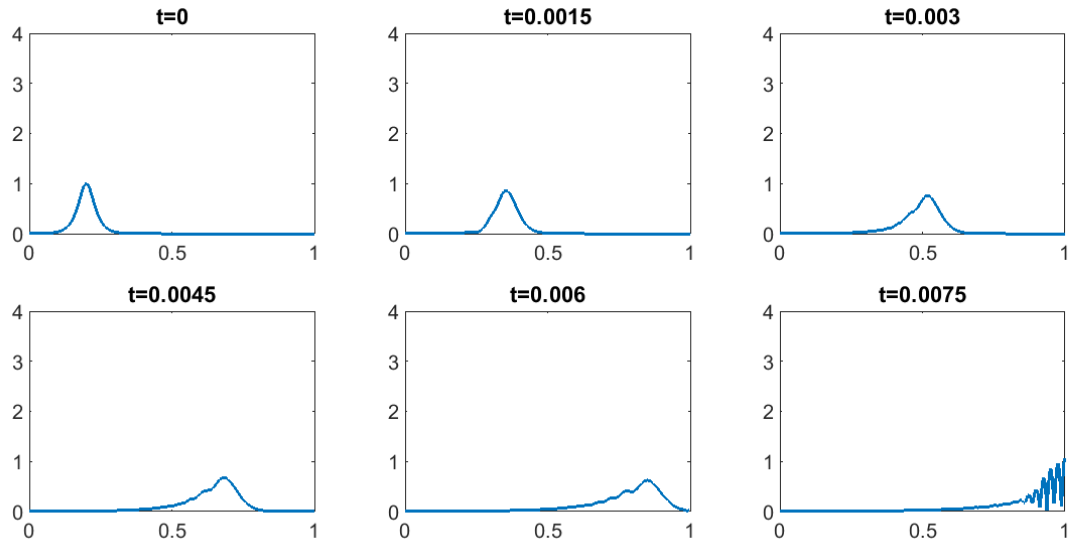
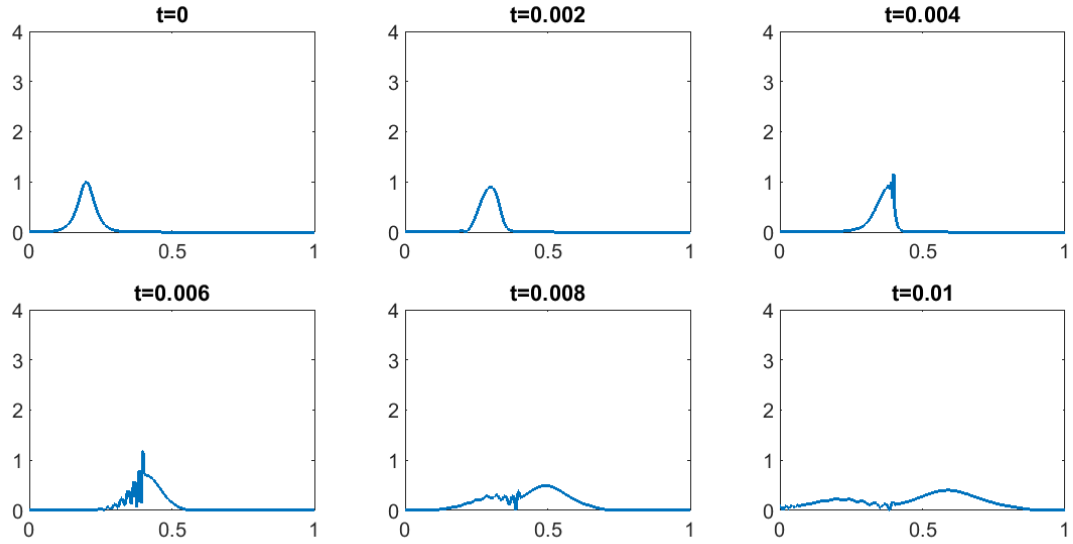
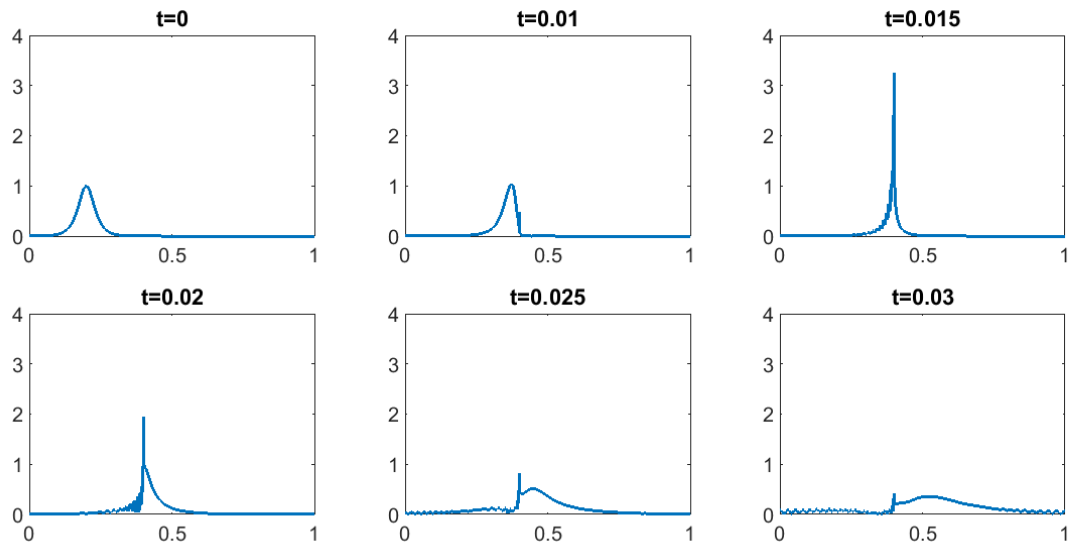
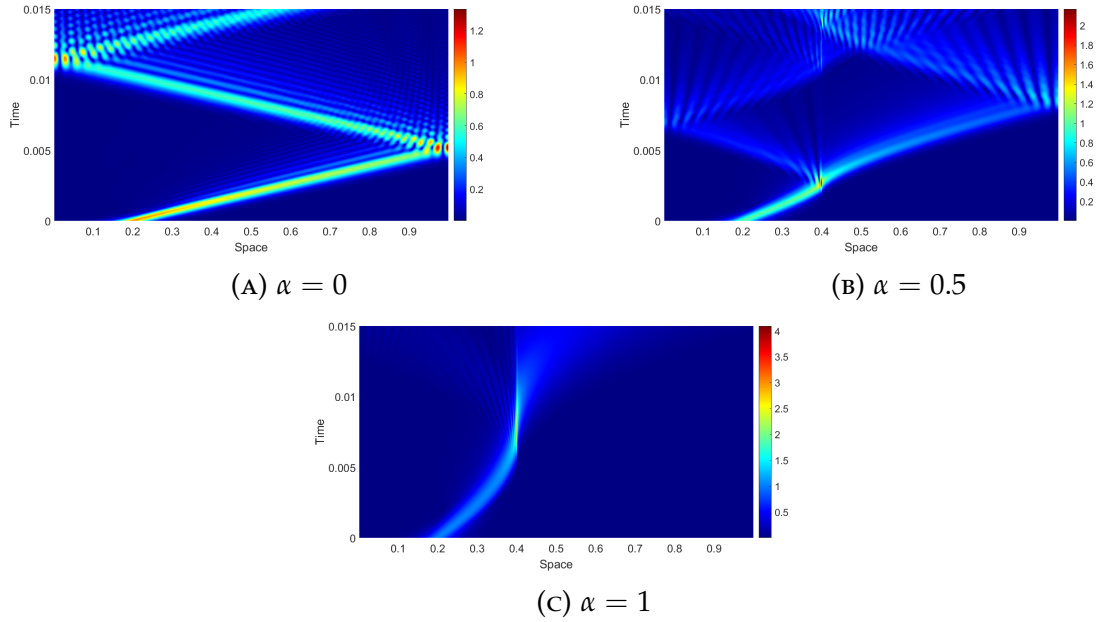


