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Topological degree methods for a Neumann problem governed by nonlinear elliptic equation

Adil Abbassi[1], Chakir Allalou [2] and Abderrazak Kassidi[3]

Abstract. In this paper, we will use the topological degree, introduced by Berkovits, to prove existence of weak solutions to a Neumann boundary value problems for the following nonlinear elliptic equation

$$-div\, a(x,u,\nabla u) = b(x)|u|^{p-2}u + \lambda H(x,u,\nabla u),$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$.

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1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 1)$ with Lipschitz boundary if $N \geq 2$, and let p be a real number such that $2 < p < \infty$. The purpose of this work is to investigate the existence

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The Author(s): [1,2,3] *Laboratory LMACS, FST of Beni Mellal, Sultan Moulay Slimane University, Morocco.*
[1] *e-mail: adil.abbassi@usms.ma*
[2] *e-mail: chakir.allalou@yahoo.fr*
[3] *e-mail: abderrazakassidi@gmail.com* .

of weak solutions to the Neumann boundary value problems given by

$$\begin{cases} -div\, a(x,u,\nabla u) = b(x)|u|^{p-2}u + \lambda H(x,u,\nabla u) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \eta} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.1}$$

where $\frac{\partial u}{\partial \eta}$ is the outer unit normal derivative, λ is a real parameter, $a(x,u,\nabla u)$ is a Carathéodory's functions defined from $\Omega \times \mathbb{R} \times \mathbb{R}^N$ into $\mathbb{R}$, (measurable with respect to x in Ω for every (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$ and continuous with respect to (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω). We mention the precise hypotheses on the functions a are listed in the assumptions $(A_1) - (A_3)$ below, allow us to choose

$$a(x,s,\xi) = |\xi|^{p-2}|\xi|, \text{ where } \quad p \geq 2,$$

we get the p-Laplace operator $\triangle_p u = div(|\nabla u|^{p-2}\nabla u)$, this is an important matter since, see for example [2, 26, 30]. At the same time, when taking

$$a(x,s,\xi) = \left(1 + |\xi|^2\right)^{(p-2)/2}|\xi|,$$

we obtain the mean curvature operator $div\left((1 + |\nabla u|^2)^{(p-2)/2}\nabla u\right)$.

The term $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory's functions verified a suitable growth condition (H_1) is required. Finally, we mention that the $b \in L^\infty(\Omega)$, $b(x) > 0$ a.e. in Ω.

When $a(x,u,\nabla u) = A(x,\nabla u)$ and $H(x,u,\nabla u) = f(x,\nabla u) + s(x)|u|^{p^*-1}u$, he et al have proved in [21] for all large λ, the problem has at least three nontrivial smooth solutions, two of constant sign (one positive, the other negative) and the third nodal. By combining variational methods based on critical point theory, with suitable truncation techniques and flow invariance arguments. Some important and interesting results can be found in, for example [1, 3, 14, 16, 20, 27]. Moreover, S. Liu [25] by using Morse theory, have established the existence of weak solutions in the case $a(x,u,\nabla u) = |\nabla u|^{p-2}u$ and $H(x,u,\nabla u) = f(x,\nabla u)$ with Dirichlet boundary conditions. Motivated by the results of the above cited papers, we shall attempt to treat problem (1.1) and extend the results of the literature [9, 21]. for the physical interpretation of the equation (1.1) and some intriguing applications in models fluid mechaincs (see [6, 7, 19]), nonlinear diffusion(see [29]) and nonlinear elasticity (see [5, 22]).

Throughout this paper, we assume that the function a satisfying the following assumptions, for a. e. in $x \in \Omega$ and all $\xi, \xi' \in \mathbb{R}^N$, $(\xi \neq \xi')$ and $s \in \mathbb{R}$.

(A_1) $\quad a(x,s,\xi)\,\xi \geq \alpha\ |\xi|^p\,,$

(A_2) $\quad |a(x,s,\xi)| \leq \beta\left(k(x) + |s|^{p-1} + |\xi|^{p-1}\right),$

(A_3) $\quad (a(x,s,\xi) - a(x,s,\xi'))(\xi - \xi') > 0,$

where α, β are some positive constants and $k(x)$ is a positive function in $L^{p'}(\Omega)$ (p' is the conjugate exponent of p).

