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Renormalized solutions for a $p(\cdot)$-Laplacian equation with Neumann nonhomogeneous boundary condition involving diffuse measure data and variable exponent

M. B. Benboubker[1], E. Nassouri[2], S. Ouaro[3], U. Traoré[4]

Abstract. In this paper we prove the existence of at least one renormalized solution for the p(x)-Laplacian equation associated with a maximal monotone operator and Radon measure data. The functional setting involves Sobolev spaces with variable exponent $W^{1,p(\cdot)}(\Omega)$.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$ such that meas$(\partial\Omega) > 0$ and η the unit outward normal vector on $\partial\Omega$. In this paper, we are interested in the following

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[1]*Ecole Supérieure de Technologie de Fès,*
Université Sidi Mohamed Ben Abdellah, Fès, Maroc.
e-mail: simo.ben@hotmail.com (Corresponding Author)

[2]*Université Thomas SANKARA, Burkina Faso*
e-mail: nassouristella@yahoo.fr

[3]*Université Joseph Ki-Zerbo, Burkina Faso*
e-mail: ouaro@yahoo.fr

[4]*Université Joseph Ki-Zerbo, Burkina Faso*
e-mail: urbain.traore@yahoo.fr .

nonlinear multivalued elliptic problem:

$$(P_\mu)\begin{cases} -\triangle_{p(x)}u + |u|^{p(x)-2}u + \alpha(u)|\nabla u|^{p(x)} \ni \mu & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\nabla u.\eta + \beta(u) = g & \text{on } \partial\Omega, \end{cases}$$

where α is maximal monotone graph on $\mathbb{R}$ with $\alpha(0) = 0$, μ is a Radon diffuse measure such that $\mu = \mu\lfloor\Omega$ and the operator $-\triangle_{p(x)}u := -div(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(\cdot)$-Laplacian, which is a natural extension of the p-Laplace operator, with p being a positive constant.
The existence of renormalized solutions for the problem (P_μ) where α is a real function and μ is L^1 or Radon measure data has been studied in [3, 4]. Recently in [8], problem (P_μ) was studied with $\mu \in L^1(\Omega)$. In [3, 4, 8], under some assumptions the authors proved that the operator associated to the approximated problem is of type (M). So, by some a priori estimates, they obtained the convergence of the approximate sequence to a renormalized solution of the initial problem. In this work, we extend the approach developed in [3, 4, 8] to the case of maximal monotone graph or and Radon diffuse measure data.
Elliptic problems with variable exponent has been extensively studied in recent years (see [3, 4, 5, 6, 7, 9, 18, 19, 20, 22, 23]) and the references therein. The interest of studying problems with variable exponent is due to the fact they can model phenomena which arise in mathematical physics such that elastic mechanics, electro-rheological fluid dynamics and image processing (see [20, 23]).
Note that the functional spaces used to study these problems are the generalized Lebesgue spaces $L^{p(\cdot)}(\Omega)$ and the generalized Lebesgue-Sobolev spaces $W^{k,p(\cdot)}(\Omega)$.
In our paper we study existence of renormalized solutions for problem (P_μ). Recall that the notion of renormalized solutions was first introduced by DiPerna and Lions [16] for the study of the Boltzmann equation. It was then adapted to the study of some nonlinear elliptic or parabolic problems and evolution problems in fluid mechanics. We refer the reader to [11, 12] for details. In the case of variable exponent, renormalized solution was studied for the first time by wittbold and Zimmerman in [22] where they considered an homogeneous Dirichlet boundary condition. Since in our study we consider an inhomogeneous Neumann boundary condition and Radon diffuse measure, it is not a surprise to meet new difficulties.
One of the major difficulties which appear is the application of the decomposition result of the measure. This is due to the fact that the Neumann condition of the problem (P_μ) forces us to work in $W^{1,p(\cdot)}(\Omega)$ while the second member of the decomposition of the measure is in $W^{-1,p(\cdot)}(\Omega)$. To overcome this difficulty, we use the same ideas as authors in [4, 9, 18, 19]. We consider a suitable domain Ω in order to work in $W_0^{1,p(\cdot)}(\Omega)$ and then go back to space in $W^{1,p(\cdot)}(\Omega)$. More precisely, we suppose that Ω is an extension domain (see [13]), so, we can fix bounded subset U_Ω of $\mathbb{R}^N$ such that $\overline{\Omega} \subset U_\Omega$ and we define a bounded linear operator $E : W^{1,p(\cdot)}(\Omega) \to W_0^{1,p(\cdot)}(U_\Omega)$ who satisfies the following properties.

(a) $E(u) = u$ a.e. in Ω for each $u \in W^{1,p(\cdot)}(\Omega)$,
(b) $\|E(u)\|_{W_0^{1,p(\cdot)}(U_\Omega)} \leq \lambda \|E(u)\|_{W^{1,p(\cdot)}(\Omega)}$ where λ is a constant depending only on Ω.

Let $\mathcal{M}_b^{p(\cdot)}(\Omega)$ a space of bounded Radon measure in variable exponent setting and $\mathfrak{M}_b^{p(\cdot)}(\Omega) = \{\mu \in \mathcal{M}_b^{p(\cdot)}(\Omega) \text{ such that : } \mu \text{ is concentred on } \Omega\}$.
For $u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and $\mu \in \mathfrak{M}_b^{p(\cdot)}(\Omega)$, we have

$$\langle \mu, E(u) \rangle = \int_\Omega u d\mu. \tag{1.1}$$

As μ is diffuse, there exist $f \in L^1(U_\Omega)$ and $F \in (L^{p'(\cdot)}(U_\Omega))^N$ such that $\mu = f - \operatorname{div}(F)$ in $\mathcal{D}(\Omega)$. So, (1.1) implies that

$$\begin{aligned} \langle \mu, E(u) \rangle &= \int_\Omega u d\mu \\ &= \int_{U_\Omega} f E(u) dx + \int_{U_\Omega} F_R . \nabla E(u) dx, \end{aligned} \tag{1.2}$$

with $F_R = \chi_\Omega F$.
Recall that $\mu \in \mathcal{M}_b^{p(\cdot)}(\Omega)$ is diffuse with respect to the capacity $W^{1,p(\cdot)}(\Omega)$ if $\mu(E) = 0$ for every set E such that $Cap_{p(\cdot)}(E, \Omega) = 0$ with

$$S_{p(\cdot)}(E) = \left\{ u \in W^{1,p(\cdot)}(\Omega) : u \geq 1 \text{ in an open set containing E and } u \geq 1 \text{ in } \Omega \right\}$$

and

$$Cap_{p(\cdot)}(E, \Omega) = \inf_{u \in S_{p(\cdot)}(E)} \left\{ \int_\Omega \left(|u|^{p(x)} + |\nabla u|^{p(x)} \right) dx \right\}.$$

In the case where $S_{p(\cdot)}(E) = 0$ we have $Cap_{p(\cdot)}(E, \Omega) = +\infty$.
This work can be seen as a generalization of [8]. Our aim is to prove the existence of renormalized solutions to the general elliptic problem (P_μ).
The plan of the paper is as follows. In Section 2, we introduce notations, useful functional spaces for our study and the main result. In Section 3, we give the proof of the main result introduce in Section 2.

