sciendo

CrossMark

Moroccan J. of Pure and Appl. Anal. (MJPAA)
Volume 8(2), 2022, Pages 212–227
ISSN: Online 2351-8227 - Print 2605-6364
DOI: 10.2478/mjpaa-2022-0015

Least energy sign-changing solutions for a nonlocal anisotropic Kirchhoff type equation

Mohammed Rahmani[1], Mostafa Rahmani[1], Aomar Anane [1], Mohammed Massar [2]

Abstract. In this paper, we investigate the existence of sign-changing solutions for the following class of fractional Kirchhoff type equations with potential

$$\left(1+b[u]_{\alpha}^{2}\right)\left((-\Delta_x)^{\alpha}u-\Delta_y u\right)+V(x,y)u=f(x,y,u),\ (x,y)\in\mathbb{R}^N=\mathbb{R}^n\times\mathbb{R}^m,$$

where $[u]_{\alpha}=\left(\int_{\mathbb{R}^N}\left(|(-\Delta_x)^{\frac{\alpha}{2}}u|^2+|\nabla_y u|^2\right)dxdy\right)^{\frac{1}{2}}$. Based on variational approach and a variant of the quantitative strain lemma, for each $b>0$, we show the existence of a least energy nodal solution u_b. In addition, a convergence property of u_b as $b\searrow 0$ is established.

Mathematics Subject Classification (2020). 35R11; 35D30
Keywords and phrases. Kirchhoff equations, BO-ZK operator, Sign-changing solutions, Deformation lemma.

1. Introduction and main result

Consider the following Kirchhoff equation involving BO-ZK operator

$$\left(1+b[u]_{\alpha}^{2}\right)\left((-\Delta_x)^{\alpha}u-\Delta_y u\right)+V(x,y)u=f(x,y,u),\ (x,y)\in\mathbb{R}^N=\mathbb{R}^n\times\mathbb{R}^m, \tag{1.1}$$

Received July 02, 2021 - Accepted: February 10, 2022.
© The Author(s) 2022. This article is published with open access by Sidi Mohamed Ben Abdallah University.

[1] *Department of Mathematics,Oriental Applied Mathematics Laboratory(LAMO), FSO,Mohamed first University, Morocco. e-mail: med.med.ja@gmail.com (M.Rahmani(Corresponding Author)), mo.rahmani@ump.ac.ma(Mos.Rahmani), ananeomar@gmail.com (A.Anane)*

[2]*Department of Mathematics, FTSH, Abdelmalek Essaadi University, Morocco. e-mail: massarmed@hotmail.com (M.Massar) .*

where $[u]_\alpha = \left(\int_{\mathbb{R}^N}\left(|(-\Delta_x)^{\frac{\alpha}{2}}u|^2 + |\nabla_y u|^2\right)dxdy\right)^{\frac{1}{2}}$, $\alpha \in (0,1)$, $n,m \geq 1$, $(-\Delta_x)^\alpha$ denotes the fractional Laplacian in x which is defined via the Fourier transform by $\widehat{(-\Delta_x)^\alpha u}(X,Y) = |X|^{2\alpha}\widehat{u}(X,Y)$, where $\widehat{u}$ is the Fourier transform of u; if u is smooth enough, it can be expressed by

$$(-\Delta_x)^\alpha u(x,y) = C_{N,s}P.V.\int_{\mathbb{R}^n}\frac{u(x,y)-u(z,y)}{|x-z|^{n+2\alpha}}dz,\ (x,y)\in\mathbb{R}^N,$$

with $C_{N,\alpha}$ is a normalization positive constant and P.V. stands for the Cauchy principal value; $V \in C(\mathbb{R}^N,\mathbb{R})$ and $f \in C^1(\mathbb{R}^N\times\mathbb{R},\mathbb{R})$.

Because of the presence of the coefficient $(1+b[u]_\alpha^2)$, equation (1.1) is nonlocal in the sense that is not a pointwise identity. This phenomenon produces new mathematical complications and so makes the study of such type of equations especially interesting. Furthermore, equation (1.1) is closely a fractional version related to the following hyperbolic equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

which was initially introduced by Kirchhoff [16] as a generalization of the classical D'Alembert's wave equation taking into consideration the change of the length of the strings produced by transverse vibrations. For additional discussions and physical phenomena described by nonlinear vibration theory, we mention [21].

On the other hand, for $b=0$, $V\equiv 1$ and $f(u)=u^p$, equation (1.1) goes back to the study of solitary waves of the generalized Benjamin-Ono-Zakharov-Kuznetsov equation

$$u_t + \partial_{x_1}\left((-\Delta_x)^\alpha u - \Delta_y u + u^p\right) = 0\ (x,y)\in\mathbb{R}^n\times\mathbb{R}^m, \tag{1.3}$$

see [25] for some local and global well-posedness results on this equation in the case $m=n=1$. The anisotropic operator $(-\Delta_x)^\alpha u - \Delta_y u$ has appeared in the study of toy models parabolic equations for which local diffusions occur only in certain directions and nonlocal diffusions. It models diffusion sensible to the direction in the Brownian and Lévy-Itô processes. Some results concerning regularity and rigidity properties of this operator can be found in [2, 11]. Recently, equation (1.1) was studied in [9] when $b=0$ and $f(x,y,u)=f(u)$. Exploiting some original arguments in [3, 4, 5], the author obtained a least energy sign-changing solution for (1.1) via the constrained minimization on a suitable Nehari manifolds.

Differential equations involving fractional Laplacian (isotropic) operator have attracted special attention due to their significant applications in many fields such as the thin obstacle problem, soft thin films, optimization, finance, phase transitions, anomalous diffusion, stratified materials, crystal dislocation, semipermeable membranes, flame propagation and ultra-relativistic limits of quantum mechanics, we refer to [1, 6, 7, 12, 17]. In particular, several studies have explored the existence and multiplicity of solutions for fractional problems through the use of different tools and techniques of nonlinear analysis, see e.g., [13, 20, 14, 27].

During the last few decades, the fractional of Kirchhoff-type problems have been investigated by many researchers, we can list here [15, 22, 24, 26]. However, as far as we know, except [18, 19], there are no papers that consider the existence of solutions for a Kirchhoff-type equation with an operator containing local and nonlocal diffusions. Influenced by the

works mentioned above, in the present paper we first establish the existence of a least energy sign-changing solution for any parameter $b > 0$, making use of a minimization method on the nodal Nehari manifold and a modified Miranda's theorem. After that, we give a convergence property of solution obtained as $b \to 0$.

To state our results, we assume that $f \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and

(f_1) $\lim_{t\to 0} \frac{f(x,y,t)}{|t|^3} = 0$ uniformly in $(x,y) \in \mathbb{R}^N$;

(f_2) $\lim_{|t|\to\infty} \frac{f(x,y,t)}{|t|^{p-1}} = 0$ uniformly in $(x,y) \in \mathbb{R}^N$, for some $p \in (2, p^*)$ where $p^* := \frac{2(n+m\alpha)}{n+(m-2)\alpha}$;

(f_3) $\lim_{|t|\to\infty} \frac{F(x,y,t)}{t^4} = +\infty$ uniformly in $(x,y) \in \mathbb{R}^N$, where $F(x,y,t) = \int_0^t f(x,y,\tau)d\tau$;

(f_4) For all $(x,y) \in \mathbb{R}^N$, the function $t \mapsto \frac{f(x,y,t)}{|t|^3}$ is nondecreasing in $\mathbb{R}\backslash\{0\}$.

