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Existence of solutions for 4p-order PDES

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Abstract. In this paper, we study the following nonlinear eigenvalue problem:

$$\begin{cases} \Delta^{2p}u = \lambda m(x)u & \text{in } \Omega, \\ u = \Delta u = ... = \Delta^{2p-1}u = 0 & \text{on } \partial\Omega. \end{cases}$$

Where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 1$, $p \in \mathbb{N}^*$, $m \in L^\infty(\Omega)$, $\mu\{x \in \Omega : m(x) > 0\} \neq 0$, and $\Delta^{2p}u := \Delta(\Delta...(\Delta u))$, $2p$ times the operator Δ.
Using the Szulkin's theorem, we establish the existence of at least one non decreasing sequence of nonnegative eigenvalues.

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1. Introduction

This work deals with a 4p-order problem under Dirichlet boundary conditions:

$$\begin{cases} \Delta^{2p}u = \lambda m(x)u \quad \text{in } \Omega, \\ u = \Delta u = ... = \Delta^{2p-1}u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 1$, $p \in \mathbb{N}^*$, $\lambda \in \mathbb{R}$, $m \in L^\infty(\Omega)$, $\mu\{x \in \Omega : m(x) > 0\} \neq 0$, and $\Delta^{2p}u := \Delta(\Delta...(\Delta u))$, $2p$ times the operator Δ. We establish the existence of at least one non decreasing sequence of nonnegative eigenvalues.

Partial Differential Equations (PDEs) introducing fourth order operators have a very important role in all fields of engineering and physics. For example, the biharmonic $\Delta^2 u = \Delta(\Delta u)$ appears in various problems of physics such as those related to fluid dynamics like flow through porous media, flow of stokes and nonlinear elasticity problems [1, 4, 5, 8, 10, 11].

Several works treated under different conditions this type of problems: G. Sweers [12] discuss boundary conditions for a slender beam and for the thin elastic plate. In [9], the authors studied the existence and the L^p estimates of solutions for some problems modelling static equilibrium of an elastic body, and in [3] (1994), R. Dalmasso investigated the existence of positive radial solutions of a semiilinear elliptic equations of order 2p in annular domain.

From the works described above, we do not see any existence result of solutions to problem (1.1) in general case. The main results of this study are stated in the following theorems:

Theorem 1.1. *The problem (1.1) has at least one non decreasing sequence of nonnegative eigenvalues* $(\lambda_n)_n$ *defined as :*

$$\lambda_n = inf_{K\in\Gamma_n} sup_{u\in K} G(u),$$

and satisfying $\lambda_n \longrightarrow +\infty$ *as* $n \longrightarrow +\infty$,
where

$$G(u) := \frac{1}{2}\int_\Omega |\Delta^p u|^2 \, dx,$$

$$\Gamma_n = \{K \subset M : K \text{ is compact symetric and } \gamma(K) \geq n\}$$

and $M = \left\{u \in W^{2p,2}(\Omega) : u = \Delta u = ... = \Delta^{p-1}u = 0 \quad \text{on } \partial\Omega \text{ and } \frac{1}{2}\int_\Omega mu^2 dx = 1\right\}$.

Theorem 1.2. *The functional*

$$\theta_p(u) := \sum_{0\leq k\leq p} \left\|\Delta^k u\right\|_{L^2}$$

is a norm equivalent to the usual norm on $W^{2p,2}(\Omega)$ *:*

$$\|u\|_{W^{2p,2}} = \sum_{|\alpha|\leq 2p} \|D^\alpha u\|_{L^2}.$$

The proof of the Theorem 1.1 is based on the Szulkin's theorem [13] :

Theorem 1.3. *Suppose that M is a closed symmetric C^1 -submanifold of a real Banach space X and 0 does not belong to M. Suppose that $f \in C^1(M, \mathbb{R})$ is even and bounded below. Define*

$$c_n = inf_{A\in\Gamma_n} sup_{x\in A} f(x).$$

If $\Gamma_k \neq \emptyset$ for some $k \geq 1$ and if f satisfies the Palais Smale condition for all $c = c_n, n = 1, ..., k$, then f has at least k distinct pairs of critical points.