2. Preliminaries and mains results

In order to study problem (P_μ), we need some results for the generalized Lebesgue-Sobolev spaces.
For an open set $\Omega \subset \mathbb{R}^N$, $N \geq 3$, we write

$$C^+(\overline{\Omega}) = \{\text{continuous functions } p(\cdot) : \overline{\Omega} \to [0, +\infty) \text{ such that } 1 < p_- \leq p_+ < +\infty\}$$

where $p_- := \inf_{x \in \Omega} p(x)$ and $p_+ := \sup_{x \in \Omega} p(x)$.
For $p(\cdot) \in C^+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable such that } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}.$$

endowed with the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \ \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1 \text{ for all } x \in \Omega\right\}$$

with

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)} dx.$$

The space $\left(L^{p(\cdot)}(\Omega), \|.\|_{p(\cdot)}\right)$ is a Banach space. Moreover, the set $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where, $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, for any $x \in \Omega$.
For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the Hölder type inequality

$$\left|\int_\Omega uv dx\right| \leq \left(\frac{1}{p_-} + \frac{1}{p_+}\right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)} \tag{2.1}$$

holds true.
We define also the variable Sobolev space

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : \ |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}$$

which is a Banach space equipped with the following norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The space $\left(W^{1,p(\cdot)}(\Omega), \|.\|_{1,p(\cdot)}\right)$ is a separable and reflexive Banach space.
The modular $\rho_{p(\cdot)}$ of the space $L^{p(\cdot)}(\Omega)$ play an important role in the generalized Lebesgue and Sobolev spaces. We introduce the following generalization for the modular convex $\rho_{p(\cdot)}$ for all measurable function $u : \Omega \to \mathbb{R}$:

$$\rho_{1,p(\cdot)}(u) = \int_\Omega |u|^{p(x)} dx + \int_\Omega |\nabla u|^{p(x)} dx.$$

We define the following function

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)} & \text{if } p(x) < N, \\ \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

Let us introduce the following notation: given two bounded continuous functions $p(x)$, $q(x) : \Omega \to \mathbb{R}$, we write

$$q(x) \ll p(x) \text{ if } \inf_{x\in\Omega}(p(x) - q(x)) > 0.$$

Lemma 2.1. (see [21, 17])
Let $(v_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions in Ω. If (v_n) converges in measure to v and is uniformly bounded in $L^{p(\cdot)}(\Omega)$ for some $1 \ll p(\cdot) \in L^\infty(\Omega)$, then, v_n strongly converges to v in $L^1(\Omega)$.

Lemma 2.2. *Let $(\alpha_n)_{n\geq 1}$ be a sequence of maximal monotone graphs such that $\alpha_n \to \alpha$ in the sense of graph i.e. for $(x,y) \in \alpha$, there exists $(x_n, y_n) \in \alpha_n$ such that $x_n \to x$ and $y_n \to y$. We consider two sequences $(z_n)_{n\geq 1} \subset L^1(\Omega)$ and $(w_n)_{n\geq 1} \subset L^1(\Omega)$. We assume that $\forall n \geq 1$, $w_n \in \alpha(z_n)$, $(w_n)_{n\geq 1}$ is bounded in $L^1(\Omega)$ and $z_n \to z$ in $L^1(\Omega)$. Then $z \in dom(\alpha)$.*

Let us introduce some functions that will be frequently used in this study.
For any given $l > 0$, we define the function h_l by $h_l(r) = min((l + 1 - |r|), 1)$. Let $h_{l_0} > 0$, we define $h_0 = h_{l_0}$ by

$$\begin{cases} h_0 \in C_c^1(\mathbb{R}),\ h_0(r) \geq 0\ \forall r \in \mathbb{R}, \\ \\ h_0(r) = 1 \text{ if } |r| \leq l_0 \text{ and } h_0(r) = 0 \text{ if } |r| \geq l_0 + 1. \end{cases}$$

Also for all $k > 0$, we define the truncation function $T_k : \mathbb{R} \to \mathbb{R}$ by

$$T_k(s) := \max\{-k, \min\{k, s\}\} = \begin{cases} k & \text{if} \quad s \geq k, \\ \\ s & \text{if} \quad |s| < k, \\ \\ -k & \text{if} \quad s \leq -k. \end{cases}$$

To give our notion of solution for the problem (P), we begin by recalling the following definition: for any $u \in W^{1,p(\cdot)}(\Omega)$, we denote by $\tau(u)$ the trace of u on $\partial\Omega$ in the usual sense. In the sequel, we will identify at the boundary u and $\tau(u)$.

We denote (see [5, 9])

$$\mathcal{T}^{1,p(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} :\ u \text{ measurable such that } \ T_k(u) \in W^{1,p(\cdot)}(\Omega) \text{ for any } \ k > 0 \right\}.$$

Next we define the very weak gradient of a measurable function $u \in \mathcal{T}^{1,p(\cdot)}(\Omega)$. The proof follows from Lemma 2.1 of [10] due to the fact that $W^{1,p(\cdot)} \subset W^{1,p_-}$ and Ω is bounded.

Proposition 2.1. *Let $u \in \mathcal{T}^{1,p(\cdot)}(\Omega)$. Then, there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that $\nabla T_k(u) = v\chi_{\{|u|<k\}}$, for all $k > 0$. The function v is denoted by ∇u. Moreover, if $u \in W^{1,p(\cdot)}(\Omega)$ then $v \in (L^{p(\cdot)}(\Omega))^N$ and $v = \nabla u$ in the usual sense.*

We denote by $\mathcal{T}_{tr}^{1,p(\cdot)}(\Omega)$ (see [1, 2]) the set of functions $u \in \mathcal{T}^{1,p(\cdot)}(\Omega)$ such that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,p(\cdot)}(\Omega)$ satisfying the following conditions:

(C1) $u_n \to u$ a.e. in Ω.
(C2) $\nabla T_k(u_n) \to \nabla T_k(u)$ in $(L^1(\Omega))^N$ for any $k > 0$.
(C3) There exists a measurable function v on $\partial\Omega$, such that $u_n \to v$ a.e. on $\partial\Omega$.