For the potential V, we assume that $V : \mathbb{R}^N \to \mathbb{R}$ is continuous and verifies:

(V_1) $\inf_{(x,y)\in\mathbb{R}^N} V(x,y) =: V_0 > 0$. Moreover, there exists $d > 0$ such that

$$\lim_{|z|\to+\infty} |\{(x,y) \in B(z,d) : V(x,y) \le M\}| = 0 \text{ for all } M > 0.$$

Remark 1.1. *By (f_1) and (f_4), $F(x,y,t) \ge 0$ for all $(x,y,t) \in \mathbb{R}^N \times \mathbb{R}$.*

Due the dependence of potential, let us first recall our suitable working Sobolev space. Let $\mathcal{H}^\alpha(\mathbb{R}^N)$ be the fractional Sobolev-Liouville space

$$\mathcal{H}^\alpha(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left(|(-\Delta_x)^{\frac{\alpha}{2}} u|^2 + |\nabla_y u|^2 + u^2 \right) dxdy < \infty \right\}$$

with the norm

$$||u||_{\mathcal{H}^\alpha} = \left(\int_{\mathbb{R}^N} \left(|(-\Delta_x)^{\frac{\alpha}{2}} u|^2 + |\nabla_y u|^2 + u^2 \right) dxdy \right)^{\frac{1}{2}}.$$

The homogeneous fractional Sobolev-Liouville space $\mathcal{D}^\alpha(\mathbb{R}^N)$ is defined by

$$\mathcal{D}^\alpha(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : [u]_\alpha < \infty\},$$

where

$$[u]_\alpha = \left(\int_{\mathbb{R}^N} \left(|(-\Delta_x)^{\frac{\alpha}{2}} u|^2 + |\nabla_y u|^2 \right) dxdy \right)^{\frac{1}{2}}.$$

We search the solutions in the following subspace

$$E_\alpha = \left\{ u \in \mathcal{D}^\alpha(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x,y)dxdy < \infty \right\},$$

endowed with the inner product

$$(u,v)_{E_\alpha} = \int_{\mathbb{R}^N} \left((-\Delta_x)^{\frac{\alpha}{2}} u (-\Delta_x)^{\frac{\alpha}{2}} v + \nabla_y u \nabla_y v \right) dxdy + \int_{\mathbb{R}^N} V(x,y)uvdxdy$$

and its related norm

$$||u||_{E_\alpha} = (u,u)_{E_\alpha}^{\frac{1}{2}}.$$

A function $u \in E_\alpha$ is called weak solution of (1.1) if

$$\left(1 + b[u]_\alpha^2\right) \int_{\mathbb{R}^N} \left((-\Delta_x)^{\frac{\alpha}{2}} u (-\Delta_x)^{\frac{\alpha}{2}} v + \nabla_y u \nabla_y v\right) dxdy + \int_{\mathbb{R}^N} V(x,y)uvdxdy$$
$$= \int_{\mathbb{R}^N} f(x,y,u)vdxdy, \text{ for all } v \in E_\alpha,$$

A sign-changing solution of (1.1) is a weak solution $u \in E_\alpha$ satisfying $u^\pm \neq 0$, where

$$u^+ = \max(u,0) \quad \textit{and} \quad u^- = \min(u,0).$$

The main results can be described by the following two theorems.

Theorem 1.1. *Assume that (V_1) and $(f_1)-(f_4)$ hold. Then equation (1.1) admits a least energy sign-changing solution u_b.*

Theorem 1.2. *Assume that (V_1) and $(f_1)-(f_4)$ hold. Then, for any real sequence $\{b_n\}$ satisfying $b_n \to 0$, up to a subsequence $u_{b_n} \to u_0$ in E_α, where u_0 is a least energy sign-changing solution for the following equation*

$$(-\Delta_x)^\alpha u - \Delta_y u + V(x,y)u = f(x,y,u), \ (x,y) \in \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m, \tag{1.4}$$

2. Auxiliary Results and Proof of main theorems

In this section, we give some preliminary results for the proof of Theorems 1.1 and 1.2.

Lemma 2.1 ([10]). *The fractional Sobolev-Liouville space $\mathcal{H}^\alpha(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$ for $q \in [2,p^*]$; and is compactly embedded into $L^q(\mathbb{R}^N)$ for $q \in (2,p^*)$.*

According to the previous lemma, we have the next result (see [9, Lemma 2.1]).

Lemma 2.2 ([9]). *Assume that (V_1) holds. Then E_α is continuously embedded into $L^q(\mathbb{R}^N)$ for $q \in [2,p^*]$; and compactly embedded into $L^q(\mathbb{R}^N)$ for $q \in [2,p^*)$;*

Obviously, the energy functional associated to (1.1) defined on E_α by

$$I_b(u) = \frac{1}{2}||u||_{E_\alpha}^2 + \frac{b}{4}[u]_\alpha^4 - \int_{\mathbb{R}^N} F(x,y,u)dxdy,$$

is of class C^1 and its critical points are solutions of (1.1). When $b = 0$, $I_b = I_0$ is the corresponding functional to equation (1.4).

To discus the existence of sign-changing solutions, we follow some original arguments in [3, 4, 5]. More precisely, we will minimize the functional I_b on the nodal set

$$\mathcal{M}_b = \left\{u \in \mathcal{N}_b : u^\pm \neq 0, \ \langle I_b'(u), u^+\rangle = \langle I_b'(u), u^-\rangle = 0\right\},$$

where

$$\mathcal{N}_b = \left\{u \in E_\alpha \backslash \{0\} : \langle I_b'(u), u\rangle = 0\right\}.$$

The following modified Miranda's theorem [23] (see[8, Lemma 2.3]) will be used to establish some results on $\mathcal{M}_b$.

Lemma 2.3 ([9]). *Let $\mathcal{G} = \{z = (z_1, z_2, \dots, z_N) \in \mathbb{R}^N : r < z_i < R \text{ for } i = 1, \dots, N\}$. Assume that the mapping $G = (G_1, G_2, \dots, G_N) : \overline{\mathcal{G}} \to \mathbb{R}^N$ is continuous such that $G(z) \neq 0$ for all $z \in \partial\mathcal{G}$, and*

$$G_i(z_1, z_2, \dots, z_{i-1}, r, z_{i+1}, \dots, z_N) \geq 0 \text{ for } i = 1, \dots, N,$$
$$G_i(z_1, z_2, \dots, z_{i-1}, R, z_{i+1}, \dots, z_N) \leq 0 \text{ for } i = 1, \dots, N.$$

Then equation $G(z) = 0$ has a solution in $\mathcal{G}$.

Lemma 2.4. *Assume that (V_1) and $(f_1) - (f_4)$ hold. Then for any $u \in E_\alpha$ with $u^\pm \neq 0$, there exists a unique pair of positive reals (t_u, s_u) such that $t_u u^+ + s_u u^- \in \mathcal{M}_b$.*

Proof. For $u \in E_\alpha$ with $u^\pm \neq 0$, define vector field $\Psi : [0, \infty)^2 \to \mathbb{R}^2$ by

$$\Psi(t, s) = \left(\langle I_b'(tu^+ + s^-u^-), tu^+\rangle, \langle I_b'(tu^+ + s^-u^-), su^-\rangle\right).$$

We notice that

$$[tu^+ + su^-]_\alpha^2 = t^2[u^+]_\alpha^2 + s^2[u^-]_\alpha^2 + 2tsZ(u^+, u^-) \tag{2.1}$$

and

$$\int_{\mathbb{R}^{2n+m}} \frac{((tu^+ + su^-)(x,y) - (tu^+ + su^-)(z,y))t(u^+(x,y) - u^+(z,y))}{|x-z|^{n+2\alpha}} dxdzdy$$
$$+ \int_{\mathbb{R}^N} |\nabla_y(tu^+)|^2 dxdy = t^2[u^+]_\alpha^2 + 2tsZ(u^+, u^-), \tag{2.2}$$

where

$$Z(u^+, u^-) = -\int_{\mathbb{R}^{2n+m}} \frac{u^+(x,y)u^-(z,y) + u^-(x,y)u^+(z,y)}{|x-z|^{n+2\alpha}} dxdzdy \geq 0,$$

By a direct computation, we derive

$$\begin{aligned}
\langle I_b'(tu^+ + s^-u^-), tu^+\rangle &= b\left(t^2[u^+]_\alpha^2 + s^2[u^-]_\alpha^2 + 2tsZ(u^+, u^-)\right)\left(t^2[u^+]_\alpha^2 + 2tsZ(u^+, u^-)\right) \\
&\quad + t^2[u^+]_\alpha^2 + 2tsZ(u^+, u^-) + \int_{\mathbb{R}^N} V(x,y)(tu^+)^2 dxdy \\
&\quad - \int_{\mathbb{R}^N} f(x, y, tu^+)tu^+ dxdy \\
&= bt^4[u^+]_\alpha^4 + 3bt^3s[u^+]_\alpha^2 Z(u^+, u^-) + bt^2s^2[u^+]_\alpha^2[u^-]_\alpha^2 + 4bt^2s^2Z(u^+, u^-)^2 \\
&\quad + t^2[u^+]_\alpha^2 + 2bts^3[u^-]_\alpha^2 Z(u^+, u^-) + 2tsZ(u^+, u^-) \\
&\quad + t^2\int_{\mathbb{R}^N} V(x,y)(u^+)^2 dxdy - \int_{\mathbb{R}^N} f(x, y, tu^+)tu^+ dxdy.
\end{aligned} \tag{2.3}$$