For this end, we need some preliminary results that we state in section 2. In particular, we define a solution of problem (1.1) and the reflexif subspace:

$$X_p = \left\{ \varphi \in W^{2p,2}(\Omega) : \varphi = \Delta\varphi = ... = \Delta^{p-1}\varphi = 0 \quad \text{on } \partial\Omega \right\},$$

where the solutions are considered. We also define the energy functional associated to the problem (1.1).
The third section contains some special norms in $W^{2p,2}$ and in X_p that we prove equivalents to the usual norm. The last section is devoted to the detailed proof of Theorem 1.1.

2. Position of problem

We say that u is a solution of the problem (1.1) if:

$$\forall v \in W^{2p,2}(\Omega) : \langle \Delta^{2p}u, v\rangle = \lambda \int_\Omega m(x)uvdx. \tag{2.1}$$

By using the Green formula, we find :

$$\begin{aligned} &\forall n \in \mathbb{N}^*, \forall 0 \leq k \leq n, \forall (u, v) \in (W^{n,2}(\Omega))^2 : \\ &\langle \Delta^n u, v\rangle = \langle \Delta^{n-k}u, \Delta^k v\rangle + \sum_{i=1}^{k} g(\Delta^{n-i}u, \Delta^{i-1}v), \end{aligned} \tag{2.2}$$

when

$$g(u, v) = \int_{\partial\Omega} (\frac{\partial u}{\partial \nu}v - \frac{\partial v}{\partial \nu}u)\partial\sigma, \tag{2.3}$$

with $\frac{\partial u}{\partial \nu}$ is the outer normal derivative.
Then, for $n = 2p$ and $k = p$ in (2.2), we have :

$$\langle \Delta^{2p}u, v\rangle = \langle \Delta^p u, \Delta^p v\rangle + \sum_{i=1}^{p} g(\Delta^{2p-i}u, \Delta^{i-1}v), \tag{2.4}$$

with $\forall i \in \{1, ..., p\}$

$$g(\Delta^{2p-i}u, \Delta^{i-1}v) = \int_{\partial\Omega} (\frac{\partial(\Delta^{2p-i}u)}{\partial \nu}\Delta^{i-1}v - \frac{\partial(\Delta^{i-1}v)}{\partial \nu}\Delta^{2p-i}u)\partial\sigma. \tag{2.5}$$

Put

$$X_p = \left\{ \varphi \in W^{2p,2}(\Omega) : \varphi = \Delta\varphi = ... = \Delta^{p-1}\varphi = 0 \quad \text{on } \partial\Omega. \right\} = \bigcap_{k=0}^{p-1} E_k,$$

such that

$$E_k = \left\{ \varphi \in W^{2p,2}(\Omega) : \Delta^k \varphi = 0 \quad \text{on} \ \ \partial\Omega. \right\}$$

Proposition 2.1. *The set X_p is a closed subspace of $W^{2p,2}(\Omega)$.*

Proof. To prove the above proposition, we must show that ($\forall 0 \leq k \leq p-1$) the operator

$$\begin{array}{rccc} T^k : & W^{2p,2}(\Omega) & \to & L^2(\partial\Omega) \\ & u & \longmapsto & (\Delta^k u)|_{\partial\Omega} \end{array}$$

is continuous from $\left(W^{2p,2}(\Omega), \|.\|_{W^{2p,2}}\right)$ to $\left(L^2(\partial\Omega), |.|_{L^2(\partial\Omega)}\right)$.