The function v is the trace of u in the generalized sense (see [1, 2]). In the sequel, the trace of $u \in \mathcal{T}_{tr}^{1,p(\cdot)}(\Omega)$ on $\partial\Omega$ will be denoted by $tr(u)$. If $u \in W^{1,p(\cdot)}(\Omega)$, $tr(u)$ coincides with $\tau(u)$ in the usual sense. Moreover, for $u \in \mathcal{T}_{tr}^{1,p(\cdot)}(\Omega)$ and for every $k > 0$, $\tau(T_k(u)) = T_k(tr(u))$ and if $\varphi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ then $(u - \varphi) \in \mathcal{T}_{tr}^{1,p(\cdot)}(\Omega)$ and $tr(u - \varphi) = tr(u) - tr(\varphi)$.

Theorem 2.1. (see [18]) *Let $p(\cdot) : \overline{X} \subset \Omega \to \mathbb{R}$ with $1 < p(\cdot) < \infty$ be a continuous function. Let μ be an element of $\mathcal{M}_b(\Omega)$. Then, $\mu \in L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$ if and only if $\mu \in \mathcal{M}_b^{p(\cdot)}(\Omega)$.*

We denote

$\mathcal{J}_0(\Omega) = \{j : \Omega \to [0,\infty)$ such that j is measurable, convex, l.s.c. function that verify $j(0) = 0\}$.

For any $j \in \mathcal{J}_0(\Omega)$, we define the function $\mathcal{J} : W^{1,p(\cdot)}(\Omega) \to [0,\infty]$ by

$$\mathcal{J}(z) = \begin{cases} \displaystyle\int_\Omega j(z)|\nabla u|^{p(x)}dx & \text{if} \quad j(z)|\nabla u|^{p(x)} \in L^1(\Omega) \\ \\ \infty & \text{if} \quad j(z)|\nabla u|^{p(x)} \notin L^1(\Omega). \end{cases}$$

We have

$$\nu \in \partial\mathcal{J}(v) \Longleftrightarrow \begin{cases} v \in W^{1,p(\cdot)}(\Omega),\ \nu \in (W^{1,p(\cdot)}(\Omega))^* \\ \\ \text{and } \mathcal{J}(z) \geq \mathcal{J}(v) + \langle \nu, z - v\rangle\,,\ \forall\, z \in W^{1,p(\cdot)}(\Omega). \end{cases}$$

Definition 2.1. *For any $\mu \in \mathfrak{M}_b^{p(\cdot)}(\Omega)$, $g \in L^1(\partial\Omega)$, a renormalized solution of the elliptic problem (P_μ) is a couple of functions $(u,\nu) \in \mathcal{T}_{tr}^{1,p(\cdot)}(\Omega) \times \mathcal{M}_b(\Omega)$ such that $\nu \in \partial\mathcal{J}(u)$ a.e. in Ω,*

$$\begin{aligned} &\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla(h(u)\varphi)dx + \int_\Omega |u|^{p(x)-2}uh(u)\varphi dx + \int_{\partial\Omega}\beta(u)h(u)\varphi ds \\ &+ \int_\Omega h(u)\varphi d\nu = \int_\Omega h(u)\varphi d\mu + \int_{\partial\Omega} gh(u)\varphi ds, \end{aligned} \tag{2.2}$$

for all $\varphi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and for any smooth function with compact support h in $\mathbb{R}$ and

$$\lim_{n\to+\infty} \int_{\{n\leq|u|\leq n+1\}} |\nabla u|^{p(x)}dx = 0. \tag{2.3}$$

Now we introduce the main result of this section.

Theorem 2.2. *Assume that $\mu \in \mathfrak{M}_b^{p(\cdot)}(\Omega)$, $g \in L^1(\partial\Omega)$. Then, problem (P_μ) admits at least one renormalized solution.*

3. Proof of Theorem 2.2

The proof of the above theorem is subdivided in four steps and use standard techniques.

3.1. The approximate problem. For each $\varepsilon > 0$, α_ε denotes the Yosida approximation of α (see [14]) and is a nondecreasing and Lipschitz-continuous function and verify $\alpha_\varepsilon(0) = 0$.
α_ε is defined as follows.

$$\alpha_\varepsilon = \frac{1}{\varepsilon}\left(I - (I + \varepsilon\alpha)^{-1}\right).$$

Since $\mu \in \mathcal{M}_b^{p(\cdot)}(U_\Omega)$, we have

$$\mu = f + div(F) \tag{3.1}$$

with $f \in L^1(U_\Omega)$ and $F \in \left(L^{p'(.)}(U_\Omega)\right)^N$. For the rest we denote $F_R = \chi_\Omega F$.
In the aim to regularize μ, we define the function

$$f_\varepsilon = T_{\frac{1}{\varepsilon}}(f)\chi_\Omega \text{ for all } x \in U_\Omega \text{ and for any } \varepsilon > 0.$$

We have now

$$\mu_\varepsilon = f_\varepsilon + div(F_R) \text{ for any } \varepsilon > 0.$$

For any $\varepsilon > 0$, one has $\mu_\varepsilon \in \mathfrak{M}_b^{p(\cdot)}(\Omega)$, $\mu_\varepsilon \rightharpoonup \mu \in \mathcal{M}_b(U_\Omega)$ and $\mu_\varepsilon \in L^\infty(\Omega)$. Furthermore, for any $k > 0$ and any $\xi \in \mathcal{T}^{1,p(\cdot)}(\Omega)$ we have

$$\left| \int_\Omega T_k(\xi) d\mu_\varepsilon \right| \leq k\lambda_0(\mu, \Omega). \tag{3.2}$$

For the rest, we consider also the following functions $\beta_\varepsilon = T_{\frac{1}{\varepsilon}}(\beta)$ and $g_\varepsilon = T_{\frac{1}{\varepsilon}}(g)$. The functions $(f_\varepsilon)_{\varepsilon>0}$ and $(g_\varepsilon)_{\varepsilon>0}$ are strongly converging to $f \in L^1(U_\Omega)$, $g \in L^1(\partial\Omega)$ respectively and verifies $\|f_\varepsilon\|_{L^1(U_\Omega)} \leq \|f\|_{L^1(U_\Omega)}$ and $\|g_\varepsilon\|_{L^1(\partial\Omega)} \leq \|g\|_{L^1(\partial\Omega)}$.
In order to get the existence of weak solutions, we introduce the approximate problem to (P_μ), namely

$$(P_{\mu_\varepsilon}) \begin{cases} -\triangle_{p(x)} u_\varepsilon + |u_\varepsilon|^{p(x)-2}u_\varepsilon + \alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)} = \mu_\varepsilon & \text{in} \quad \Omega, \\ \\ |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon.\eta + \beta_\varepsilon(u_\varepsilon) = g_\varepsilon & \text{on} \quad \partial\Omega. \end{cases} \tag{3.3}$$

In [4] authors have studied problem (P_{μ_ε}) and showed the following result.