From $(f_1) - (f_2)$, for given ε, there exists $C_\varepsilon > 0$ such that

$$|f(x, y, t)t| \leq \varepsilon t^2 + C_\varepsilon |t|^{p^*} \text{ for all } t \in \mathbb{R}. \tag{2.4}$$

Choosing $\varepsilon < \frac{[u^+]_\alpha^2}{2||u^+||_{L^2(\mathbb{R}^N)}^2}$, it follows from (2.3)-(2.4) that

$$\begin{aligned}
\langle I_b'(tu^+ + su^-), tu^+\rangle &> bt^4[u^+]_\alpha^4 + 3bt^3s[u^+]_\alpha^2 Z(u^+, u^-) + bt^2s^2[u^+]_\alpha^2[u^-]_\alpha^2 + 4bt^2s^2Z(u^+, u^-)^2 \\
&\quad + 2bts^3[u^-]_\alpha^2 Z(u^+, u^-) + 2tsZ(u^+, u^-)
\end{aligned}$$

$$+ t^2 \int_{\mathbb{R}^N} V(x,y)(u^+)^2 dxdy + \frac{t^2}{2}[u^+]_\alpha^2 - C_1 t^{p*}||u^+||^{p^*}_{L^{p*}(\mathbb{R}^N)}$$
$$> \frac{t^2}{2}[u^+]_\alpha^2 - C_1 t^{p*}||u^+||^{p^*}_{L^{p*}(\mathbb{R}^N)}. \tag{2.5}$$

On the other hand, for some $\rho > 0$, we have $|\Omega_\rho^+| > 0$, where $\Omega_\rho^+ := \{(x,y) \in \mathbb{R}^N : u^+(x,y) > \rho\}$. By (f_3), for $A > \frac{2b[u^+]_\alpha^4}{\int_{\Omega_\rho^+}(u^+)^4 dxdy}$, there exists $T > 0$ such that

$$\frac{f(x,y,t)}{t^3} > A \text{ for all } t > T \text{ and } (x,y) \in \mathbb{R}^N.$$

This and (f_4) imply that for all $t > \frac{T}{\rho}$,

$$\int_{\mathbb{R}^N} f(x,y,tu^+)tu^+ dxdy \geq \int_{\Omega_\rho^+} \frac{f(x,y,tu^+)}{(tu^+)^3}(tu^+)^4 dxdy > 2bt^4[u^+]_\alpha^4.$$

Therefore, by (2.3) we get

$$\langle I_b'(tu^+ + su^-), tu^+\rangle < -bt^4[u^+]_\alpha^4 + 3bt^3 s[u^+]_\alpha^2 Z(u^+,u^-) + bt^2s^2[u^+]_\alpha^2[u^-]_\alpha^2 + 4bt^2s^2Z(u^+,u^-)^2$$
$$+ t^2[u^+]_\alpha^2 + 2bts^3[u^-]_\alpha^2 Z(u^+,u^-) + 2tsZ(u^+,u^-)$$
$$+ t^2 \int_{\mathbb{R}^N} V(x,y)(u^+)^2 dxdy. \tag{2.6}$$

Similarly, for some $C_2 > 0$ we have

$$\langle I_b'(tu^+ + su^-), su^-\rangle > \frac{s^2}{2}[u^-]_\alpha^2 - C_2 s^{p*}||u^-||^{p^*}_{L^{p*}(\mathbb{R}^N)}. \tag{2.7}$$

and for s sufficiently large,

$$\langle I_b'(tu^+ + su^-), su^-\rangle < -bs^4[u^-]_\alpha^4 + 3bts^3[u^-]_\alpha^2 Z(u^+,u^-) + bt^2s^2[u^+]_\alpha^2[u^-]_\alpha^2 + 4bt^2s^2Z(u^+,u^-)^2$$
$$+ s^2[u^-]_\alpha^2 + 2bt^3 s[u^+]_\alpha^2 Z(u^+,u^-) + 2tsZ(u^+,u^-)$$
$$+ s^2 \int_{\mathbb{R}^N} V(x,y)(u^-)^2 dxdy. \tag{2.8}$$

By (2.5)-(2.8), we can find $0 < r < R$ such that

$$\langle I_b'(ru^+ + su^-), ru^+\rangle > 0, \quad \langle I_b'(tu^+ + ru^-),\ ru^-\rangle > 0 \text{ for all } t,s \in [r,R] \tag{2.9}$$

and

$$I_b'(Ru^+ + su^-), Ru^+\rangle < 0, \quad I_b'(tu^+ + Ru^-), Ru^-\rangle < 0 \text{ for all } t,s \in [r,R]. \tag{2.10}$$

Hence, applying Lemma 2.3, there exist $t_u, s_u \in [r,R]$ such that $\Psi(t_u,s_u) = 0$, which means that $t_u u^+ + s_u u^- \in \mathcal{M}_b$. Let us now show the uniqueness of (t_u,s_u).

Case 1: $u \in \mathcal{M}_b$. Since $u = u^+ + u^-$,

$$b\left([u^+]_\alpha^2 + [u^-]_\alpha^2 + 2Z(u^+,u^-)\right)\left([u^+]_\alpha^2 + 2Z(u^+,u^-)\right) + [u^+]_\alpha^2 + 2Z(u^+,u^-)$$
$$+ \int_{\mathbb{R}^N} V(x,y)(u^+)^2 dxdy = \int_{\mathbb{R}^N} f(x,y,u^+)u^+ dxdy \tag{2.11}$$

and

$$b\left([u^+]_\alpha^2+[u^-]_\alpha^2+2Z(u^+,u^-)\right)\left([u^-]_\alpha^2+2Z(u^+,u^-)\right)+[u^-]_\alpha^2+2Z(u^+,u^-)$$
$$+\int_{\mathbb{R}^N}V(x,y)(u^-)^2dxdy=\int_{\mathbb{R}^N}f(x,y,u^-)u^-dxdy. \tag{2.12}$$

Claim 1. Let $t_0,s_0>0$ such that $t_0u^+ + s_0u^- \in \mathcal{M}_b$. Then $t_0=s_0=1$.

In fact, we have

$$b\left(t_0^2[u^+]_\alpha^2+s_0^2[u^-]_\alpha^2+2t_0s_0Z(u^+,u^-)\right)\left(t_0^2[u^+]_\alpha^2+2t_0s_0Z(u^+,u^-)\right)+t_0^2[u^+]_\alpha^2+2t_0s_0Z(u^+,u^-)$$
$$+\int_{\mathbb{R}^N}V(x,y)(t_0u^+)^2dxdy=\int_{\mathbb{R}^N}f(x,y,t_0u^+)t_0u^+dxdy \tag{2.13}$$

and

$$b\left(t_0^2[u^+]_\alpha^2+s_0^2[u^-]_\alpha^2+2t_0s_0Z(u^+,u^-)\right)\left(s_0^2[u^-]_\alpha^2+2t_0s_0Z(u^+,u^-)\right)+s_0^2[u^-]_\alpha^2+2t_0s_0Z(u^+,u^-)$$
$$+\int_{\mathbb{R}^N}V(x,y)(s_0u^-)^2dxdy=\int_{\mathbb{R}^N}f(x,y,s_0u^-)s_0u^-dxdy. \tag{2.14}$$

If we assume for example that $t_0\le s_0$, it follows from (2.13) that

$$bt_0^4\left([u^+]_\alpha^2+[u^-]_\alpha^2+2Z(u^+,u^-)\right)\left([u^+]_\alpha^2+2Z(u^+,u^-)\right)+t_0^2\left([u^+]_\alpha^2+2Z(u^+,u^-)\right)$$
$$+t_0^2\int_{\mathbb{R}^N}V(x,y)(u^+)^2dxdy\le\int_{\mathbb{R}^N}f(x,y,t_0u^+)t_0u^+dxdy. \tag{2.15}$$

Combining (2.11) and (2.15), we immediately obtain

$$\left(1-\frac{1}{t_0^2}\right)\left([u^+]_\alpha^2+2Z(u^+,u^-)+\int_{\mathbb{R}^N}V(x,y)(u^+)^2dxdy\right)$$
$$\ge\int_{\mathbb{R}^N}\left(\frac{f(x,y,u^+)}{(u^+)^3}-\frac{f(x,y,t_0u^+)}{(t_0u^+)^3}\right)(u^+)^4dxdy. \tag{2.16}$$

If $t_0<1$, by (f_4) and (2.16), we get a contradiction. Then $s_0\ge t_0\ge 1$. Analogously, from (2.12), (2.14) and (f_4), we have $t_0\le s_0\le 1$. Then, we deduce that $t_0=s_0=1$.