FIGURE 1. Test 1: Evolution of $|u(x, t)|$ in the space x with $\alpha = 0$

If one takes a glance in the these illustrations, we can see that a swings of the wave solution arise at the point of degeneracy contrary to the non degenerate state. This amounts to say that

FIGURE 2. Test 1: Evolution of $|u(x, t)|$ in the space x with $\alpha = 0.5$ FIGURE 3. Test 1: Evolution of $|u(x, t)|$ in the space x with $\alpha = 1$

the wave does not perpetuate its smooth form and this show clearly the noisy effect caused by the degeneracy phenomenon, we can observe that in figures 2 and 3 where the noise is more important when α goes to 1 and the wave is split on two sub-waves at point x_0 . Moreover, in the light of the figure 4, we can see that in the case of $\alpha = 0$ (non degenerate effects) the wave reaches the bound $x = 1$ and come back to $x = 0$ during the time T , this reflection on the bounds is due to the Dirichlet boundary conditions, where in the case of $\alpha = 0.5$ and $\alpha = 1$,

FIGURE 4. Test 1: Evolution of $|u(x, t)|$ in the plane (x, t)

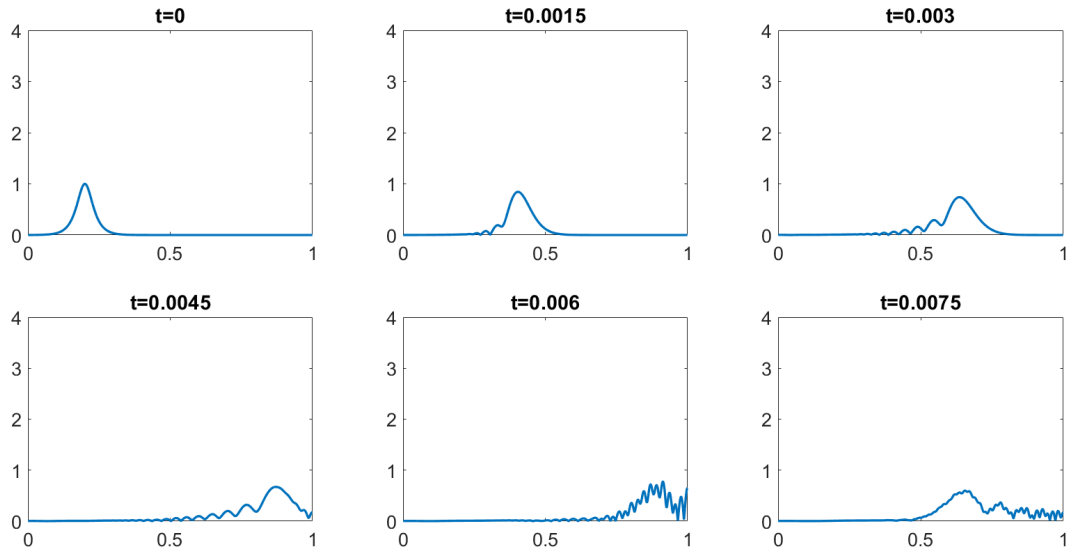
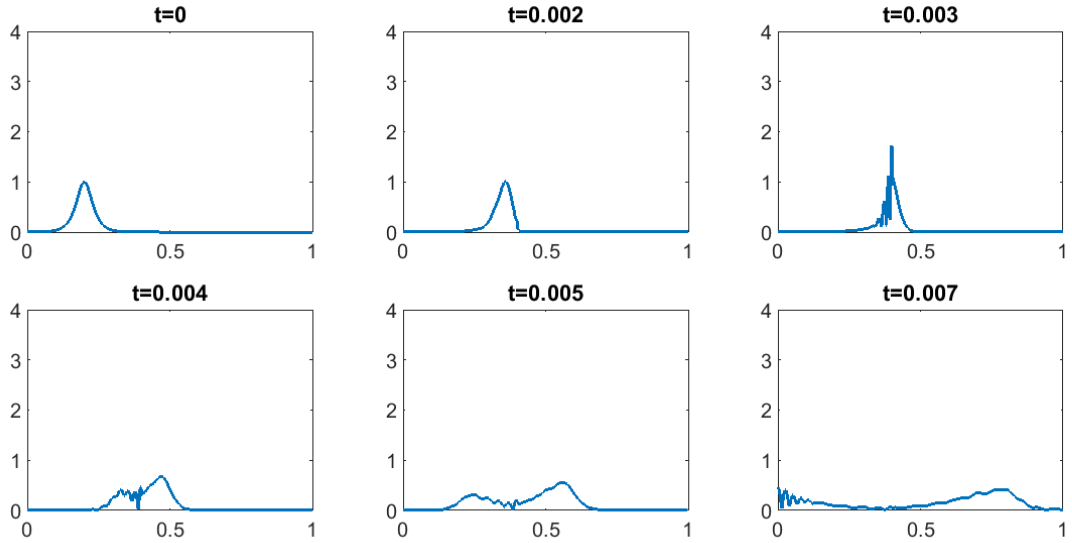
we conclude that the time needed to outreach the degenerate point x_0 is important as α goes to 1 and beyond x_0 becomes more important to reach the bounds. To sum up, the degeneracy phenomenon plays the role of an obstacle in front of the wave.

Test 2: For the second test, we consider a simple case of the problem where $p(t, x) = 10$, $\sigma = 10$, $\beta = 20$ and $\vartheta(x) = \exp(ix^2)$. The numerical results are presented in the figures 5, 6, 7 and 8 following the same process in Test 1 and consider the same cases ($\alpha = 0$, $\alpha = 0.5$ and $\alpha = 1$).