Lemma 3.1. *Problem (P_{μ_ε}) admits at least one weak solution u_ε in the sense that $u_\varepsilon \in W^{1,p(\cdot)}(\Omega)$ and for all $\varphi \in W^{1,p(\cdot)}(\Omega)$,*

$$\begin{aligned} &\int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \nabla\varphi dx + \int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon\varphi dx + \int_\Omega \alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}\varphi dx \\ &+ \int_{\partial\Omega} \beta_\varepsilon(u_\varepsilon)\varphi ds = \int_\Omega \varphi d\mu_\varepsilon + \int_{\partial\Omega} g_\varepsilon\varphi ds \end{aligned} \tag{3.4}$$

and

$$\rho_{1,p(\cdot)}(T_k(u_\varepsilon)) \leq 1 + (k\lambda_1 + \lambda_2)^{\frac{1}{p_-}} \tag{3.5}$$

where $\lambda_1 = const(f, g, p_-, p_+)$ and $\lambda_2 = const(F, p_-, p_+)$.

3.2. A priori estimates.

Lemma 3.2. *Let u_ε be a weak solution of (P_{μ_ε}). Then, there exists a positive constant C_1 depending of μ and F such that for any $k > 0$,*

$$\int_\Omega |\nabla T_k(u_\varepsilon)|^{p(x)} dx \leq kC_1; \tag{3.6}$$

$$\int_\Omega \alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)} T_k(u_\varepsilon) dx \leq kC_1 \tag{3.7}$$

and

$$\int_\Omega |u_\varepsilon|^{p(x)-1}dx \leq C_1. \tag{3.8}$$

Proof. We take $\varphi = T_k(u_\varepsilon)$ as test function in (3.4) to get

$$\int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \nabla T_k(u_\varepsilon)dx + \int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon T_k(u_\varepsilon)dx + \int_\Omega \alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}T_k(u_\varepsilon)dx$$
$$+ \int_{\partial\Omega} \beta_\varepsilon(u_\varepsilon)T_k(u_\varepsilon)ds = \int_\Omega T_k(u_\varepsilon)d\mu_\varepsilon + \int_{\partial\Omega} g_\varepsilon T_k(u_\varepsilon)ds. \tag{3.9}$$

Using inequality (3.2) and the fact that $\int_{\partial\Omega} g_\varepsilon T_k(u_\varepsilon)ds \leq k\|g\|_{L^1(\partial\Omega)}$ we get that

$$\int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \nabla T_k(u_\varepsilon)dx + \int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon T_k(u_\varepsilon)dx + \int_\Omega \alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}T_k(u_\varepsilon)dx$$
$$+ \int_{\partial\Omega} \beta_\varepsilon(u_\varepsilon)T_k(u_\varepsilon)ds \leq k\left(\lambda_0 + \|g\|_{L^1(\partial\Omega)}\right). \tag{3.10}$$

It is obvious that

$$\int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon T_k(u_\varepsilon)dx \geq 0. \tag{3.11}$$

We also have that

$$\begin{aligned}\int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \nabla T_k(u_\varepsilon)dx &= \int_{\{|u_\varepsilon|\leq k\}} |\nabla u_\varepsilon|^{p(x)}dx \\ &= \int_{\{|u_\varepsilon|\leq k\}} |\nabla T_k(u_\varepsilon)|^{p(x)}dx \\ &= \int_\Omega |\nabla T_k(u_\varepsilon)|^{p(x)}dx.\end{aligned} \tag{3.12}$$

Since α_ε and β_ε are non-decreasing functions with $\alpha_\varepsilon(0) = 0$ and $\beta_\varepsilon(0) = 0$, we get that

$$\int_\Omega \alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}T_k(u_\varepsilon)dx \geq 0 \text{ and } \int_{\partial\Omega} \beta_\varepsilon(u_\varepsilon)T_k(u_\varepsilon)ds \geq 0.$$

From (3.10), dropping some non negative terms and using (3.12) we get

$$\int_\Omega |\nabla T_k(u_\varepsilon)|^{p(x)}dx \leq k\left(\lambda_0 + \|g\|_{L^1(\partial\Omega)}\right), \tag{3.13}$$

which implies (3.6).
Also from (3.10) we get

$$\int_\Omega \alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)})T_k(u_\varepsilon)dx \leq k\left(\lambda_0 + \|g\|_{L^1(\partial\Omega)}\right),$$

which implies (3.7).
From (3.10) and the above results, we get that

$$\int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon T_k(u_\varepsilon)dx \leq k\left(\lambda_0 + \|g\|_{L^1(\partial\Omega)}\right). \tag{3.14}$$

Dividing (3.14) by k and passing to the limit as $k \to 0$ by using the Lebesgue Dominated Convergence Theorem, we obtain relation (3.8).

Lemma 3.3. *For any $\varepsilon > 0$, we have.*

(i) *The sequence $(\alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)})_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$.*

(ii) *The sequence $(\alpha_\varepsilon(T_k(u_\varepsilon))|\nabla T_k(u_\varepsilon)|^{p(x)})_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$.*

Proof. **(i)** According to (3.7), we have

$$\int_\Omega \alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}\frac{1}{k}T_k(u_\varepsilon)dx \leq C_1.$$

Letting k goes to 0 in the above inequality, we get that

$$\int_\Omega |\alpha(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}|dx \leq C_1.$$

(ii) The proof of (ii) is trivial and follows from the fact that the sequence $(\alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)})_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$.

3.3. Convergence results. The aim of this section is to introduce the convergence results which are necessary for the proof of the existence of the solution.

Lemma 3.4. *For any $h \in C_c^1(\mathbb{R})$ and $\varphi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, we have*

$$\lim_{\varepsilon\to 0}\int_\Omega h(u_\varepsilon)\varphi d\mu_\varepsilon = \int_\Omega h(u)\varphi d\mu.$$

Proof. We have

$$\begin{aligned}
\int_\Omega h(u_\varepsilon)\varphi d\mu_\varepsilon &= \int_\Omega E(h(u_\varepsilon)\varphi))d\mu_\varepsilon \\
&= \langle \mu_\varepsilon, E(h(u_\varepsilon)\varphi)\rangle \\
&= \int_{U_\Omega} f_\varepsilon E(h(u_\varepsilon)\varphi)dx + \int_{U_\Omega} F_R.\nabla E(h(u_\varepsilon)\varphi)dx \qquad (3.15)\\
&= \int_{U_\Omega} T_{\frac{1}{\varepsilon}}(f)\chi_\Omega E(h(u_\varepsilon)\varphi)dx + \int_{U_\Omega} F(\chi_\Omega).\nabla E(h(u_\varepsilon)\varphi)dx \\
&= \int_{U_\Omega} T_{\frac{1}{\varepsilon}}(f)E(\chi_\Omega h(u_\varepsilon)\varphi dx + \int_{U_\Omega} F.\nabla E(\chi_\Omega h(u_\varepsilon)\varphi dx \\
&= \int_\Omega T_{\frac{1}{\varepsilon}}(f)E(h(u_\varepsilon)\varphi)dx + \int_{U_\Omega} F.\nabla E(\chi_\Omega h(u_\varepsilon)\varphi)dx.
\end{aligned}$$