Case 2: $u\notin\mathcal{M}_b$. Let (t_1,s_1) and (t_2,s_2) be two pair of positive reals such that

$$v_1:=t_1u^+ + s_1u^-\in\mathcal{M}_b \quad\text{and}\quad v_2:=t_2u^+ + s_2u^-\in\mathcal{M}_b.$$

Then $v_1^\pm\neq 0$ and

$$\frac{t_2}{t_1}v_1^+ + \frac{s_2}{s_1}v_1^- = v_2\in\mathcal{M}_b.$$

Since $v_1\in\mathcal{M}_b$, it follows from Claim 1 that $\frac{t_2}{t_1}=\frac{s_2}{s_1}=1$, i.e., $t_1=t_2$ and $s_1=s_2$. The uniqueness is now achieved.

For $u\in E_\alpha$ with $u^\pm\neq 0$, we define the function $\varphi_u:[0,\infty)\times[0,\infty)\to\mathbb{R}$ as follows

$$\varphi_u(t,s)=I_b(tu^+ + su^-).$$

Lemma 2.5. *Assume that (V_1) and $(f_1)-(f_4)$ hold. For any $u \in E_\alpha$ with $u^\pm \neq 0$, the function φ_u has a unique global maximum $(t_+, s_-) = (t_+(u), s_-(u)) := (t_u, s_u)$, where the pair (t_u, s_u) is given by Lemma 2.4.*

Proof. Let $\Omega_0^\pm := \{(x,y) \in \mathbb{R}^N : u^\pm(x,y) \neq 0\}$. Then $|\Omega_0^\pm| > 0$ and

$$\begin{aligned}
\varphi_u(t,s) &= \frac{1}{2}||tu^+ + su^-||^2_{E_\alpha} + \frac{b}{4}[tu^+ + su^-]^4_\alpha - \int_{\mathbb{R}^N} F(x,y,tu^+ + su^-)dxdy \\
&= \frac{t^2}{2}||u^+||^2_{E_\alpha} + \frac{s^2}{2}||u^-||^2_{E_\alpha} + tsZ(u^+,u^-) + \frac{b}{4}[tu^+ + su^-]^4_\alpha - \int_{\mathbb{R}^N} (F(x,y,tu^+ + su^-)dxdy \\
&= \frac{t^2}{2}||u^+||^2_{E_\alpha} + \frac{s^2}{2}||u^-||^2_{E_\alpha} + tsZ(u^+,u^-) + \frac{b}{4}\left(t^2[u^+]^2_\alpha + s^2[u^-]^2_\alpha + 2tsZ(u^+,u^-)\right)^2 \\
&- \int_{\Omega_0^+} F(x,y,tu^+)dxdy - \int_{\Omega_0^-} F(x,y,su^-)dxdy \\
&\leq (t^2+s^2)\left(\frac{1}{2}||u^+||^2_{E_\alpha} + \frac{1}{2}||u^-||^2_{E_\alpha} + Z(u^+,u^-)\right) \\
&+ \frac{b(t^2+s^2)^2}{4}\left([u^+]^2_\alpha + [u^-]^2_\alpha + Z(u^+,u^-)\right)^2 \\
&- \int_{\Omega_0^+} F(x,y,tu^+)dxdy - \int_{\Omega_0^-} F(x,y,su^-)dxdy.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\varphi_u(t,s)}{(t^2+s^2)^2} \leq\ & \frac{1}{t^2+s^2}\left(\frac{1}{2}||u^+||^2_{E_\alpha} + \frac{1}{2}||u^-||^2_{E_\alpha} + Z(u^+,u^-)\right) + \frac{b}{4}\left([u^+]^2_\alpha + [u^-]^2_\alpha + Z(u^+,u^-)\right)^2 \\
&- \frac{t^4}{(t^2+s^2)^2}\int_{\Omega_0^+}\frac{F(x,y,tu^+)}{(tu^+)^4}(u^+)^4dxdy - \frac{s^4}{(t^2+s^2)^2}\int_{\Omega_0^-}\frac{F(x,y,su^-)}{(su^-)^4}(u^-)^4dxdy.
\end{aligned} \tag{2.17}$$

Observing that

$$\frac{t^4}{(t^2+s^2)^2} \geq \frac{1}{4} \text{ if } t \geq s \quad \text{and} \quad \frac{s^4}{(t^2+s^2)^2} \geq \frac{1}{4} \text{ if } t \leq s,$$

it follows from (2.17), (f_3) and Fatou's lemma that

$$\begin{aligned}
\limsup_{\substack{|(t,s)|\to\infty \\ t\geq s}} \frac{\varphi_u(t,s)}{(t^2+s^2)^2} \leq\ & \frac{b}{4}\left([u^+]^2_\alpha + [u^-]^2_\alpha + Z(u^+,u^-)\right)^2 \\
& - \frac{1}{4}\int_{\Omega_0^+}\liminf_{t\to+\infty}\frac{F(x,y,tu^+)}{(tu^+)^4}(u^+)^4dxdy = -\infty
\end{aligned}$$

and

$$\limsup_{\substack{|(t,s)|\to\infty \\ s\geq t}} \frac{\varphi_u(t,s)}{(t^2+s^2)^2} \leq \frac{b}{4}\left([u^+]^2_\alpha + [u^-]^2_\alpha + Z(u^+,u^-)\right)^2$$

$$-\frac{1}{4}\int_{\Omega_0^-}\liminf_{s\to+\infty}\frac{F(x,y,tu^-)}{(su^-)^4}(u^-)^4dxdy=-\infty.$$

Hence, $\varphi_u(t,s)\to-\infty$ as $|(t,s)|\to\infty$. By the continuity, we conclude that φ_u admits a global maximum $(\hat{t},\hat{s})\in[0,\infty)^2$. Note that $[tu^+]_\alpha^2+[su^-]_\alpha^2\leq[tu^++su^-]_\alpha^2$ and

$$\int_{\mathbb{R}^N}F(x,y,tu^++su^-)dxdy=\int_{\mathbb{R}^N}F(x,y,tu^+)dxdy+\int_{\mathbb{R}^N}F(x,y,su^-)dxdy.$$

Thus

$$\varphi_u(t,0)+\varphi_u(0,s)=I_b(tu^+)+I_b(su^-)\leq I_b(tu^++su^-)=\varphi_u(t,s)\text{ for all }t,s\geq0.$$

This together with the fact that $\max_{t\geq0}\varphi_u(t,0),\max_{s\geq0}\varphi_u(0,s)>0$ gives

$$\max_{t\geq0}\varphi_u(t,0)<\max_{t,s\geq0}\varphi_u(t,s)=\varphi_u(\hat{t},\hat{s})\quad\text{and}\quad\max_{s\geq0}\varphi_u(0,s)<\max_{t,s\geq0}\varphi_u(t,s)=\varphi_u(\hat{t},\hat{s}),$$

which proves that $\hat{t},\hat{s}>0$. Consequently $(\hat{t},\hat{s})$ is a critical point with positive coordinates. From Lemma 2.4, we conclude that $(\hat{t},\hat{s})=(t_u,s_u)$, and hence $(t_+,s_-):=(t_u,s_u)$ is a unique global maximum, the proof is complete.