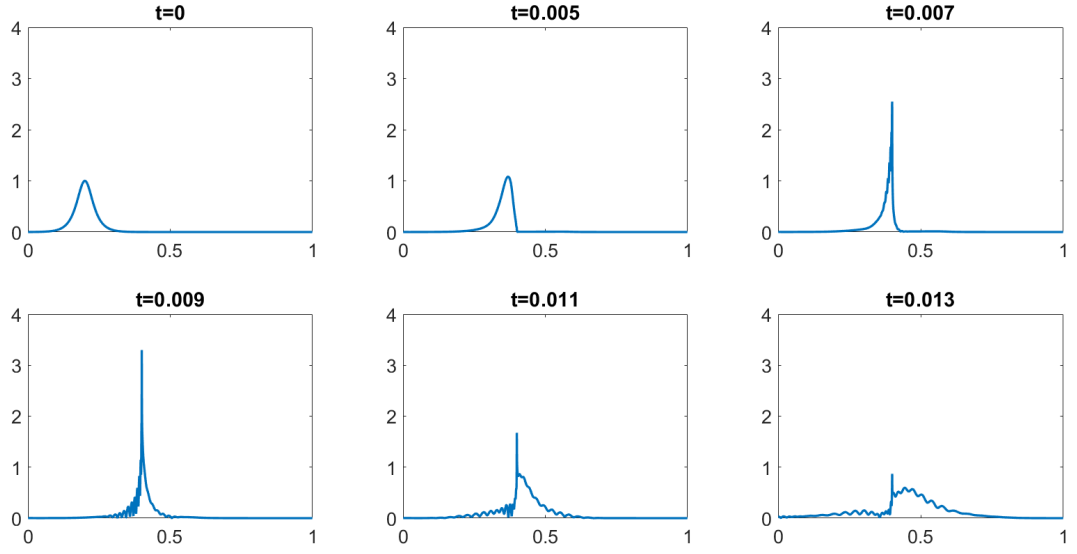
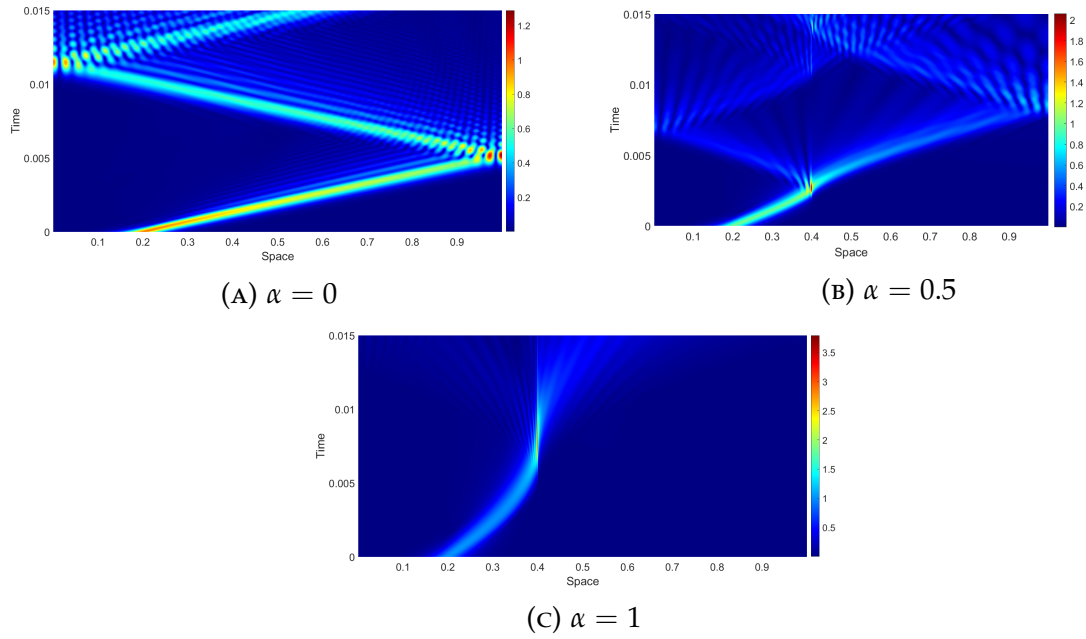
The second test, in addition to degeneracy, concerns the nonlinear case of the problem (1.1) with an external force. We can see via figures 6 and 7 that the wave hardly crosses the degeneracy point barrier when $\alpha = 0.5$ or 1 compared with the case of $\alpha = 0$ in figure 5. On the other hand, the oscillatory behavior of the reflected waves and the decay, in the same illustrations, can be explained by the reflection from the boundaries and the damping terms considered in example. Actually, the velocity of the wave solution and the parameter of degeneracy α are proportionally inevitable when α grows to 1, the spread of the wave becomes more and more slow if x is a neighbor of the point of degeneracy x_0 , which can be observed in figure 8.

Test 3: In this test, we consider a simple case of the problem where $p(t, x) = 10$, $\sigma = 10$, $\beta = 20$ and $\vartheta(x) = \exp(-\frac{x-0.4}{200})\frac{1}{10\sqrt{2\pi}}$. The numerical results are presented in the figures 9, 10, 11 and 12 following the same process of the previous tests and consider the same cases ($\alpha = 0$, $\alpha = 0.5$ and $\alpha = 1$).