First, the operator

$$\begin{array}{rccc} \Delta^k : & W^{2p,2}(\Omega) & \to & W^{2p-2k,2}(\Omega) \\ & u & \longmapsto & \Delta^k u \end{array}$$

is linear and continuous from $\left(W^{2p,2}(\Omega), \|.\|_{W^{2p,2}}\right)$ to $\left(W^{2p-2k,2}(\Omega), \|.\|_{W^{2p-2k,2}}\right)$.
Indeed, we have :

$$\begin{aligned} \left\|\Delta^k u\right\|_{W^{2p-2k,2}} &= \sum_{|\alpha|\leq 2p-2k} \left\|D^\alpha(\Delta^k u)\right\|_{L^2} \\ &\leq \sum_{|\beta|\leq 2p} \left\|D^\beta(u)\right\|_{L^2} = \|u\|_{W^{2p,2}} \end{aligned}$$

On the other hand, according to J.F.Bonder and J.D.Rossi in [2], there exists a positive constant F such that :

$$\forall p \geq 1, \forall u \in W^{1,p}(\Omega) : |u|_{L^p(\partial\Omega)} \leq F \|u\|_{W^{1,p}}.$$

Furthermore,

$$\forall r \in \mathbb{N}^*, \|u\|_{W^{1,p}} \leq \|u\|_{W^{r,p}}.$$

So,

$$\forall u \in W^{r,p}(\Omega) : |u|_{L^p(\partial\Omega)} \leq F \|u\|_{W^{r,p}},$$

this means that, the operator :

$$\begin{array}{rccc} Tr : & W^{2p-2k,2}(\Omega) & \to & L^2(\partial\Omega) \\ & u & \longmapsto & u|_{\partial\Omega} \end{array}$$

is continuous from $\left(W^{2p-2k,2}(\Omega), \|.\|_{W^{2p-2k,2}}\right)$ to $\left(L^2(\partial\Omega), |.|_{L^2(\partial\Omega)}\right)$.
Finally, $\forall 0 \leq k \leq p-1$, the operator $T^k = Tr \circ \Delta^k$ is continuous, and the set

$$E_k = (T^k)^{-1}\{0\},$$

is closed.
Consequently, X_p is a closed subspace of $W^{2p,2}(\Omega)$.

Remark 2.1. *The subspace X_p is reflexive, separable and uniformly convex.*

We say that u is a weak solution for the problem (1.1) if $u \in X_p$ and

$$\int_\Omega \Delta^p u \, \Delta^p v dx = \lambda \int_\Omega m(x) uv dx \quad \forall v \in X_p,$$

since $\forall i \in \{1, ..., p\}$, $g(\Delta^{2p-i}u, \Delta^{i-1}v) = 0$.

It is clear that the energy functional associated to the problem(1.1) is defined by :

$$\Psi(u) = \frac{1}{2}\int_\Omega |\Delta^p u|^2 dx - \frac{\lambda}{2}\int_\Omega m(x)u^2 dx.$$

3. A special norms in $W^{2p,2}(\Omega)$ and in X_p

In this section, we will prove that θ_p and $\xi_p := \|\Delta^p .\|_{L^2}$ are , successively , norms in $W^{2p,2}(\Omega)$ and in X_p equivalents to the usual norm.

Proof of theorem 1.2 :

(i) First, θ_p is a positive functional for all $p \in \mathbb{N}^*$.

(ii) We have

$$\theta_p(u) = 0 \iff \left\|\Delta^k u\right\|_{L^2} = 0, \ \forall k \in \{0, ..., p\}.$$

Then,

$$\|u\|_{L^2} = 0 \implies u = 0.$$

(iii) Let $\alpha \in \mathbb{R}$, we have:

$$\theta_p(\alpha u) = \sum_{0 \le k \le p} \left\|\Delta^k(\alpha u)\right\|_{L^2} = |\alpha| \, \theta_p(u).$$

(iv) θ_p satisfies the triangular inequality.
Indeed, $\forall 0 \le k \le p$, $\forall (u, v) \in (W^{2p,2}(\Omega))^2$ we have

$$\left\|\Delta^k(u+v)\right\|_{L^2} \leqslant \left\|\Delta^k u\right\|_{L^2} + \left\|\Delta^k v\right\|_{L^2}$$

In the other hand, we establish the equivalence between the usual norm and θ_p by recurrence on p.