Lemma 2.6. *Assume that (V_1) and $(f_1)-(f_4)$. For any $\{u_n\}\subset\mathcal{M}_b$, if $u_n\rightharpoonup u$ weakly in E_α, then $u^\pm\neq0$.*

Proof. For all $v\in\mathcal{M}_b$, we have

$$||v^\pm||_{E_\alpha}^2\leq\int_{\mathbb{R}^N}f(x,y,v^\pm)v^\pm dxdy$$

By $(f_1)-(f_2)$ and Lemma 2.2, for $\varepsilon>0$, there exists C_ε, such that

$$||v^\pm||_{E_\alpha}^2\leq\varepsilon V_0^{-1}||v^\pm||_{E_\alpha}^2+CC_\varepsilon||v^\pm||_{E_\alpha}^{p^*}.$$

Letting $\varepsilon<V_0$, we get

$$||v^\pm||_{E_\alpha}\geq\delta:=\left(\frac{1-\varepsilon V_0^{-1}}{CC_\varepsilon}\right)^{\frac{1}{p^*-2}}.$$

Let $\{u_n\}\subset\mathcal{M}_b$ such that $u_n\rightharpoonup u$ weakly in E_α. Then

$$\int_{\mathbb{R}^N}f(x,y,u_n^\pm)u_n^\pm dxdy\geq\delta\text{ for all }n\in\mathbb{N}.$$

According to $(f_1)-(f_2)$, we see that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}f(x,y,u_n^\pm)u_n^\pm dxdy=\int_{\mathbb{R}^N}f(x,y,u^\pm)u^\pm dxdy,\tag{2.18}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}F(x,y,u_n^\pm)dxdy=\int_{\mathbb{R}^N}F(x,y,u^\pm)dxdy.\tag{2.19}$$

Thus

$$\int_{\mathbb{R}^N}f(x,y,u^\pm)u^\pm dxdy\geq\delta>0,$$

which yields that $u^\pm\neq0$.

Proof of Theorem 1.1. Let $\{u_n\} \subset \mathcal{M}_b$ be a minimizing sequence of I_b, that is

$$\lim_{n\to\infty} I_b(u_n) = \inf_{v\in\mathcal{M}_b} I_b(v) = c_\infty^{\mathcal{M}_b} > 0. \tag{2.20}$$

Let us first show that $\{u_n\}$ is bounded in E_α. Arguing indirectly, up to a subsequence $||u_n||_{E_\alpha} \to \infty$ and set $v_n = \frac{u_n}{||u_n||_{E_\alpha}}$. Then $\{v_n\}$ is bounded in E_α. So, in view of Lemma 2.2, we may assume that

$$v_n \rightharpoonup v \text{ in } E_\alpha,\ v_n \to v \text{ a.e. } \mathbb{R}^N \text{ and } v_n \to v \text{ in } L^q(\mathbb{R}^N) \text{ for } q \in [2, p^*). \tag{2.21}$$

Since $||u_n||_{E_\alpha} v_n^+ + ||u_n||_{E_\alpha} v_n^- = u_n \in \mathcal{M}_b$, by Lemma 2.5,

$$t_+(v_n) = s_-(v_n) = ||u_n||_{E_\alpha}.$$

Therefore, for all $t > 0$ and $n \in \mathbb{N}$,

$$\begin{aligned} I_b(u_n) &= I_b(||u_n||_{E_\alpha} v_n) = I_b(t_+(v_n)v_n^+ + s_-(v_n)v_n^-) = \varphi_{v_n}(t_+(v_n), s_-(v_n)) \\ &\geq \varphi_{v_n}(t,t) = I_b(tv_n) = \frac{t^2}{2} + \frac{bt^4}{4}[v_n]_\alpha^4 - \int_{\mathbb{R}^N} F(x,y,tv_n)dxdy \\ &\geq \frac{t^2}{2} - \int_{\mathbb{R}^N} F(x,y,tv_n)dxdy. \end{aligned} \tag{2.22}$$

If $v = 0$, then (2.20)-(2.22) and Fatou's lemma imply

$$c_\infty^{\mathcal{M}_b} \geq \frac{t^2}{2} \text{ for all } t > 0, \tag{2.23}$$

which is an absurd and hence $v \neq 0$. Thus $|\Omega_0| > 0$, where $\Omega_0 = \{(x,y) : v(x,y) \neq 0\}$.

Dividing $I_b(u_n)$ by $||u_n||_{E_\alpha}^4$ and using (2.21), (f_3) and Fatou's lemma, we entail

$$0 = \lim_{n\to\infty} \frac{I_b(u_n)}{||u_n||_{E_\alpha}^4} \leq \frac{b}{4} - \int_{\Omega_0} \liminf_{n\to\infty} \frac{F(x,y,||u_n||_{E_\alpha} v_n)}{(||u_n||_{E_\alpha} v_n)^4} v_n^4 dxdy = -\infty.$$

which is impossible and consequently $\{u_n\}$ is bounded in E_α. Then $u_n \rightharpoonup u_b$ in E_α. From Lemma 2.6, we infer that $u_b^\pm \neq 0$. By Lemma 2.4, there exist $t_+, s_- > 0$ such that $t_+u_b^+ + s_-u_b^- \in \mathcal{M}_b$ and

$$\langle I_b'(t_+u_b^+ + s_-u_b^-), u_b^\pm \rangle = 0.$$

Since $u_n \rightharpoonup u_b$ and $u_n \in \mathcal{M}_b$, from (2.18) and Fatou's Lemma, we deduce that

$$\langle I_b'(u_b), u_b^\pm \rangle \leq \liminf_{n\to\infty} \langle I_b'(u_n), u_n^\pm \rangle = 0.$$

Similar to the analysis above, we see that $t_+, s_- \leq 1$. It follows from (f_4) and (2.18)-(2.19) that

$$\begin{aligned} c_\infty^{\mathcal{M}_b} &\leq I_b(t_+u_b^+ + s_-u_b^-) \\ &= I_b(t_+u_b^+ + s_-u_b^-) - \frac{1}{4}\langle I_b'(t_+u_b^+ + s_-u_b^-), t_+u_b^+ + s_-u_b^- \rangle \\ &= \frac{1}{4}||t_+u_b^+ + s_-u_b^-||_{E_\alpha}^2 + \int_{\mathbb{R}^N} \left(\frac{1}{4} f(x,y,t_+u_b^+)t_+u_b^+ - F(x,y,t_+u_b^+) \right) dxdy \\ &\quad + \int_{\mathbb{R}^N} \left(\frac{1}{4} f(x,y,s_-u_b^-)s_-u_b^- - F(x,y,s_-u_b^-) \right) dxdy \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{4}||u_b||^2_{E_\alpha} + \int_{\mathbb{R}^N}\left(\frac{1}{4}f(x,y,u_b^+)u^+ - F(x,y,u_b^+)\right)dxdy\\
&\quad + \int_{\mathbb{R}^N}\left(\frac{1}{4}f(x,y,u_b^-)u_b^- - F(x,y,u_b^-)\right)dxdy\\
&= \frac{1}{4}||u_b||^2_{E_\alpha} + \int_{\mathbb{R}^N}\left(\frac{1}{4}f(x,y,u_b)u_b - F(x,y,u_b)\right)dxdy\\
&\leq \liminf_{n\to\infty}\left[\frac{1}{4}||u_n||^2_{E_\alpha} + \int_{\mathbb{R}^N}\left(\frac{1}{4}f(x,y,u_n)u_n - F(x,y,u_n)\right)dxdy\right]\\
&= \liminf_{n\to\infty}\left[I_b(u_n) - \frac{1}{4}\langle I_b'(u_n),u_n\rangle\right] = c_\infty^{\mathcal{M}_b}.
\end{aligned}$$

This yields $I_b(t_+u_b^+ + s_-u_b^-) = c_\infty^{\mathcal{M}_b}$. Arguing as in the proof of Claim 1, we derive that $t_+ = s_- = 1$, and so

$$I_b(u_b) = \inf_{v\in\mathcal{M}_b} I_b(v).$$

Now, we show that $I_b'(u_b) = 0$. Suppose by contradiction that $I_b'(u_b) \neq 0$. Then, by the continuity of I_b', there exist $\nu, \varrho > 0$ such that

$$||I_b'(v)|| \geq \nu \text{ for all } v \in E_\alpha \text{ with } ||v-u_b||_{E_\alpha} \leq 3\varrho. \tag{2.24}$$