Thanks to the Lebesgue dominated convergence theorem, we have

$$\lim_{\varepsilon\to 0}\int_\Omega T_{\frac{1}{\varepsilon}}(f)E(h(u_\varepsilon)\varphi)dx = \int_\Omega fE(h(u)\varphi)dx. \qquad (3.16)$$

Since the sequence $(\chi_\Omega h(u_\varepsilon)\varphi)$ is bounded in $W^{1,p(\cdot)}(\Omega)$, thanks to the properties (b) of E, we can deduce that $(E(\chi_\Omega h(u_\varepsilon)\varphi))$ is bounded in $W_0^{1,p(x)}(\Omega)$. Moreover, $E(\chi_\Omega h(u_\varepsilon)\varphi) = \chi_\Omega h(u_\varepsilon)\varphi$ a.e. in U_Ω and

$$\chi_\Omega h(u_\varepsilon)\varphi \to \chi_\Omega h(u)\varphi \text{ a.e. in } U_\Omega \text{ as } \varepsilon \to 0.$$

Therefore,

$$E(\chi_\Omega h(u_\varepsilon)\varphi) \to E(\chi_\Omega h(u)\varphi) \text{ a.e. in } U_\Omega \text{ as } \varepsilon \to 0.$$

Consequently,

$$\nabla E(\chi_\Omega h(u_\varepsilon)\varphi) \rightharpoonup \nabla E(\chi_\Omega h(u)\varphi) \text{ a.e. in } (L^{p(\cdot)}(U_\Omega))^N \text{ as } \varepsilon \to 0.$$

Since $F \in (L^{p'(\cdot)}(U_\Omega))^N$, it comes that

$$\lim_{\varepsilon\to 0}\int_\Omega F.\nabla E(\chi_\Omega h(u_\varepsilon)\varphi)dx = \int_\Omega F.\nabla E(\chi_\Omega h(u)\varphi)dx. \tag{3.17}$$

Letting $\varepsilon \to 0$ in (3.15) and thanks to (3.16) and (3.17) we get

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_\Omega h(u_\varepsilon)\varphi d\mu_\varepsilon &= \int_\Omega fE(h(u)\varphi)dx + \int_\Omega F.\nabla E(\chi_\Omega h(u)\varphi)dx \\
&= \int_{U_\Omega} fE(\chi_\Omega h(u)\varphi)dx + \int_\Omega F.\nabla E(\chi_\Omega h(u)\varphi)dx \\
&= \langle \mu, E(h(u)\varphi)\rangle \\
&= \int_\Omega h(u)\varphi d\mu.
\end{aligned}$$

Lemma 3.5. *Assume that $\mu \in \mathfrak{M}_b^{p(\cdot)}(\Omega)$, $g \in L^1(\partial\Omega)$. Then, there exists a measurable function $u : \Omega \to \mathbb{R}$ such that $u \in dom(\alpha)$ a.e. in Ω and $u_\varepsilon \to u$ in measure and a.e. in Ω as $\varepsilon \to 0$.*

Proof. For the proof we refer to [4] (see also [3]).

As for $k > 0$, T_k is continuous, then $T_k(u_\varepsilon) \to T_k(u)$ a.e. in Ω.

$$\begin{aligned}
&T_k(u_\varepsilon) \rightharpoonup T_k(u) \text{ as } \varepsilon \to 0 \text{ in } W^{1,p-}(\Omega), \\
&T_k(u_\varepsilon) \to T_k(u) \text{ as } \varepsilon \to 0 \text{ in } L^{p-}(\Omega), \text{ and a.e. in } \Omega.
\end{aligned} \tag{3.18}$$

From Lemma 2.2, we get that for all $k > 0$, $T_k(u) \in \text{dom}(\alpha)$ a.e. in Ω and as $\text{dom}(\alpha)$ is bounded, we deduce that $u \in \text{dom}(\alpha)$ a.e. in Ω.

Using same methods as in the proof of the assertions 2 and 3 in [3], we obtain the following lemma.

Lemma 3.6. *Assume that $\mu \in \mathfrak{M}_b^{p(\cdot)}(\Omega)$, $g \in L^1(\partial\Omega)$, then*

(i) $(\nabla T_k(u_\varepsilon))_{\varepsilon>0}$ converges strongly to $\nabla T_k(u)$ in $(L^{p(\cdot)}(\Omega))^N$.
(ii) $(u_\varepsilon)_{\varepsilon>0}$ converges a.e on $\partial\Omega$ to some function u.

Remark 3.1. *By Lemma 3.5 and Lemma 3.6, we deduce that $u \in \mathcal{T}_{tr}^{1,p(\cdot)}(\Omega)$.*

3.4. Existence result. Step 1: Passing to the limit as $\varepsilon \to 0$

Let $\varphi \in W^{1,p(\cdot)}(\Omega) \cap L^{\infty}(\Omega)$ and $h \in C_c^1(\mathbb{R})$. In (3.4) we take $h(u_\varepsilon)\varphi$ as test function to get

$$\int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \nabla(h(u_\varepsilon)\varphi)dx + \int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon h(u_\varepsilon)\varphi dx + \int_\Omega \alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}h(u_\varepsilon)\varphi dx$$
$$+ \int_{\partial\Omega} \beta_\varepsilon(u_\varepsilon)h(u_\varepsilon)\varphi ds = \int_\Omega h(u_\varepsilon)\varphi d\mu_\varepsilon + \int_{\partial\Omega} g_\varepsilon h(u_\varepsilon)\varphi ds. \tag{3.19}$$

Since the function h has compact support, then there exists a positive real number M such that $Supp(h) \subset [-M, M]$ which leads to

$$\begin{aligned}\int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \nabla(h(u_\varepsilon)\varphi)dx &= \int_\Omega |\nabla T_M(u_\varepsilon)|^{p(x)-2}\nabla T_M(u_\varepsilon)\nabla(h(u_\varepsilon)\varphi)dx.\\ &= \int_\Omega |\nabla T_M(u_\varepsilon)|^{p(x)-2}\nabla T_M(u_\varepsilon)h(u_\varepsilon)\nabla\varphi dx\\ &+ \int_\Omega |\nabla T_M(u_\varepsilon)|^{p(x)}h'(u_\varepsilon)\varphi dx.\end{aligned} \tag{3.20}$$