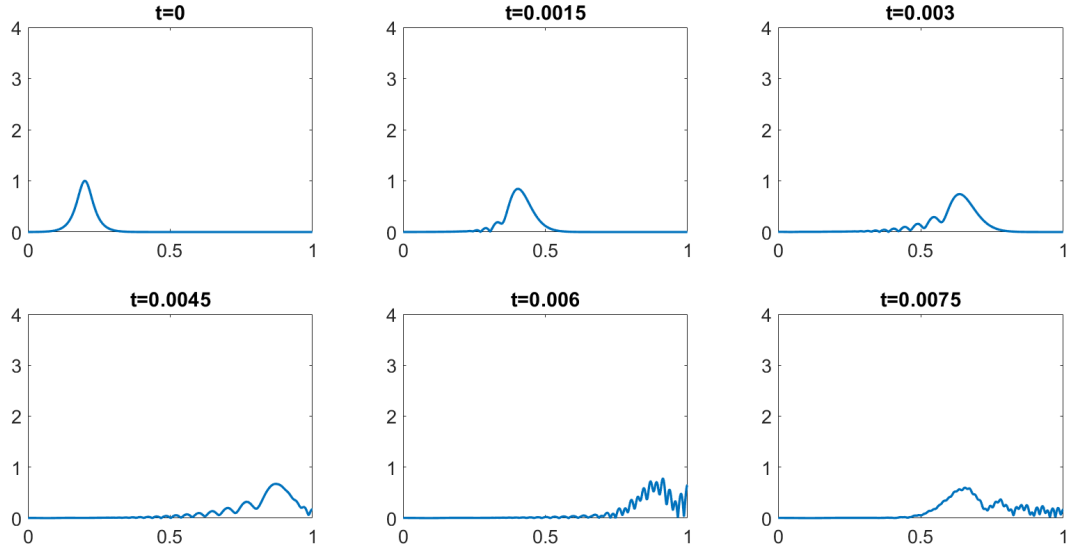
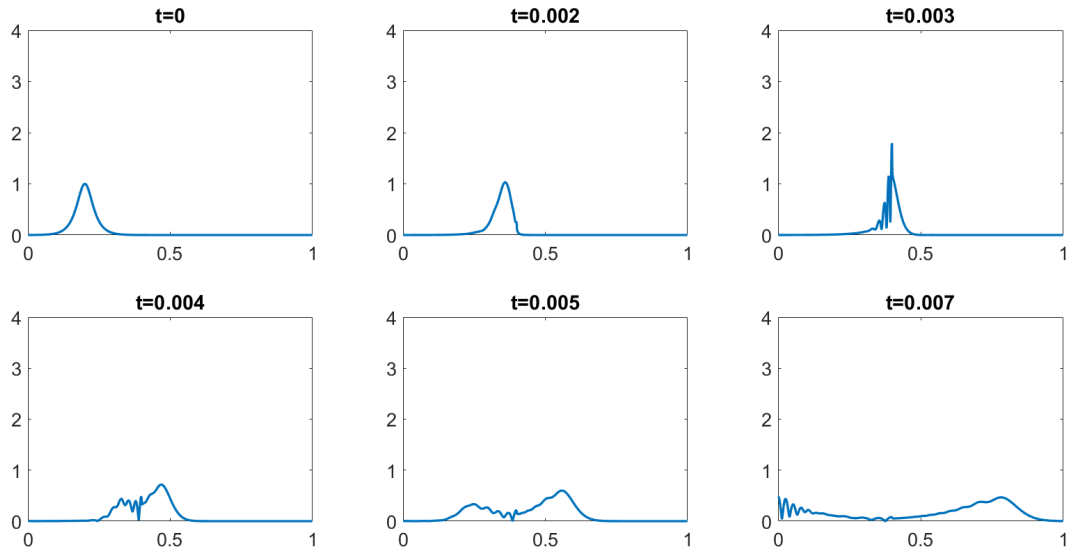
In the third test, we can observe that the results obtained are similar to the second one even if we change herein the source term to the Gaussian form. In the end, we conclude that ϑ has no influence on the behavior of the wave compared to the effect of the degeneracy phenomenon.

FIGURE 5. Test 2: Evolution of $|u(x, t)|$ in the space x with $\alpha = 0$ FIGURE 6. Test 2: Evolution of $|u(x, t)|$ in the space x with $\alpha = 0.5$