Furthermore, the Carathéodory's functions H is satisfies only the growth condition, for all $t \in \mathbb{R}^N$, $s \in \mathbb{R}$ and a.e. $x \in \Omega$.

(H_1) $\quad |H(x,s,t)| \leq \varrho(e(x) + |s|^{p-1} + |t|^{p-1}),$

where ϱ is a positive constant, $e(x)$ is a positive function in $L^{p'}(\Omega)$.

In this paper, we propose a topological degree theory developed by Berkovits in [8] for operators of generalized monotone type. Roughly speaking, the class of operators for the extended

degree is essentially obtained by replacing the compact perturbation by a composition of operators of monotone type, based on the Leray-Schauder degree [24]. The topological degree theory is introduce the first time by Leray-Schauder [24] in their study of the nonlinear equations for compact perturbations of the identity in infinite-dimensional Banach spaces. Browder [10] constructed a topological degree for operators of class (S_+) in reflexive Banach spaces with the Galerkin method, see also [17, 18, 32, 31]. Among many examples, we refer the reader to do classical works[11, 12, 13, 15, 23, 33] for more details.

This paper is organised as follows. In section 2 we discuss some basic preliminaries and we recall some classes of mappings of generalized (S_+) type and the recent Berkovits degree. In the third section, we give some related lemmas to prepare for the proof of the main theorem. Section 4 is devoted to the main results of the paper.

2. Preliminaries

In order to discuss the problem (1.1), we recall some necessary definitions and basic results required for further developments.

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 1)$, p be a real number such that $2 < p < \infty$. We will work in the Sobolev space $W^{1,p}(\Omega)$ endowed with the norm

$$\|u\| = \Big(\int_\Omega |u|^p + |\nabla u|^p dx \Big)^{1/p}.$$

The norm in $L^p(\Omega)$ will be denoted by $\|u\|_p = \big(\int_\Omega |u|^p dx \big)^{1/p}$.

Now, we give some results and properties from the theory of topological degree. We start by defining some classes of mappings. In what follows, let X be a real separable reflexive Banach space with dual X^* and with continuous dual pairing $\langle \cdot, \cdot \rangle$ between X^* and X in this order, and given a nonempty subset Ω of X, let $\overline{\Omega}$ and $\partial\Omega$ denote the closure and the boundary of Ω in X, respectively. The symbol $\to$ $(\rightharpoonup)$ stands for strong (weak) convergence.

Definition 2.1. *Let Y be another real Banach space. A operator $F : \Omega \subset X \to Y$ is said to be*

(1) bounded, if it takes any bounded set into a bounded set.
(2) demicontinuous, if for any sequence $(u_n) \subset \Omega$, $u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$.
(3) compact, if it is continuous and the image of any bounded set is relatively compact.

Definition 2.2. *A mapping $F : \Omega \subset X \to X^*$ is said to be*

(1) of type (S_+), if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$ and $\limsup_{n\to\infty} \langle Fu_n, u_n - u\rangle \leq 0$, we have $u_n \to u$.
(2) quasimonotone, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, we have $\limsup_{n\to\infty} \langle Fu_n,\ u_n - u\rangle \geq 0$.

Definition 2.3. *Let $T : \Omega_1 \subset X \to X^*$ be a bounded operator such that $\Omega \subset \Omega_1$. For any operator $F : \Omega \subset X \to X$, we say that*

(1) F satisfies condition $(S_+)_T$, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $\limsup_{n\to\infty} \langle Fu_n,\ y_n - y\rangle \leq 0$, we have $u_n \to u$.

(2) *F has the property $(QM)_T$, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$, we have* $\limsup_{n\to\infty} \langle Fu_n, y - y_n\rangle \geq 0$.

Let $\mathcal{O}$ be the collection of all bounded open set in X. For any $\Omega \subset X$, we consider the following classes of operators:

$\mathcal{F}_1(\Omega) := \{F : \Omega \to X^* \setminus F$ is bounded, demicontinuous and satifies condition$(S_+)\}$,

$\mathcal{F}_{T,B}(\Omega) := \{F : \Omega \to X \setminus F$ is bounded, demicontinuous and satifies condition$(S_+)_T\}$,

$\mathcal{F}_T(\Omega) := \{F : \Omega \to X \setminus F$ is demicontinuous and satifies condition$(S_+)_T\}$,

$\mathcal{F}_B(X) := \{F \in \mathcal{F}_{T,B}(\overline{E}) \setminus E \in \mathcal{O},\ \mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{E}})\}$.