Since

$$|\nabla T_M(u_\varepsilon)|^{p(x)-2}\nabla T_M(u_\varepsilon) \rightharpoonup |\nabla T_M(u)|^{p(x)-2}\nabla T_M(u) \text{ weakly in } (L^{p'(\cdot)}(\Omega))^N$$

and $h(u_\varepsilon)\nabla\varphi \rightharpoonup h(u)\nabla\varphi$ strongly in $L^{p(\cdot)}(\Omega)$.
Hence,

$$\int_\Omega |\nabla T_M(u_\varepsilon)|^{p(x)-2}\nabla T_M(u_\varepsilon)h(u_\varepsilon)\nabla\varphi dx \to \int_\Omega |\nabla T_M(u)|^{p(x)-2}\nabla T_M(u)h(u)\nabla\varphi dx$$

and as

$$|\nabla T_M(u_\varepsilon)|^{p(x)} \to |\nabla T_M(u)|^{p(x)} \text{ in } L^1(\Omega),$$

it follows that

$$\int_\Omega |\nabla T_M(u)|^{p(x)}h'(u)\varphi dx \to \int_\Omega |\nabla T_M(u)|^{p(x)}h'(u)\varphi dx.$$

Then,

$$\lim_{\varepsilon\to 0}\int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon\nabla(h(u_\varepsilon)\varphi)dx = \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla(h(u)\varphi)dx. \tag{3.21}$$

We have

$$|u_\varepsilon|^{p(x)-2}u_\varepsilon h(u_\varepsilon)\varphi \to |u|^{p(x)-2}uh(u)\varphi \text{ as } \varepsilon \to 0$$

and

$$|u_\varepsilon|^{p(x)-2}u_\varepsilon h(u_\varepsilon)\varphi \le C_4(h)|u_\varepsilon|^{p(x)-1}\|\varphi\|_\infty \in L^1(\Omega).$$

Then, by the Lebesgue Generalized Convergence Theorem, we have

$$\lim_{\varepsilon\to 0}\int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon h(u_\varepsilon)\varphi dx = \int_\Omega |u|^{p(x)-2}uh(u)\varphi dx. \tag{3.22}$$

In the same way we get that

$$\int_{\partial\Omega} g_\varepsilon h(u_\varepsilon)\varphi ds \to \int_{\partial\Omega} gh(u)\varphi ds \text{ as } \varepsilon \to 0 \tag{3.23}$$

and

$$\int_{\partial\Omega} \beta_\varepsilon(u_\varepsilon)h(u_\varepsilon)\varphi ds \to \int_{\partial\Omega} \beta(u)h(u)\varphi ds \text{ as } \varepsilon \to 0. \tag{3.24}$$

From Lemma 3.3, we know that the sequence $(\alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)})_{0<\varepsilon<1}$ is bounded in $L^1(\Omega)$. So, there exists $\nu \in \mathcal{M}_b(\Omega)$ such that

$$\alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)} \overset{*}{\rightharpoonup} \nu \text{ in } \mathcal{M}_b(\Omega) \text{ as } \varepsilon \to 0.$$

Moreover, for any $\varphi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, we have

$$\begin{aligned}
\int_\Omega \varphi d\nu &= \lim_{\varepsilon\to 0} \int_\Omega \alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}\varphi dx \\
&= \lim_{\varepsilon\to 0} \left(\int_\Omega \varphi d\mu_\varepsilon + \int_{\partial\Omega} g_\varepsilon \varphi ds - \int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \nabla\varphi dx \right) \\
&- \lim_{\varepsilon\to 0} \int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon \varphi dx - \lim_{\varepsilon\to 0} \int_{\partial\Omega} \beta_\varepsilon(u_\varepsilon)\varphi ds. \\
&= \int_\Omega \varphi d\mu + \int_{\partial\Omega} g\varphi ds - \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi dx \\
&- \int_\Omega |u|^{p(x)-2}u\varphi dx - \int_{\partial\Omega} \beta(u)\varphi ds.
\end{aligned}$$

Therefore, $\nu \in \mathcal{M}_b^{p(\cdot)}(\Omega) \cap \left((W^{1,p(\cdot)}(\Omega)^* + (L^1(\Omega))^* \right)$.
Letting $\varepsilon \to 0$ in (3.3), thanks to (3.21), (3.22), (3.23), (3.24) and Lemma 3.4 we get that

$$\begin{aligned}
&\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla(h(u)\varphi)dx + \int_\Omega |u|^{p(x)-2}uh(u)\varphi dx + \int_\Omega h(u)\varphi d\nu \\
&+ \int_{\partial\Omega} \beta(u)h(u)\varphi ds = \int_\Omega h(u)\varphi d\mu + \int_{\partial\Omega} gh(u)\varphi ds. \qquad (3.25)
\end{aligned}$$

Step 2: Subdifferential argument
Now we must show that $\nu \in \partial\mathcal{J}(u)$.
Since α is a maximal monotone graph, there exists $j \in \mathcal{J}_0(\Omega)$ such that for all $r \in \mathbb{R}$,

$$\alpha(r) = \partial j(r).$$

According to [14], for $0 < \varepsilon \leq 1$, the functional $j_\varepsilon : \mathbb{R} \to [0, +\infty)$ defined by $j_\varepsilon(r) = \int_0^r \alpha_\varepsilon(t)dt$ have the following properties.

(i) For any $0 < \varepsilon \leq 1$, j_ε is a convex Frechet-differentiable function and $j_\varepsilon'(r) = \alpha_\varepsilon(r)$ for all $r \in \mathbb{R}$;
(ii) $j_\varepsilon \uparrow j$ pointwise in $\mathbb{R}$ as $\varepsilon \downarrow 0$ and a.e. in Ω.