(1) For $p = 1$, the authors in [8] shows that the norm $u \longmapsto \|u\|_{L^2} + \|\Delta u\|_{L^2}$ is an equivalent norm to the usual one in $W^{2,2}(\Omega)$. So, there exists two positives constants A and B such that :

$$\forall v \in W^{2,2}(\Omega), A\theta_1(v) \le \|v\|_{W^{2,2}} \le B\theta_1(v), \tag{3.1}$$

where,

$$\theta_1(v) = \|v\|_{L^2} + \|\Delta v\|_{L^2}.$$

(2) Let $p \ge 1$, suppose that θ_p is an equivalent norm to the usual's one in $W^{2p,2}(\Omega)$ and let us show that θ_{p+1} is an equivalent norm to the usual's one in $W^{2p+2,2}(\Omega)$.
For $u \in W^{2p+2,2}(\Omega)$, we have :

$$\begin{aligned}\|u\|_{W^{2p+2,2}} &= \sum_{|\alpha|\leq 2p+2} \|D^\alpha u\|_{L^2} \\ &= \|u\|_{W^{2p,2}} + \sum_{2p+1\leq|\alpha|\leq 2p+2} \|D^\alpha u\|_{L^2}.\end{aligned}$$

By the recurrence's hypothesis, there exists two positives constants C and D such that :

$$\forall v \in W^{2p,2}(\Omega), C\theta_p(v) \leq \|v\|_{W^{2p,2}} \leq D\theta_p(v).$$

It follows that

$$C\theta_p(u) \leq \|u\|_{W^{2p,2}} \leq D\theta_p(u). \tag{3.2}$$

Furthermore,

$$\sum_{2p+1\leq|\alpha|\leq 2p+2} \|D^\alpha u\|_{L^2} = \sum_{|\alpha|=2p,1\leq|\beta|\leq 2} \left\|D^\beta D^\alpha u\right\|_{L^2}.$$

It's clear that $\forall\, |\alpha| = 2p, D^\alpha u \in W^{2,2}(\Omega)$. So,

$$\sum_{1\leq|\beta|\leq 2} \left\|D^\beta D^\alpha u\right\|_{L^2} \leq \|D^\alpha u\|_{W^{2,2}},$$

and according to (3.1), we have :

$$\|D^\alpha u\|_{W^{2,2}} \leq B(\|D^\alpha u\|_{L^2} + \|\Delta(D^\alpha u)\|_{L^2}),\ \forall\, |\alpha| = 2p.$$

We deduct that:

$$\begin{aligned}\sum_{|\alpha|=2p}\Big(\sum_{1\leq|\beta|\leq 2} \left\|D^\beta D^\alpha u\right\|_{L^2}\Big) &\leq B \sum_{|\alpha|=2p} (\|D^\alpha u\|_{L^2} + \|\Delta(D^\alpha u)\|_{L^2}) \\ &\leq B(\|u\|_{W^{2p,2}} + \|\Delta u\|_{W^{2p,2}}).\end{aligned}$$

Applying (3.2) to u and Δu we find:

$$\begin{aligned}\|u\|_{W^{2p+2,2}} &= \|u\|_{W^{2p,2}} + \sum_{2p+1\leq|\alpha|\leq 2p+2} \|D^\alpha u\|_{L^2} \\ &\leq (B+1)\|u\|_{W^{2p,2}} + B\|\Delta u\|_{W^{2p,2}} \\ &\leq D[(B+1)\theta_p(u) + B\theta_p(\Delta u)].\end{aligned}$$

Using the following results,

$$\begin{aligned}\theta_p(\Delta u) = \sum_{1\leq k\leq p+1} \left\|\Delta^k u\right\|_{L^2} &= \theta_{p+1}(u) - \|u\|_{L^2} \\ &\leq \theta_{p+1}(u),\end{aligned}$$

and $\theta_p(u) \leq \theta_{p+1}(u)$, we end up with $\|u\|_{W^{2p+2,2}} \leq D(1+2B)\theta_{p+1}(u)$.
The inequality, $\|u\|_{W^{2p+2,2}} \geq \theta_{p+1}(u)$. is immediate.
This acheives the proof of theorem 1.2.