Let $0 < a_1 < 1 < a_2$ with $a_2 - a_1 < \frac{\varrho}{\sqrt{2}||u_b||_{E_\alpha}}$, and $D = (a_1,a_2)\times(a_1,a_2)$. Define the function $\gamma_b : \overline{D} \to E_\alpha$ by

$$\gamma_b(t,s) = tu_b^+ + su_b^-.$$

Then $I(\gamma_b(1,1)) = c_\infty^{\mathcal{M}_b}$, and by Lemma 2.5, $I_b(\gamma_b(t,s)) < c_\infty^{\mathcal{M}_b}$ for all $(t,s) \in \overline{D}\backslash\{(1,1)\}$. Therefore

$$\varpi := \max_{(t,s)\in\partial D} I_b(\gamma_b(t,s)) < c_\infty^{\mathcal{M}_b}. \tag{2.25}$$

In view of (2.24) and [28, Lemma 2.3], for $\varepsilon < \min\left\{\frac{\nu\varrho}{8}, \frac{c_\infty^{\mathcal{M}_b}-\varpi}{2}\right\}$, there exists $\eta \in C([0,1]\times E_\alpha, E_\alpha)$ verifying

$\eta_1)$ $\eta(t,v) = v$ if $v \notin I_b^{-1}([c_\infty^{\mathcal{M}_b} - 2\varepsilon, c_\infty^{\mathcal{M}_b} + 2\varepsilon])$;
$\eta_2)$ $I_b(\eta(1,v)) \leq c_\infty^{\mathcal{M}_b} - \varepsilon$ for all $v \in E_\alpha$ with $||v-u_b||_{E_\alpha} \leq \varrho$ and $I_b(v) \leq c_\infty^{\mathcal{M}_b} + \varepsilon$;
$\eta_3)$ $I_b(\eta(1,v) \leq I_b(\eta(0,v)) = I_b(v)$ for all $v \in E_\alpha$.

We claim that

$$\max_{(t,s)\in\overline{D}} I_b(\eta(1,\gamma_b(t,s)) < c_\infty^{\mathcal{M}_b}. \tag{2.26}$$

Note that

$$I_b(\gamma_b(t,s)) < c_\infty^{\mathcal{M}_b} + \varepsilon \text{ for all } (t,s) \in \overline{D}$$

and

$$\begin{aligned}
||\gamma_b(t,s) - u_b||^2_{E_\alpha} &= ||(t-1)u_b^+ + (s-1)u_b^+||^2_{E_\alpha}\\
&\leq 2(a_2-a_1)^2||u_b||^2_{E_\alpha} < \varrho^2,
\end{aligned}$$

which yields $||\gamma_b(t,s)-u_b||_{E_\alpha}<\varrho$ for all $(t,s)\in\overline{D}$. Therefore, (η_2) implies $I_b(\eta(1,\gamma_b(t,s)))\leq c_\infty^{\mathcal{M}_b}-\varepsilon$, and hence (2.26) holds.

Now, we show that

$$\eta(1,\gamma_b(D))\cap\mathcal{M}_b\neq\emptyset. \tag{2.27}$$

Let

$$\begin{aligned}\zeta_b(t,s)&=\eta(1,\gamma_b(t,s)),\\ \phi_0(t,s)&=\left(\langle I_b'(\gamma_b(t,s)),u_b^+\rangle,\langle I_b'(\gamma_b(t,s)),u_b^-\rangle\right)\\&=:\left(\phi_0^1(t,s),\phi_0^2(t,s)\right)\end{aligned}$$

and

$$\phi_1(t,s)=\left(\frac{1}{t}\langle I_b'(\zeta_b(t,s)),(\zeta_b(t,s))^+\rangle,\frac{1}{s}\langle I_b'(\zeta_b(t,s)),(\zeta_b(t,s))^-\rangle\right).$$

By direct computation

$$\begin{aligned}\frac{\partial\phi_0^1}{\partial t}(1,1)&=||u_b^+||_{E_\alpha}^2+3b[u_b^+]_\alpha^4+8bZ(u_b^+,u_b^-)[u_b^+]_\alpha^2+b[u_b^+]_\alpha^2[u_b^-]_\alpha^2+4bZ(u_b^+,u_b^-)^2\\&\quad-\int_{\mathbb{R}^N}f'(x,y,u_b^+)(u_b^+)^2dxdy\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}\frac{\partial\phi_0^1}{\partial s}(1,1)&=4bZ(u_b^+,u_b^-)[u_b^+]_\alpha^2+6bZ(u_b^+,u_b^-)[u_b^-]_\alpha^2+2b[u_b^+]_\alpha^2[u_b^-]_\alpha^2+8bZ(u_b^+,u_b^-)^2\\&\quad+2Z(u_b^+,u_b^-).\end{aligned} \tag{2.29}$$

From $\langle I_b'(u_b^++u_b^-),u_b^+\rangle=0$, we entail

$$\begin{aligned}\int_{\mathbb{R}^N}f(x,y,u_b^+)u_b^+dxdy&=||u_b^+||_{E_\alpha}^2+b[u_b^+]_\alpha^4+4bZ(u_b^+,u_b^-)[u_b^+]_\alpha^2+2bZ(u_b^+,u_b^-)[u_b^-]_\alpha^2\\&\quad+b[u_b^+]_\alpha^2[u_b^-]_\alpha^2+4bZ(u_b^+,u_b^-)^2+2Z(u_b^+,u_b^-).\end{aligned} \tag{2.30}$$

By (f_4), we see that

$$f'(x,y,t)t^2>3f(x,y,t)t \text{ for all } (x,y)\in\mathbb{R}^N \text{ and } t\neq0. \tag{2.31}$$

Combining (2.28) and (2.30)-(2.31), we obtain

$$\begin{aligned}-\frac{\partial\phi_0^1}{\partial t}(1,1)&>2||u_b^+||_{E_\alpha}^2+4bZ(u_b^+,u_b^-)[u_b^+]_\alpha^2+6bZ(u_b^+,u_b^-)[u_b^-]_\alpha^2+2b[u_b^+]_\alpha^2[u_b^-]_\alpha^2\\&\quad+8bZ(u_b^+,u_b^-)^2+6Z(u_b^+,u_b^-).\end{aligned} \tag{2.32}$$

Similarly, we have

$$\begin{aligned}\frac{\partial\phi_0^2}{\partial t}(1,1)&=6bZ(u_b^+,u_b^-)[u_b^+]_\alpha^2+4bZ(u_b^+,u_b^-)[u_b^-]_\alpha^2+2b[u_b^+]_\alpha^2[u_b^-]_\alpha^2+8bZ(u_b^+,u_b^-)^2\\&\quad+2Z(u_b^+,u_b^-)\end{aligned} \tag{2.33}$$

and

$$-\frac{\partial\phi_0^2}{\partial s}(1,1) > 2||u_b^-||_{E_\alpha}^2 + 6bZ(u_b^+,u_b^-)[u_b^+]_\alpha^2 + 4bZ(u_b^+,u_b^-)[u_b^-]_\alpha^2 + 2b[u_b^+]_\alpha^2[u_b^-]_\alpha^2 + 8bZ(u_b^+,u_b^-)^2 + 6Z(u_b^+,u_b^-). \tag{2.34}$$

From (2.29) and (2.32)-(2.34), we deduce

$$-\frac{\partial\phi_0^1}{\partial t}(1,1) > \frac{\partial\phi_0^1}{\partial s}(1,1) \quad \text{and} \quad -\frac{\partial\phi_0^2}{\partial s}(1,1) > \frac{\partial\phi_0^2}{\partial t}(1,1).$$

Therefore

$$\mathrm{Jac}(\phi_0)(1,1) > 0.$$

Note that $(1,1)$ is the unique pre-image of $(0,0)$ by ϕ_0, thus $(1,1)$ is the unique regular point of ϕ_0. By the Brouwer degree theory,

$$\deg(\phi_0, D, (0,0)) = \mathrm{sign}(\mathrm{Jac}(\phi_0)(1,1)) = 1. \tag{2.35}$$

By (2.25), we know that for all $(t,s) \in \partial D$,

$$I_b(\gamma_b(t,s)) \leq \varpi < c_\infty^{\mathcal{M}_b} - 2\varepsilon.$$

This together with (η_1) gives $\eta(1,\gamma_b(t,s))) = \gamma_b(t,s)$ for all $(t,s) \in \partial D$. Hence

$$\deg(\phi_1, D, (0,0)) = \deg(\phi_0, D, (0,0)) = 1, \tag{2.36}$$

from which $\phi_1(t_0,s_0) = (0,0)$ for some $(t_0,s_0) \in D$. Consequently

$$\eta(1,\gamma_b(t_0,s_0)) = \zeta_u(t_0,s_0) \in \mathcal{M}_b,$$

but this contradicts (2.26). We conclude that u_b is a critical point of I_b and so it is a sign-changing solution of (1.1).