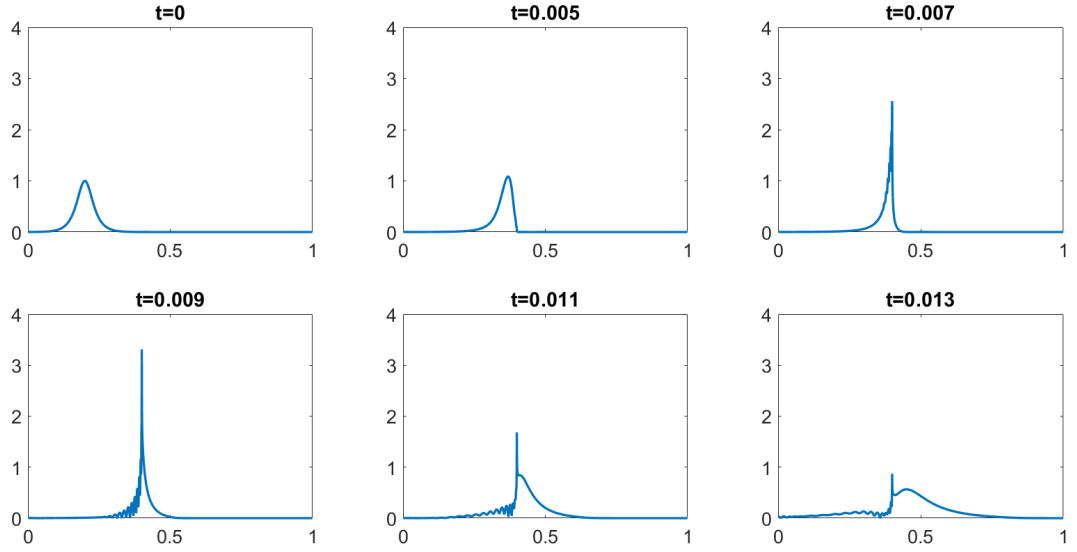
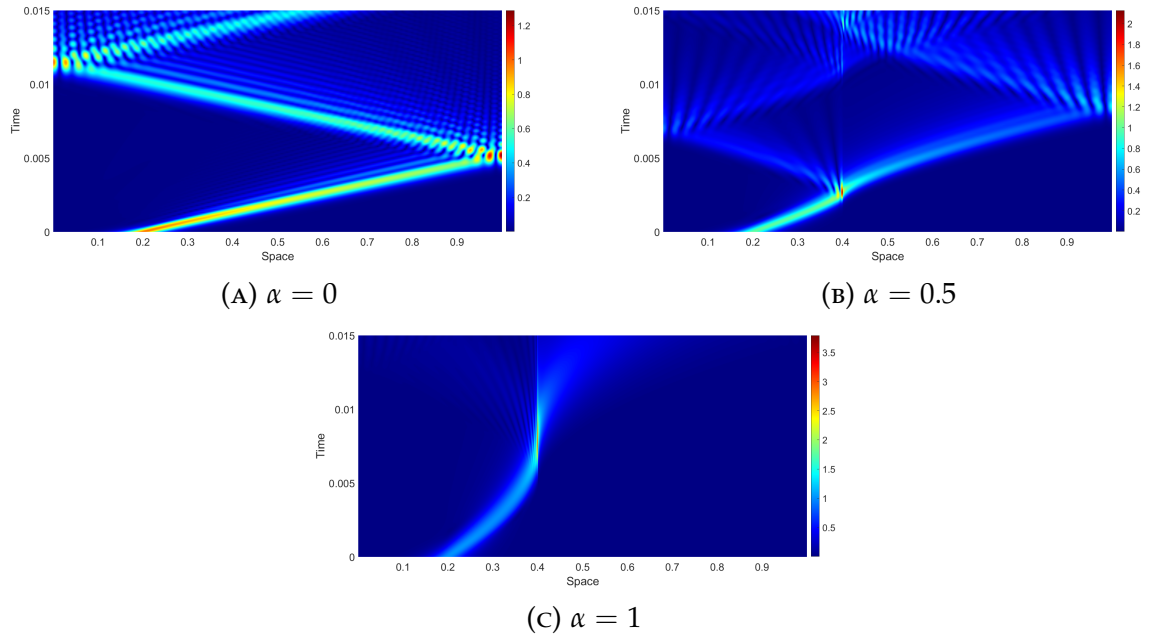
Test 4: For the last test, we consider a simple case of the problem where $p(t, x) = 0$, $\sigma = 0.1$, $\beta = 20$ and $\vartheta(x) = 0$. The numerical results are presented in the figures 13, 14, 15 and 16 considering the same cases ($\alpha = 0$, $\alpha = 0.5$ and $\alpha = 1$) as in the previous tests. Comparing with the second and the third tests, it seems that the curvature of the wave solution in the three values of α changes very slightly and this leads us to say that the external forces applied on the