Proposition 3.1. *On X_p, the functional*

$$\xi_p(u) := \|\Delta^p u\|_{L^2}$$

is a norm equivalent to the norm $\|.\|_{W^{2p,2}}$.

Proof of proposition 3.1 :

(i) It is clear that ξ_p is a positive functional.

(ii) Let us show that : $\xi_p(u) = 0 \Longleftrightarrow u = 0$.
Let $u \in X_p$, we have :

$$\xi_p(u) = \|\Delta^p u\|_{L^2(\Omega)} = 0 \iff \left\|\Delta(\Delta^{p-1}u)\right\|_{L^2(\Omega)} = 0.$$

Put $v = \Delta^{p-1}u$. Since $u \in X_p$ then $v \in W_0^{2,2}(\Omega)$.
It is well known that the functional

$$\varrho(w) := \|\Delta w\|_{L^2(\Omega)}$$

is a norm on $W_0^{2,2}(\Omega)$.
So, $\|\Delta v\|_{L^2(\Omega)} = 0 \implies v = 0$ *on* Ω, and, it is clear that :
$v = 0 \implies \|v\|_{L^2(\Omega)} = 0 = \left\|\Delta^{p-1}u\right\|_{L^2(\Omega)}$.
It results that :

$$\|\Delta^p u\|_{L^2(\Omega)} = 0 \Longrightarrow \left\|\Delta^{p-1}u\right\|_{L^2(\Omega)} = 0. \tag{3.3}$$

By applying the implication (3.3) several times, we obtain :

$$\|\Delta^p u\|_{L^2(\Omega)} = 0 \Longrightarrow u = 0 \text{ on } \Omega.$$

(iii) Let $\alpha \in \mathbb{R}$,

$$\xi_p(\alpha u) = \|\Delta^p(\alpha u)\|_{L^2(\Omega)} = |\alpha|\,\xi_p(u).$$

(iv) ξ_p satisfies the triangular inequality.
Indeed, $\forall (u,v) \in X_p^2$,

$$\begin{aligned} \xi_p(u+v) &= \|\Delta^p(u+v)\|_{L^2(\Omega)} = \|\Delta^p u + \Delta^p v\|_{L^2(\Omega)} \\ &\leq \|\Delta^p u\|_{L^2(\Omega)} + \|\Delta^p v\|_{L^2(\Omega)} \\ &\leq \xi_p(u) + \xi_p(v). \end{aligned}$$

Hence, ξ_p is a norm in X_p.
Thanks to the Theorem 1.2, it suffices to show that ξ_p is equivalent to θ_p .
By recurrence's demonstration on p, we have:

(*) For $p = 1$, the authors in [8] shows that the norm $u \longmapsto \|u\|_{L^2} + \|\Delta u\|_{L^2}$ is an equivalent norm to the norm $u \longmapsto \|\Delta u\|_{L^2}$ in $W^{2,2}(\Omega) \bigcap W_0^{1,2}(\Omega)$. So, ξ_1 is an equivalent norm to θ_1 on X_1.

(**) Let $p \geq 1$, suppose that ξ_p is an equivalent norm to θ_p in X_p and let us show that ξ_{p+1} is an equivalent norm to θ_{p+1} in X_{p+1} .
Let $u \in X_{p+1} = \{\varphi \in W^{2p+2,2}(\Omega) : \varphi = \Delta\varphi = ... = \Delta^{p-1}\varphi = \Delta^p\varphi = 0 \text{ on } \partial\Omega.\}$. It's clear that:

$$\xi_{p+1}(u) := \left\|\Delta^{p+1}(u)\right\|_{L^2(\Omega)} \leq \theta_{p+1}(u) := \sum_{0\leq k\leq p+1} \left\|\Delta^k u\right\|_{L^2}.$$