Proof of Theorem 1.2. We split the proof into three steps.

Step 1. For given sequence of positive reals $b_n \searrow 0$, the sequence $\{u_{b_n}\}$ is bounded in E_α. In fact, choose $\xi_0 \in C_0^\infty(\mathbb{R}^N)$ such that $\xi_0^\pm \neq 0$. Then, using $(f_3)-(f_4)$, for any $b \in (0,1]$, we can find $t_1, s_1 > 0$ independent of b such that

$$\langle I_b'(t_1\xi_0^+ + s_1\xi_0^-), t_1\xi_0^+\rangle < 0 \quad \text{and} \quad \langle I_b'(t_1\xi_0^+ + s_1\xi_0^-), s_1\xi_0^-\rangle < 0. \tag{2.37}$$

Moreover, in view of Lemma 2.4, there is a unique pair of positive reals (t_b,s_b) such that

$$\xi_b := t_b t_1\xi_0^+ + s_b s_1\xi_0^- \in \mathcal{M}_b. \tag{2.38}$$

By (2.37) and similar arguments as those used in Claim 1, we can show that $t_b, s_b \in (0,1]$. According to $(f_1)-(f_2)$, for some $C_1, C_2 > 0$ we have

$$|f(x,y,t)t|,\ |F(x,y,t)| \leq C_1|t|^2 + C_2|t|^p \text{ for all } (x,y,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.39}$$

Furthermore, using (f_4), we can infer that

$$\frac{1}{4}f(x,y,t)t - F(x,y,t) \geq 0 \text{ for all } (x,y,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Therefore

$$\frac{1}{4}||u_{b_n}||_{E_\alpha}^2 \leq I_{b_n}(u_{b_n}) - \frac{1}{4}\langle I_{b_n}'(u_{b_n}), u_{b_n}\rangle = I_{b_n}(u_{b_n})$$

$$\begin{aligned}
&\leq I_{b_n}(\xi_{b_n}) = I_{b_n}(\xi_{b_n}) - \frac{1}{4}\langle I'_{b_n}(\xi_{b_n}), \xi_{b_n}\rangle \\
&= \frac{1}{4}||\xi_{b_n}||^2_{E_\alpha} + \int_{\mathbb{R}^N} \frac{1}{4} f(x,y,\xi_{b_n})\xi_{b_n} - F(x,y,\xi_{b_n})dxdy \\
&\leq ||t_1\xi_0^+||^2_{E_\alpha} + ||s_1\xi_0||^2_{E_\alpha} + C_1\int_{\mathbb{R}^N}\left(|t_1\xi_0^+|^2 + |s_1\xi_0^-|^2\right)dxdy \\
&\quad + C_2\int_{\mathbb{R}^N}\left(|t_1\xi_0^+|^p + |s_1\xi_0^-|^p\right)dxdy \\
&=: K_0. \qquad (2.40)
\end{aligned}$$

Hence $\{u_{b_n}\}$ is bounded in E_α.

Step 2. Equation (1.4) admits a sign-changing solution u_0. Since $\{u_{b_n}\}$ is bounded in E_α, up to a subsequence, we may assume that

$$\begin{aligned}
&u_{b_n} \rightharpoonup u_0 \text{ in } E_\alpha \\
&u_{b_n} \to u_0 \text{ in } L^q(\mathbb{R}^N) \text{ for all } q \in [2, p^*) \qquad (2.41) \\
&u_{b_n} \to u_0 \text{ a.e. in } \mathbb{R}^N.
\end{aligned}$$

Noticing that u_{b_n} is a solution of (1.1), for all $v \in C_0^\infty(\mathbb{R}^N)$ we have

$$\begin{aligned}
&\left(1 + b_n[u_{b_n}]^2_\alpha\right)\int_{\mathbb{R}^N}\left((-\Delta_x)^{\frac{\alpha}{2}}u_{b_n}(-\Delta_x)^{\frac{\alpha}{2}}v + \nabla_y u_{b_n}\nabla_y v\right)dxdy + \int_{\mathbb{R}^N} V(x,y)u_{b_n}vdxdy \\
&= \int_{\mathbb{R}^N} f(x,y,u_{b_n})vdxdy.
\end{aligned}$$

Letting $n \to \infty$, from (2.41) we get

$$\int_{\mathbb{R}^N}\left((-\Delta_x)^{\frac{\alpha}{2}}u_0(-\Delta_x)^{\frac{\alpha}{2}}v + \nabla_y u_0\nabla_y v\right)dxdy + \int_{\mathbb{R}^N} V(x,y)u_0vdxdy = \int_{\mathbb{R}^N} f(x,y,u_0)vdxdy,$$

which means that u_0 is a solution of (1.4). In addition, we have

$$\begin{aligned}
&||u_{b_n} - u_0||^2_{E_\alpha} = \langle I'_{b_n}(u_{b_n}) - I'_0(u_0), u_{b_n} - u_0\rangle \\
&-b_n[u_{b_n}]^2_\alpha\int_{\mathbb{R}^N}\left((-\Delta_x)^{\frac{\alpha}{2}}u_{b_n}(-\Delta_x)^{\frac{\alpha}{2}}(u_{b_n} - u_0) + \nabla_y u_{b_n}\nabla_y(u_{b_n} - u_0)\right)dxdy \\
&-\int_{\mathbb{R}^N}(f(x,y,u_0) - f(x,y,u_{b_n})(u_{b_n} - u_0)dxdy. \qquad (2.42)
\end{aligned}$$

Then with the aid of Lemma 2.2 and (2.39), we conclude that $||u_{b_n} - u_0||_\alpha \to 0$. By Lemma 2.6, we have $u_0^\pm \neq 0$. Hence u_0 is a sign-changing solution of (1.4).

Step 3. u_0 is a list energy solution of (1.1). Similar to the proof of Theorem 1.1, equation (1.4) admits a list energy sign-changing solution v_0. By Lemma 2.4, there exists a unique pair of positive reals $(t_{b_n}, s_{b_n}) \in (0,\infty)^2$ such that $t_{b_n}v_0^+ + s_{b_n}v_0^- \in \mathcal{M}_{b_n}$. We claim that $t_{b_n} \to 1$ and $s_{b_n} \to 1$. In fact, since $t_{b_n}v_0^+ + s_{b_n}v_0^- \in \mathcal{M}_{b_n}$,

$$\begin{aligned}
&b_n\left(t^2_{b_n}[v_0^+]^2_\alpha + s^2_{b_n}[v_0^-]^2_\alpha + 2t_{b_n}s_{b_n}Z(v_0^+, v_0^-)\right)\left(t^2_{b_n}[v_0^+]^2_\alpha + 2t_{b_n}s_{b_n}Z(v_0^+, v_0^-)\right) \qquad (2.43) \\
&+ t^2_{b_n}[v_0^+]^2_\alpha + 2t_{b_n}s_{b_n}Z(v_0^+, v_0^-) + \int_{\mathbb{R}^N} V(x,y)(t_{b_n}v_0^+)^2dxdy = \int_{\mathbb{R}^N} f(x,y,t_{b_n}v_0^+)t_{b_n}v_0^+dxdy
\end{aligned}$$

and

$$b_n \left(t_{b_n}^2 [v_0^+]_\alpha^2 + s_{b_n}^2 [v_0^-]_\alpha^2 + 2t_{b_n} s_{b_n} Z(v_0^+, v_0^-) \right) \left(s_{b_n}^2 [v_0^-]_\alpha^2 + 2t_{b_n} s_{b_n} Z(v_0^+, v_0^-) \right) \tag{2.44}$$
$$+ s_{b_n}^2 [v_0^-]_\alpha^2 + 2t_{b_n} s_{b_n} Z(v_0^+, v_0^-) + \int_{\mathbb{R}^N} V(x,y)(s_{b_n} v_0^-)^2 dxdy = \int_{\mathbb{R}^N} f(x,y,s_{b_n} v_0^-) s_{b_n} v_0^- dxdy.$$