FIGURE 7. Test 2: Evolution of $|u(x, t)|$ in the space x with $\alpha = 1$ FIGURE 8. Test 2: Evolution of $|u(x, t)|$ in the plane (x, t)

boson both the trapping potential p and the control force $\vartheta\chi_\omega$ defined by (4.2) do not have any impact on the quantum diffusion of the boson in front the one of degeneracy phenomenon.

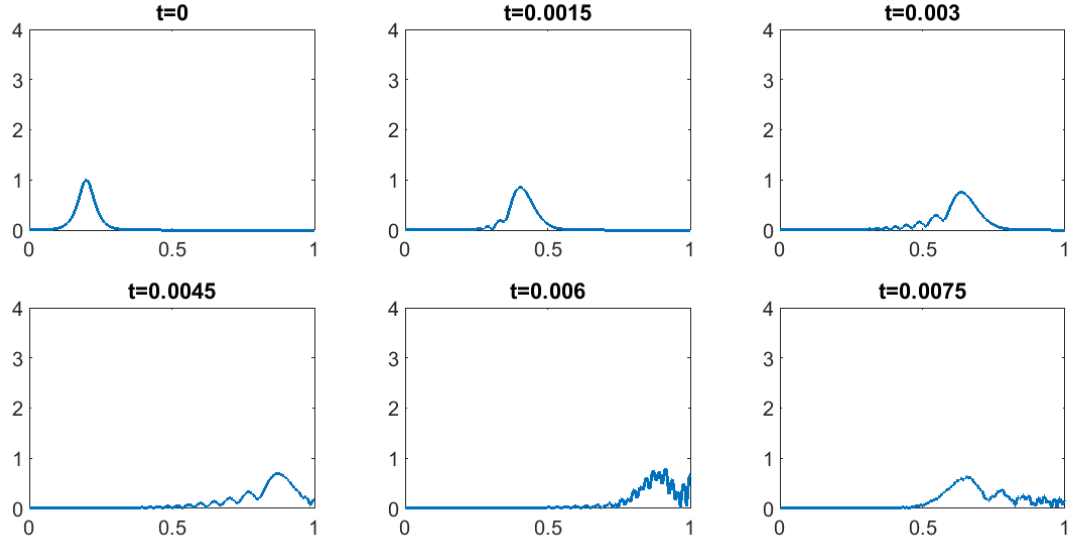
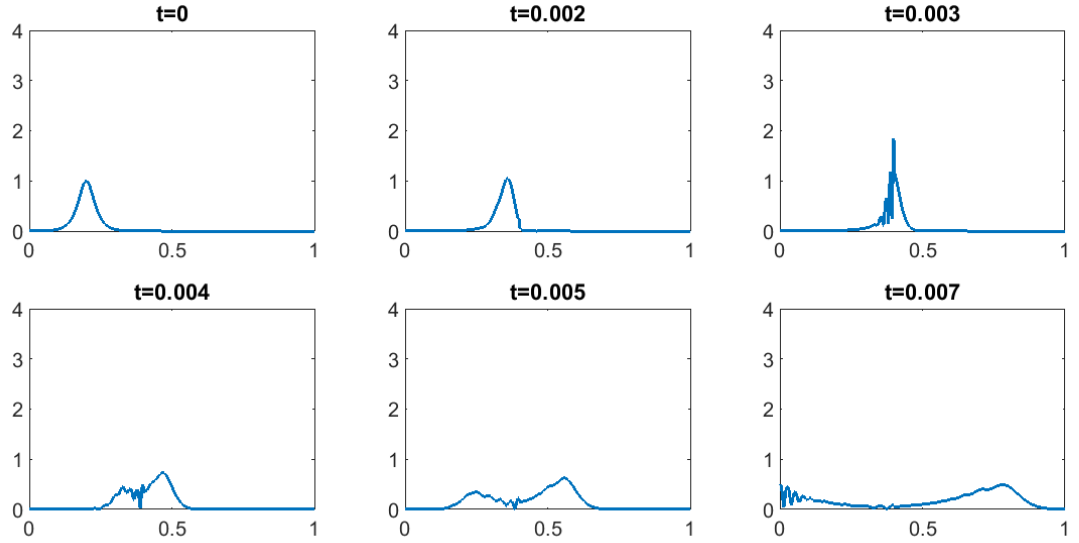
FIGURE 9. Test 3: Evolution of $|u(x, t)|$ in the space x with $\alpha = 0$ FIGURE 10. Test 3: Evolution of $|u(x, t)|$ in the space x with $\alpha = 0.5$