Now, we prove that there exists a constant $C_{p+1} > 0$ such that

$$\theta_{p+1}(u) \leq C_{p+1}\xi_{p+1}(u).$$

We have

$$\theta_{p+1}(u) = \theta_p(u) + \left\|\Delta^{p+1}u\right\|_{L^2}.$$

By the recurrence's hypothesis, there exists a constant $C_p > 0$ such that:

$$\theta_p(u) \leq C_p \xi_p(u).$$

Then,

$$\begin{aligned} \theta_{p+1}(u) &= \theta_p(u) + \left\|\Delta^{p+1}u\right\|_{L^2} \\ &\leq C_p\xi_p(u) + \left\|\Delta^{p+1}u\right\|_{L^2} \\ &\leq Max\left(C_p, 1\right)\left(\left\|\Delta^p u\right\|_{L^2} + \left\|\Delta^{p+1}u\right\|_{L^2}\right). \end{aligned}$$

Note that $v := \Delta^p u$ is in X_1, so

$$\theta_1(v) \leq C_1\xi_1(v),$$

that means that:

$$\left\|\Delta^p u\right\|_{L^2} + \left\|\Delta^{p+1}u\right\|_{L^2} \leq C_1\left\|\Delta^{p+1}u\right\|_{L^2},$$

consequently we get:

$$\theta_{p+1}(u) \leq Max\left(C_p, 1\right) C_1\left(\left\|\Delta^{p+1}u\right\|_{L^2}\right).$$

So,

$$\xi_{p+1}(u) \leq \theta_{p+1}(u) \leq C_{p+1}\xi_{p+1}(u),$$

where

$$C_{p+1} = Max\left(C_p, 1\right) C_1.$$

Finally, for all $p \geqslant 1$ there exists a constant $C_p > 0$ such that for all $u \in X_p$ we have

$$\xi_p(u) \leq \theta_p(u) \leq C_p\xi_p(u).$$

4. Existence result

We consider the functionals :

$$G(u) := \frac{1}{2}\int_\Omega \left|\Delta^p u\right|^2 dx = \frac{1}{2}\xi_p(u)^2$$

and

$$F(u) := \frac{1}{2}\int_\Omega mu^2 dx.$$

So, we have :

$$\Psi(u) = G(u) - \lambda F(u).$$

In order to apply the A.Szulkin's theorem, we should show the following proposition:

Proposition 4.1. *The functionals G and F satisfies the following properties :*

(i) G and F are C^1 functionals.
(ii) F' is completely continuous.
(iii) G' is of (S^+) type and G is an even functional and bounded below.
(iv) G satisfies the (PS) condition on $M = \left\{u \in X_p : F(u) = 1\right\} \neq \emptyset$.

(v) $\forall n \in \mathbb{N}^* : \Gamma_n \neq \emptyset$ *with*

$$\Gamma_n = \{K \subset M : K \text{ is compact symetric and } \gamma(K) \geq n\}.$$

Proof. (i) G and F are C^1 functionals: It is clear that G and F are continuous and derivable functionals on X_p and $\forall (u,v) \in X_p^2$:

$$\left\langle G^{'}(u), v \right\rangle = \int_\Omega \Delta^p u \ \Delta^p v dx$$

and

$$\left\langle F^{'}(u), v \right\rangle = \int_\Omega muvdx.$$

Moreover, the application: $u \mapsto G^{'}(u)$ is linear from X_p to $L(X_p, \mathbb{R})$ with $\forall (u,v) \in X_p^2$:

$$\begin{array}{rcl} \left|\left\langle G^{'}(u), v \right\rangle\right| & = & \left|\int_\Omega \Delta^p u \ \Delta^p v dx\right| \\ & \leq & \|\Delta^p u\|_{L^2(\Omega)} \|\Delta^p v\|_{L^2(\Omega)} \\ & \leq & \|u\|_{X_p} \|v\|_{X_p} = \xi_p(u)\xi_p(v). \end{array}$$

So,

$$sup_{\xi_p(v)\leq 1, v\neq 0} \frac{\left|\left\langle G^{'}(u), v \right\rangle\right|}{\xi_p(v)} \leq \xi_p(u) \Longrightarrow \left\|G^{'}(u)\right\|_{L(X_p,\mathbb{R})} \leq \xi_p(u).$$