Thanks to (i), for all $t \in \mathbb{R}$ and almost everywhere Ω, we have

$$j_\varepsilon(u_\varepsilon) + (t - u_\varepsilon)\alpha_\varepsilon(u_\varepsilon) \leq j_\varepsilon(t) \leq j(t). \qquad (3.26)$$

Therefore,

$$j_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)} + (t - u_\varepsilon)\alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)} \leq j_\varepsilon(t)|\nabla u_\varepsilon|^{p(x)} \leq j(t)|\nabla u_\varepsilon|^{p(x)}.$$

So,

$$j_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)} + (\xi - u_\varepsilon)\alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)} \leq j_\varepsilon(\xi)|\nabla u_\varepsilon|^{p(x)} \leq j(\xi)|\nabla u_\varepsilon|^{p(x)} \qquad (3.27)$$

holds for all $\xi \in W^{1,p(\cdot)}(\Omega)$ and for a.e. $x \in \Omega$.
Fixing ε_0 such that $0 < \varepsilon < \varepsilon_0$, for all $\xi \in C(\Omega)$(set of continuous functions on Ω), integrating over Ω above inequality and using (ii) we get

$$\begin{aligned}
&\int_\Omega j(\xi)|\nabla u_\varepsilon|^{p(x)}dx \\
&\geq \int_\Omega j_{\varepsilon_0}(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}dx + \int_\Omega (\xi - u_\varepsilon)\alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}dx, \\
&\geq \int_\Omega j_{\varepsilon_0}(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}dx + \int_\Omega (\xi - u)\alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}dx \\
&+ \int_\Omega (u - u_\varepsilon)\alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}dx
\end{aligned} \tag{3.28}$$

for all $t \in \mathbb{R}$ and all $0 < \varepsilon < \varepsilon_0$.
We have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\Omega (u - u_\varepsilon)\alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}dx \\
&= \lim_{\varepsilon\to 0}\left(\int_\Omega (u - u_\varepsilon)d\mu_\varepsilon - \int_\Omega |\nabla T_k(u_\varepsilon)|^{p(x)-2}\nabla T_k(u_\varepsilon).\nabla(u - u_\varepsilon)dx\right) \\
&+ \lim_{\varepsilon\to 0}\left(-\int_{\partial\Omega}\beta_\varepsilon(u_\varepsilon)(u - u_\varepsilon)ds + \int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(g)(u - u_\varepsilon)ds\right) \\
&- \lim_{\varepsilon\to 0}\int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon(u - u_\varepsilon)dx \\
&\geq 0.
\end{aligned} \tag{3.29}$$

Passing to the limit as $\varepsilon_0 \downarrow 0$ and $\varepsilon \downarrow 0$ in (3.28) and using Fatou's Lemma and (3.29) we get

$$\int_\Omega j(u)|\nabla u|^{p(x)}dx + \int_\Omega (\xi - u)d\nu \leq \int_\Omega j(\xi)|\nabla u|^{p(x)}dx.$$

Therefore,

$$\mathcal{J}(\xi) \geq \mathcal{J}(u) + \langle \mu, \xi - u\rangle \text{ for all } \xi \in C(\Omega).$$

Since $\nu \in \mathcal{M}_b^{p(\cdot)}(\Omega)$, inequality above is true for all $\xi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$. So $\nu \in \partial\mathcal{J}(u)$.

Step 3: Proof of 2.3

In (3.4), we take $\varphi = T_1(u_\varepsilon - T_n(u_\varepsilon))$ to get

$$\begin{aligned}
&\int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \nabla T_1(u_\varepsilon - T_n(u_\varepsilon))dx + \int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon T_1(u_\varepsilon - T_n(u_\varepsilon)) \\
&+ \int_\Omega \alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}T_1(u_\varepsilon - T_n(u_\varepsilon))dx + \int_{\partial\Omega}\beta_\varepsilon(u_\varepsilon)T_1(u_\varepsilon - T_n(u_\varepsilon))ds \\
&= \int_\Omega T_1(u_\varepsilon - T_n(u_\varepsilon))d\mu_\varepsilon + \int_{\partial\Omega} g_\varepsilon T_1(u_\varepsilon - T_n(u_\varepsilon))ds.
\end{aligned} \tag{3.30}$$

We have

$$\int_\Omega \alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}T_1(u_\varepsilon - T_n(u_\varepsilon))dx$$
$$= \int_{\Omega_1\cap\Omega_2} \alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}(u_\varepsilon - n\text{sign}(u_\varepsilon))dx + \int_{\Omega_1\cap\Omega_3} \alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}\text{sign}(u_\varepsilon - n\text{sign}(u_\varepsilon))dx,$$

with $\Omega_1 = \{|u_\varepsilon| > n\}$, $\Omega_2 = \{|u_\varepsilon - n.\text{sign}(u_\varepsilon)| \leq 1\}$ and $\Omega_3 = \{|u_\varepsilon - n.\text{sign}(u_\varepsilon)| > 1\}$.
Let $u_\varepsilon \in \Omega_1$, so $u_\varepsilon < -n$ or $u_\varepsilon > n$.
In a first time we suppose that $u_\varepsilon < -n$ i.e. $\text{sign}(u_\varepsilon) = -1$ to get

$$\begin{aligned} u_\varepsilon < -n \quad &\Leftrightarrow u_\varepsilon + n < 0 \\ &\Leftrightarrow u_\varepsilon - n.\text{sign}(u_\varepsilon) < 0 \\ &\Leftrightarrow \text{sign}(u_\varepsilon) = \text{sign}(u_\varepsilon - n.\text{sign}(u_\varepsilon)). \end{aligned}$$

In a second time we suppose that $u_\varepsilon > n$ i.e. $\text{sign}(u_\varepsilon) = 1$ to get

$$\begin{aligned} u_\varepsilon > n \quad &\Leftrightarrow u_\varepsilon - n > 0 \\ &\Leftrightarrow u_\varepsilon - n.\text{sign}(u_\varepsilon) > 0 \\ &\Leftrightarrow \text{sign}(u_\varepsilon) = \text{sign}(u_\varepsilon - n.\text{sign}(u_\varepsilon)). \end{aligned}$$

Using monotonicity of α_ε, it follows that

$$\int_{\Omega_1\cap\Omega_2} \alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}(u_\varepsilon - n\text{sign}(u_\varepsilon))dx \geq 0 \tag{3.31}$$

and

$$\int_{\Omega_1\cap\Omega_3} \alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}\text{sign}(u_\varepsilon - n\text{sign}(u_\varepsilon))dx \geq 0. \tag{3.32}$$

From (3.31) and (3.32), we get

$$\int_\Omega \alpha_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)}T_1(u_\varepsilon - T_1(u_\varepsilon))dx \geq 0. \tag{3.33}$$

Using same arguments as (3.33) we get

$$\int_{\partial\Omega} \gamma_\varepsilon(u_\varepsilon)T_1(u_\varepsilon - T_n(u_\varepsilon))dx \geq 0. \tag{3.34}$$

and

$$\int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon T_1(u_\varepsilon - T_n(u_\varepsilon))dx \geq 0. \tag{3.35}$$

From (3.30) and thanks to (3.33), (3.34) and (3.35), we deduce that

$$\int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \nabla T_1(u_\varepsilon - T_n(u_\varepsilon))dx \leq \int_\Omega T_1(u_\varepsilon - T_n(u_\varepsilon))d\mu_\varepsilon + \int_{\partial\Omega} g_\varepsilon T_1(u_\varepsilon - T_n(u_\varepsilon))ds. \tag{3.36}$$

$$\int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \nabla T_1(u_\varepsilon - T_n(u_\varepsilon))dx = \int_{\{n<|u_\varepsilon|<n+1\}} |\nabla u_\varepsilon|^{p(x)}dx,$$

since

$$T_1(s - T_n(s)) = \begin{cases} s - n\ \mathrm{sign}(s) & \text{if} \quad n < |s| < n+1, \\ n\ \mathrm{sign}(s) & \text{if} \quad |s| \geq n+1, \\ 0 & \text{if} \quad |s| \leq n. \end{cases}$$