In the tables beneath, each one represents different situation and manifests the impact of the degeneracy degree α on the probability density of finding the studied boson $\|U^n\|_{L^2(0,1)}$ and which show us clearly that our scheme is numerically stable $\forall \alpha \in [0, 1]$ and consequently the inequality (3.3) is well justified.

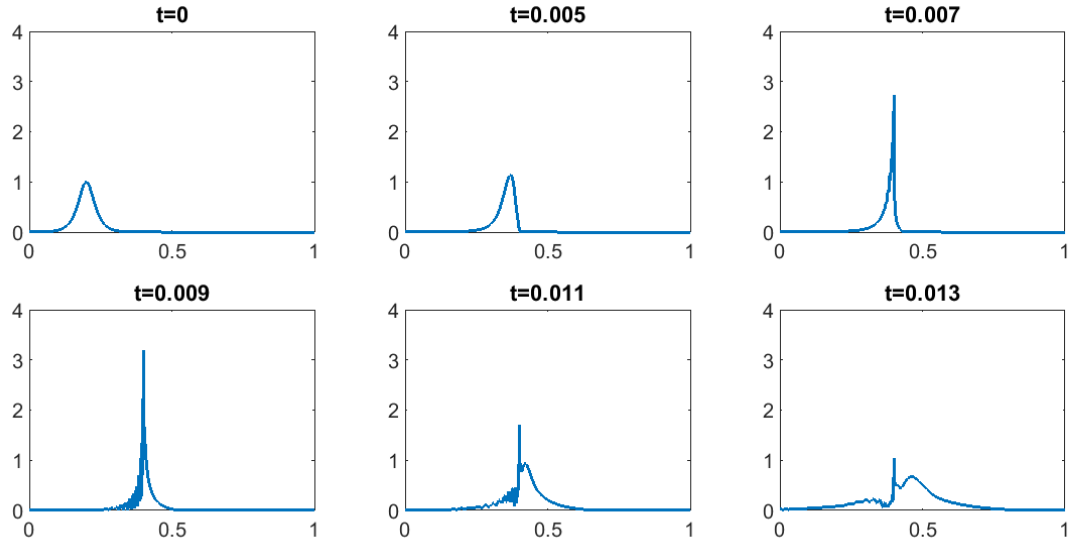
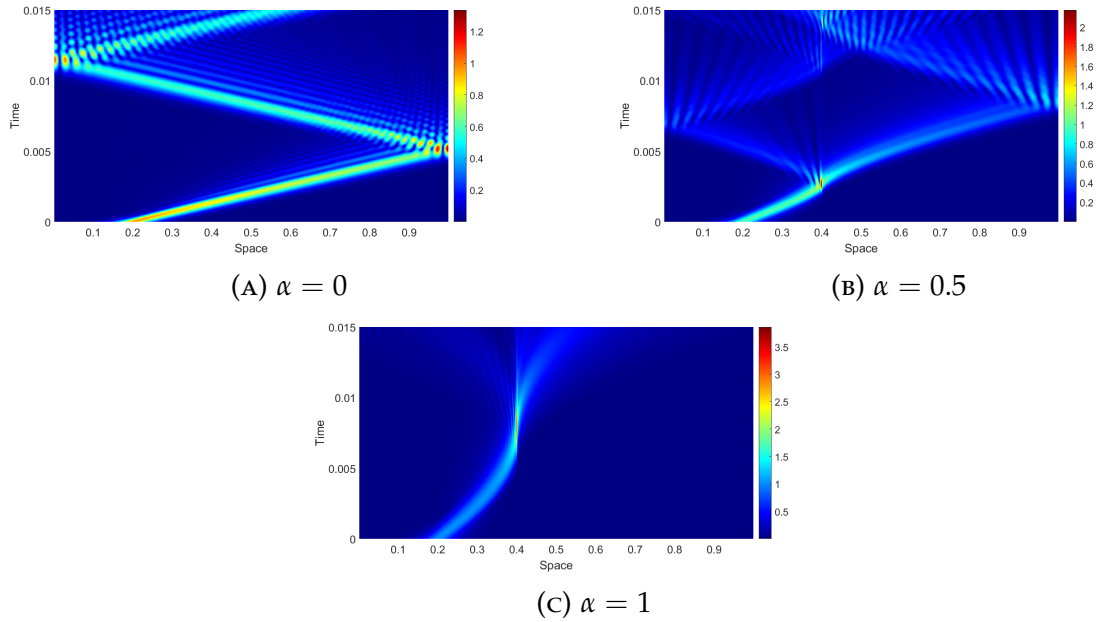
FIGURE 11. Test 3: Evolution of $|u(x, t)|$ in the space x with $\alpha = 1$ FIGURE 12. Test 3: Evolution of $|u(x, t)|$ in the plane (x, t)