This implies that $G^{'}$ is continuous and finally G is a C^1 functional.
On the other hand, for $F^{'}$, we have :

$$\begin{array}{rcl} \left|\left\langle F^{'}(u), v \right\rangle\right| & \leq & \|m\|_\infty \xi_p(u)\xi_p(v) \\ & \Longleftrightarrow & \\ \left\|F^{'}(u)\right\|_{L(X_p,\mathbb{R})} & \leq & \|m\|_\infty \xi_p(u). \end{array}$$

So, $F^{'}$ is continuous and finally F is a C^1 functional.

(ii) $F^{'}$ is completely continuous :
Let $(u_n)_n$ be a sequence of X_p such that $u_n \rightharpoonup u$ in X_p .
Let us show that :

$$sup_{v\in X_p, \xi_p(v)\leq 1} \left|\left\langle F^{'}(u_n) - F^{'}(u), v \right\rangle\right| \longrightarrow 0.$$

We have

$$\begin{array}{rcl} \left|\left\langle F^{'}(u_n) - F^{'}(u), v \right\rangle\right| & = & \left|\int_\Omega m(u_n - u)vdx\right|. \\ & \leq & \|m\|_\infty \|u_n - u\|_{L^2} \|v\|_{L^2}. \end{array} \tag{4.1}$$

Thanks to the compact embedding of $W^{2p,2}(\Omega) \bigcap W_0^{1,2}(\Omega)$ into $L^2(\Omega)$ and $X_p \subset W^{2p,2}(\Omega) \bigcap W_0^{1,2}(\Omega)$, we deduct the compact embedding of X_p into $L^2(\Omega)$.
It results that $u_n \longrightarrow u$ in $L^2(\Omega)$, and there exists a positive constant C such that :

$$\forall \varphi \in X_p : \|\varphi\|_{L^2} \leq C \|\varphi\|_{X_p} = C\xi_p(\varphi).$$

Then, it follows by (4.1) that

$$\left|\left\langle F^{'}(u_n) - F^{'}(u), v\right\rangle\right| \leq C \|m\|_{\infty} \|u_n - u\|_{L^2} \|v\|_{X_p}.$$

So,

$$sup_{v\in X_p, \xi_p(v)\leq 1} \left|\left\langle F^{'}(u_n) - F^{'}(u), v\right\rangle\right| \leq C \|m\|_{\infty} \|u_n - u\|_{L^2}.$$

Hence, it results that

$$lim_{n\longrightarrow+\infty}(sup_{v\in X_p, \xi_p(v)\leq 1} \left|\left\langle F^{'}(u_n) - F^{'}(u), v\right\rangle\right|) = 0.$$

(iii) G is an even and positive functional, so it's bounded below.
Let $(v_n)_n$ be a sequence of X_p such that $v_n \rightharpoonup v$ in X_p
and $limsup_{n\longrightarrow+\infty} \left\langle G^{'}(v_n), v_n - v\right\rangle \leq 0$.
We must show that $v_n \longrightarrow v$ in X_p.
In view of definition of the norm in X_p , we have :

$$\begin{aligned} 0 \leq \|v_n - v\|^2_{X_p} &= \|\Delta^p v_n - \Delta^p v\|^2_{L^2} \\ &= \left\langle G^{'}(v_n), v_n - v\right\rangle + \|\Delta^p v\|^2_{L^2} - \int_{\Omega} \Delta^p v_n \Delta^p v dx \\ &= \left\langle G^{'}(v_n) - G^{'}(v), v_n - v\right\rangle. \end{aligned}$$

Note that $v_n \rightharpoonup v$ in X_p implies that $lim_{n\longrightarrow+\infty} \left\langle G^{'}(v), v_n - v\right\rangle = 0$.
So,

$$limsup_{n\longrightarrow+\infty} \left\langle G^{'}(v_n) - G^{'}(v), v_n - v\right\rangle = limsup_{n\longrightarrow+\infty} \left\langle G^{'}(v_n), v_n - v\right\rangle \leq 0.$$

It means that :

$$limsup_{n\longrightarrow+\infty} \|v_n - v\|^2_{X_p} = 0.$$

Therefore, the result follows.
Consequently, $G^{'}$ satisfies the (S^+) condition.