Therefore, using (f_4) and the convergence of $\{b_n\}$, we see that $\{t_{b_n}\}$ and $\{s_{b_n}\}$ are bounded. Hence, up to a subsequences, we may assume that $t_{b_n} \to t_0$ and $s_{b_n} \to s_0$. It holds from (2.43)-(2.44) that

$$t_0^2 [v_0^+]_\alpha^2 + 2t_0 s_0 Z(v_0^+, v_0^-) + \int_{\mathbb{R}^N} V(x,y)(t_0 v_0^+)^2 dxdy = \int_{\mathbb{R}^N} f(x,y,t_0 v_0^+) t_0 v_0^+ dxdy \tag{2.45}$$

and

$$s_0^2 [v_0^-]_\alpha^2 + 2t_0 s_0 Z(v_0^+, v_0^-) + \int_{\mathbb{R}^N} V(x,y)(s_0 v_0^-)^2 dxdy = \int_{\mathbb{R}^N} f(x,y,s_0 v_0^-) s_0 v_0^- dxdy. \tag{2.46}$$

Recalling that v_0 is a solution of (1.4), we write

$$[v_0^+]_\alpha^2 + 2Z(v_0^+, v_0^-) + \int_{\mathbb{R}^N} V(x,y)(v_0^+)^2 dxdy = \int_{\mathbb{R}^N} f(x,y,v_0^+) v_0^+ dxdy \tag{2.47}$$

and

$$[v_0^-]_\alpha^2 + 2Z(v_0^+, v_0^-) + \int_{\mathbb{R}^N} V(x,y)(v_0^-)^2 dxdy = \int_{\mathbb{R}^N} f(x,y,v_0^-) v_0^- dxdy. \tag{2.48}$$

By virtue of (f_1) and (2.43)-(2.44), it can be shown that $t_0, s_0 > 0$. So, arguing as in the proof of Claim 1, from (2.45)-(2.48), we obtain $t_0 = s_0 = 1$. Then

$$I_0(v_0) \leq I_0(u_0) = \lim_{n\to\infty} I_{b_n}(u_{b_n}) \leq \lim_{n\to\infty} I_{b_n}(t_{b_n} v_0^+ + s_{b_n} v_0^-) = I_0(v_0^+ + v_0^-) = I_0(v_0).$$

Consequently $I_0(u_0) = I_0(v_0)$. The proof of Theorem 1.2 is now finished.

Funding Not applicable.

Conflict of interest The author declares that there is no conflict of interest regarding the publication of this paper.

References

[1] D. Applebaum, Lévy processes from probability to finance and quantum groups. Not. Am. Math. Soc. 51, 1336-1347 (2004).
[2] G. Barles, E. Chasseigne, C. Imbert, Hölder continuity of solutions of second-order nonlinear elliptic integro-differential equations. J. Eur. Math. Soc. 13, 1-26 (2011).
[3] T. Bartsch, T. Weth, Three nodal solutions of singularly perturbed elliptic equations on domains without topology. Ann. Inst. H. Poincaré Anal. Non Linéaire, 22 (3), 259-281 (2005).
[4] T. Bartsch, T. Weth, M. Willem, Partial symmetry of least energy nodal solutions to some variational problems, J. Anal. Math. 96, 118 (2005).
[5] T. Bartsch, Z. Liu, T. Weth, Sign changing solutions of superlinear Schrödinger equations. Comm. Partial Differential Equations, 29 (12), 25-42 (2004).
[6] L. Caffarelli, Some nonlinear problems involving non-local diffusions, in: ICIAM 07-6th International Congress on Industrial and Applied Mathematics, Eur. Math. Soc., Zürich, 43-56 (2009).

[7] L. Caffarelli, Nonlocal diffusions, drifts and games, in: Nonlinear Partial Differential Equations. Abel Symposia, 7, 37-52 (2012).
[8] K. Cheng, Q. Gao, Sign-changing solutions for the stationary Kirchhoff problems involving the fractional Laplacian in $\mathbb{R}^N$, Acta Math Sci. 38B(6), 1712-1730 (2018).
[9] A. Esfahani, Existence of solutions of class of nonlinear nonlocal problems involving the potential. Chaos, Solitons and Fractals, 119, 1-7 (2019).
[10] A. Esfahani, Anisotropic Gagliardo-Nirenberg inequality with fractional derivatives. Z. Angew. Math. Phys. 66, 3345-3356 (2015).
[11] A. Farina, E. Valdinoci, Regularity and rigidity theorems for a class of anisotropic nonlocal operators. Manuscr. Math. 153, 53-70 (2017).
[12] A. Fiscella, E. Valdinoci, A critical Kirchhoff type problem involving a nonlocal operator. Nonlinear Anal. 94, 156-170 (2014).
[13] R.L. Frank, E. Lenzmann, Uniqueness and nondegeneracy of ground states for $(-\Delta)^s q + q - q^{\alpha+1} = 0$ in $\mathbb{R}$. Acta Math. 210, 261318 (2013).
[14] T.X. Gou, H.R. Sun, Solutions of nonlinear Schrödinger equation with fractional Laplacian without the AR condition. Appl. Math. Comput. 257, 409-416 (2015).
[15] T. Isernia, Sign-changing solutions for a fractional Kirchhoff equation. Nonlinear. Anal. 190111623 (2020).
[16] G. Kirchhoff, Vorlesungen uber Mechanik. 3rd Edn., Teubner, Leipzig (1883).
[17] N. Laskin, Fractional quantum mechanics and Lévy path integrals. Phys. Lett. 268,298-305 (2000).
[18] M. Massar, Infinitely many radial solutions for a nonlocal equation involving an anisotropic operator, J. Nonlinear Funct. Anal. 2020 (2020). https://doi.org/10.23952/jnfa.2020.29.
[19] M. Massar, On a class of Kirchhoff equations involving an anisotropic operator and potential. Arab. J. Math. (2020). https://doi.org/10.1007/s40065-020-00304-y.
[20] M. Massar, M. Talbi, On a class of p-fractional Laplacian equations with potential depending on parameter. Math. Meth. Appl. Sci. 43, 2721-2734 (2020).
[21] L. Medeiros, J. Limaco, S. Menezes, Vibrations of elastic strings: Mathematical aspects, Part one, J. Comput. Anal. Appl. 4 (2) 91-127 (2002).
[22] X. Mingqi, V.D. Rădulescu, B.L. Zhang, Combined effects for fractional Schrödinger-Kirchhoff systems with critical nonlinearities. ESAIM Control Optim. Calc. Var. 24, 1249-1273 (2018).
[23] C. Miranda, Un'osservazione su un teorema di Brouwer. Boll Un Mat Ital. 3, 5-7 (1940).
[24] N. Pan, B. Zhang, J. Cao, Degenerate Kirchhoff-type diffusion problems involving the fractional p-Laplacian, Nonlinear Anal. Real World Appl., 37, 56-70 (2017)
[25] F. Ribaud, S. Vento, Local and global well-posedness results for the Benjamin-Ono-Zakharov-Kuznetsov equation. Discrete Contin. Dyn. Syst. 37, 449-483 (2017).
[26] L. Shao, H. Chen, Existence and concentration result for a class of fractional Kirchhoff equations with Hartree-type nonlinearities and steep potential well. C. R. Acad. Sci. Paris, Ser. I, 356, 489-497 (2018).
[27] Y. Wei , X. Su, Multiplicity of solutions for non-local elliptic equations drivenby the fractional Laplacian. Calc. Var. Partial Diferential Equations, 52, 95-124 (2015).
[28] M. Willem, Minimax Theorems, Birkhäuser, Basel, (1996).