5. Conclusion

In this paper, we assessed some theoretical and numerical studies of the degenerate Gross-Pitaevskii equation (1.1) where the degeneracy occurs in the interior of the space domain. The

FIGURE 13. Test 4: Evolution of $|u(x, t)|$ in the space x with $\alpha = 0$ FIGURE 14. Test 4: Evolution of $|u(x, t)|$ in the space x with $\alpha = 0.5$

degenerate GPE is showed to be well-posed in the second section whereas in the third section we investigate the stability of our scheme which is evaluated via a numerical experiments in the fourth section. In fact, in the first test wherein we consider that our boson is free (i.e $p(t, x) = 0$, $(t, x) \in (0, T) \times (0, L)$) and with no interaction with the other bosons (i.e, $\beta = 0$), the degeneracy phenomenon implies immediately that the Hamiltonian energy defined by (1.5) is null at position x_0 (case of test1) or when the position x_0 constitutes a zero point energy

FIGURE 15. Test 4: Evolution of $|u(x, t)|$ in the space x with $\alpha = 1$ FIGURE 16. Test 4: Evolution of $|u(x, t)|$ in the plane (x, t)

of our system where we have a considerable interaction with other bosons (tests 2, 3 and 4). Actually, the experiments 2, 3 and 4 show us that the impact of external forces manifesting in potential fields and electric fields does not change the fact that the point energy of the studied boson at position x_0 remains the lowest one of our quantum system (1.1). Note that in test 2, since the potential energy is taking positive then a small perturbations (may be an

Test 1	$t = 0.003$	$t = 0.006$	$t = 0.01$	$t = 0.015$
$\alpha = 0$	10	10.0112	10.0001	10.0010
$\alpha = 0.5$	10.0197	9.9998	10.0068	10.0072
$\alpha = 1$	10.0047	10.1278	10.2054	10.0080

TABLE 1. The values of probability density of finding boson over space $\|U^n\|_{L^2}$ for Test 1

Test 2	$t = 0.003$	$t = 0.006$	$t = 0.01$	$t = 0.015$
$\alpha = 0$	9.8104	9.6379	9.4465	9.4638
$\alpha = 0.5$	9.7619	9.4918	9.2344	9.1612
$\alpha = 1$	9.7236	9.5604	9.3402	9.0645

TABLE 2. The values of probability density of finding boson over space $\|U^n\|_{L^2}$ for Test 2

Test 3	$t = 0.003$	$t = 0.006$	$t = 0.01$	$t = 0.015$
$\alpha = 0$	9.8125	9.6335	9.3760	9.0781
$\alpha = 0.5$	9.7640	9.4868	9.1606	8.7612
$\alpha = 1$	9.7257	9.5556	9.2683	8.6586

TABLE 3. The values of probability density of finding boson over space $\|U^n\|_{L^2}$ for Test 3

Test 4	$t = 0.003$	$t = 0.006$	$t = 0.01$	$t = 0.015$
$\alpha = 0$	9.9971	10.0054	9.9902	9.9862
$\alpha = 0.5$	10.0173	9.9942	9.9973	9.9927
$\alpha = 1$	10.0017	10.1149	10.1987	9.9993

TABLE 4. The values of probability density of finding boson over space $\|U^n\|_{L^2}$ for Test 4

outcome of interaction or influence of electric fields) will cause the classical wave boson to grow exponentially away from the original stationary point. By the way, the importance of a zero point energy (the point wherein the energy is the lowest) in the context of boson can be seen for instance in the case of liquid helium. Liquid helium is the state of helium at very low temperature if it is at standard atmospheric pressure and liquid helium may show superfluidity. The zero point energy allows the liquid helium firstly to retain its kinetic energy and secondly to not freeze regardless of the low temperature. Finally, we emphasize that the case of the dimension 2 remains an open problem.

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