(iv) G satisfies the (PS) condition on $M = \left\{u \in X_p : F(u) = 1\right\} \neq \emptyset$:
Let $(v_n)_n$ be a Palais Smale (P.S) sequence of M for the functional G : $G(v_n)$ is bounded and $G^{'}(v_n) \longrightarrow 0$.
Let us show that there exists a subsequence of (v_n) which strongly converges in X_p.
Since $(v_n) \subset M$, so $\frac{1}{2}\int_{\Omega} m(v_n)^2 dx = 1$. Furthermore, $G(v_n)$ is bounded assure that $(\|v_n\|^2_{X_p})_n$ is bounded. Which implies that the sequence (v_n) is bounded in X_p.
The space X_p is a Banach reflexive, then there exists a subsequence noted also by (v_n) which weekly converges in X_p $v_n \rightharpoonup v$ in X_p.
Note that :

$$\left\langle G^{'}(v_n), v_n - v\right\rangle = \left\langle G^{'}(v_n), v_n\right\rangle - \left\langle G^{'}(v_n), v\right\rangle.$$

We have :

$$\left|\left\langle G^{'}(v_n), v\right\rangle\right| \leq \left\|G^{'}(v_n)\right\|_{L(X_p,\mathbb{R})} \|v\|_{X_p}$$

and

$$\begin{aligned}\left|\left\langle G^{'}(v_n), v_n\right\rangle\right| &\leq \left\|G^{'}(v_n)\right\|_{L(X_p,\mathbb{R})} \|v_n\|_{X_p} \\ &\leq \left\|G^{'}(v_n)\right\|_{L(X_p,\mathbb{R})} (sup_n \|v_n\|_{X_p}).\end{aligned}$$

It follows by the condition : $G^{'}(v_n) \longrightarrow 0$, that :
$\left\langle G^{'}(v_n), v\right\rangle \longrightarrow 0$ and $\left\langle G^{'}(v_n), v_n\right\rangle \longrightarrow 0$.
Consequently,

$$lim \left\langle G^{'}(v_n), v_n - v\right\rangle = 0 \leq 0.$$

Thus, thanks to the (S^+) condition, we deduce that : $v_n \longrightarrow v$ in X_p.

(v) $\forall n \in \mathbb{N}^* : \Gamma_n \neq \emptyset$
Note that $\forall n \in \mathbb{N}^* : \emptyset \in \Gamma_n$. So, $\forall n \in \mathbb{N}^* : \Gamma_n \neq \emptyset$.

Proof of theorem 1.1
The functional Ψ satisfies the assertions of theorem 1.3, then it has an infinite sequence of critical points u_n defined by :

$$c_n = inf_{A\in\Gamma_n} sup_{x\in A} \Psi(x) = \Psi(u_n).$$

It results that

$$\begin{aligned}\forall n \in \mathbb{N}^*, \left\langle \Psi^{'}(u_n), u_n\right\rangle = 0. &\iff G(u_n) - \lambda_n F(u_n) = 0 \\ &\iff \Psi(u_n) = 0.\end{aligned}$$

Moreover, since $F(u) = 1, \forall u \in M$, we have :

$$inf_{A\in\Gamma_n} sup_{u\in A} \Psi(u) = inf_{A\in\Gamma_n} sup_{u\in A} (G(u) - \lambda) = 0.$$

Hence, we conclude the existence of an infinite sequence of nonnegative eigenvalues $(\lambda_n)_n$ for the problem(1.1) defined as :

$$\lambda_n = inf_{K\in\Gamma_n}\ sup_{u\in K} \left(\frac{1}{2}\int_\Omega |\Delta^p u|^2\, dx\right).$$

Finally, by the same way as in [13], we deduct that $\lambda_n \longrightarrow +\infty$ as $n \longrightarrow +\infty$.

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