Then, from (3.36) it comes that

$$\int_{\{n<|u|<n+1\}} |\nabla u_\varepsilon|^{p(x)} dx \leq \int_\Omega T_1(u_\varepsilon - T_n(u_\varepsilon)) d\mu_\varepsilon + \int_{\partial\Omega} g_\varepsilon T_1(u_\varepsilon - T_n(u_\varepsilon)) ds. \tag{3.37}$$

We have (see [19])

$$\lim_{n\to+\infty} \lim_{\varepsilon\to 0} \int_\Omega T_1(u_\varepsilon - T_n(u_\varepsilon)) d\mu_\varepsilon = 0 \tag{3.38}$$

and

$$\lim_{n\to+\infty} \lim_{\varepsilon\to 0} \int_{\partial\Omega} g_\varepsilon T_1(u_\varepsilon - T_n(u_\varepsilon)) dx = 0. \tag{3.39}$$

Consequently,

$$\lim_{n\to+\infty} \lim_{\varepsilon\to 0} \int_{\{n<|u_\varepsilon|<n+1\}} |\nabla u_\varepsilon|^{p(x)} dx = 0. \tag{3.40}$$

References

[1] F. Andreu, J. M. Mazon, J. Toledo and N. Igbida; *L^1 existence and uniqueness results for quasi-linear elliptic equations with nonlinear boundary conditions*, Ann. Ins. H. Poincaré Anal. Non Lineaire **20** (2007), 61 – 89.

[2] F. Andreu, J. M. Mazon, J. Toledo and S. Segura De Leon; *Quasi-linear elliptic and parabolic equations in L^1 with nonlinear boundary conditions*, Adv. Math. Sci. Appl. **7** (1) (1997), 183 – 213.

[3] E. Azroul, A. Barbara, M. B. Benboubker and S. Ouaro; *Renormalized solutions for a p(x)-Laplacian equation with Neumann nonhomogeneous boundary conditions and L^1-data*, Annals Univ. Craiova, Math, and Comp. Sci. Ser. 40, **(1)**, (2013), 9 – 22.

[4] E. Azroul, M. B. Benboubker, R. Bouzyani and H. Chrayteh; *Renormalized solutions for some nonlinear nonhomogeneous elliptic problems with neumann boundary condition and right hand side measure*, Bol. Soc. Paran. Mat. 39 6 (2021), pp. 81–103.

[5] E. Azroul, M. B. Benboubker and S. Ouaro; *Entropy solutions for nonlinear nonhomogeneous Neumann problems involving the generalized $p(\cdot)$-Laplace operator*, J. Appl. Anal. Comput. 3, no. **2** (2013), pp. 105 – 121 .

[6] M. B. Benboubker, E. Azroul, A. Barbara; *Quasilinear elliptic problems with nonstandard growths*, Electronic J. Diff. Equ, **62** (2011), 1–16.

[7] M. B. Benboubker, H. Chrayteh, H. Hjiaj, C. Yazough; *Existence of solutions in the sense of distributions of anisotropic nonlinear elliptic equations with variable exponent*, Topol. Methods Nonlinear Anal., 46 (2) 2015, 665–693.

[8] M. B. Benboubker, E. Nassouri, S. Ouaro and U. Traoré; *Renormalized solutions for p(x)-Laplacian equation with Neumann nonhomogeneous boundary condition*, Adv. Oper. Theory 5 (2020), 1480–497.

[9] M. B. Benboubker, S. Ouaro and U. Traoré; *Entropy solution for nonlinear nonhomogeneous Neumann problems involving the generalized $p(\cdot)$-Laplace Operator and measure data*, J. Nonl. Evol. Evol. Equ. Appl. **1** (2015), pp. 1–24.

[10] P. Benilan, J. L. Vazquez, L. Boccardo, M. Pierre and T. Gallouet; *An L^1 theory of existence and uniqueness of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci **22** (1995), pp. 214 – 273.

[11] D. Blanchard and F. Murat; *Renormalised solutions of nonlinear parabolic problems with L1 data: existence and uniqueness*, Proc. Roy. Soc. Edinburgh Sect. A **127**(6) (1997), pp. 1137 – 1152.

[12] D. Blanchard, F. Murat and H. Redwane; *Existence and uniqueness of a renormalized solution for a fairly general class of nonlinear parabolic problems*, J. Differential Equations, **177**(2) (2001), pp. 331 – 374 .

[13] H. Brezis; *Analyse Fonctionnelle: Théorie et Applications*. Paris Masson, 1983 .
[14] H. Brezis; *Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert*, North-Holland Mathematics Studies, 2011.
[15] J. K. Brook and R. V. Chacon; *Continuity and compactness of measures.* Adv. Math., **37** (1980), pp. 16 − 26.
[16] R. J. DiPerna and P. L. Lions; *On the Cauchy problem for Boltzmann equations: global existence and weak stability*, Ann. of Math.(2), **130**,(1989), 321-366.
[17] P. Halmos; *Measure Theory*, D. Van Nostrand, New York, 1950.
[18] I. Nyanquini, S. Ouaro and S. Soma; *Entropy solution to nonlinear multivalued elliptic problem with variable exponents and measure data*, Annals Univ. Craiova, Math, and Comp. Sci. Ser. **40** (2), (2013), 174 − 198.
[19] S. Ouaro, A. Ouedraogo and S. Soma; *Multivalued problem with Robin boundary condition involving diffuse measure data and variable exponent*, Adv. Nonlinear Anal. 3 (4), No. 1, (2014) pp. 209 − 235.
[20] M. Rao, S. Levine and Y. Chen; *Variable exponent, linear growth functional in image restoration*, SIAM J. Appl. Math. **66**, (2006), pp. 1383 − 1406.
[21] M. Sanchon and J. M. Urbano; *Entropy solutions for the p(x)-Laplace equation*, Trans. Amer. Math. Soc., **361** (2009), 6387 − 6405.
[22] P. Wittbold and A. Zimmermann; *Existence and uniqueness of renormalized solutions to nonlinear elliptic equations with variable exponent and L^1-data*, Nonlinear Analysis: Theory, Methods and Applications **72** (2010), pp. 2990–3008.
[23] V. V. Zhikov; *On some variational problems*, Russ. J. Math. Phys. **5** (1997), pp. 105 − 116.