Throughout the paper $T \in \mathcal{F}_1(\overline{E})$ is called an essential inner map to F.

Lemma 2.1. ([8, Lemmas 2.2 and 2.4]). *Lets $T \in \mathcal{F}_1(\overline{E})$ be continuous and $S : D_S \subset X^* \to X$ be demicontinuous such that $T(\overline{E}) \subset D_s$, where E is a bounded open set in a real reflexive Banach space X. Then the following statements are true :*

(1) If S is quasimonotone, then $I + SoT \in \mathcal{F}_T(\overline{E})$, where I denotes the identity operator.
(2) If S is of class (S_+), then $SoT \in \mathcal{F}_T(\overline{E})$.

Definition 2.4. *Suppose that E is bounded open subset of a real reflexive Banach space X, $T \in \mathcal{F}_1(\overline{E})$ be continuous and let $F, S \in \mathcal{F}_T(\overline{E})$. The affine homotopy $\Lambda : [0,1] \times \overline{E} \to X$ defined by*

$$\Lambda(t,\ u) := (1-t)Fu + tSu \qquad for \qquad (t,\ u) \in [0,1] \times \overline{E}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Remark 2.1. ([8]). The above affine homotopy satisfies condition $(S_+)_T$.

Now, we introduce the Berkovits topological degree for the class $\mathcal{F}_B(X)$ for more details see [8].

Theorem 2.1. There exists a unique degree function

$$d : \left\{(F, E, h) \setminus E \in \mathcal{O},\ T \in \mathcal{F}_1(\overline{E}),\ F \in \mathcal{F}_{T,B}(\overline{E}),\ h \notin F(\partial E)\right\} \longrightarrow \mathbb{Z}$$

that satisfies the following properties*:*

(1) (Normalization*)* For any $h \in E$, we have $d(I, E, h) = 1$.
(2) (Additivity*)* Let $F \in \mathcal{F}_{T,B}(\overline{E})$. If E_1 and E_2 are two disjoint open subsets of E such that $h \notin F(\overline{E} \setminus (E_1 \cup E_2))$ then we have

$$d(F, E, h) = d(F, E_1, h) + d(F, E_2, h)\ .$$

(3) (Homotopy invariance*)* If $\Lambda : [0,1] \times \overline{E} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and h: $[0,1] \to X$ is a continuous path in X such that $h(t) \notin \Lambda(t, \partial E)$ for all $t \in [0,1]$, then the value of $d(\Lambda(t, \cdot), E, h(t))$ is constant for all $t \in [0,1]$.
(4) (Existence*)* if $d(F, E, h) \neq 0$, then the equation $Fu = h$ has a solution in E.

3. Notions of solutions and technical lemmas

In this section, we give the definition of a weak solution for Problem (1.1), and we introduce some useful technical lemmas to prove existence results.

Definition 3.1. *A measurable function u is said to be a weak solution of (1.1), if $u \in W^{1,p}(\Omega)$ such that*

$$\int_\Omega a(x,u,\nabla u)\nabla v dx = \int_\Omega b(x)|u|^{p-2}uvdx + \lambda \int_\Omega H(x,u,\nabla u)vdx, \quad \forall v \in W^{1,p}(\Omega).$$

Lemma 3.1. ([4]). *Let $g \in L^r(\Omega)$ and $g_n \subset L^r(\Omega)$ such that $\|g_n\|_r \leq C$, $1 < r < \infty$, If $g_n(x) \to g(x)$ a.e. in Ω then $g_n \rightharpoonup g$ weakly in $L^r(\Omega)$.*

Let us consider the nonlinear operator F from $W^{1,p}(\Omega)$ into its dual of the form

$$Fu = -\operatorname{div} a(x,u,\nabla u),$$

then

$$\langle Fu, v\rangle = \int_\Omega a(x,u,\nabla u)\nabla v\, dx,$$

for all v in $W^{1,p}(\Omega)$.

Lemma 3.2. *Assume that $(A_1)-(A_3)$ hold, then F is bounded, coercive, continuous and of type (S_+).*

Proof. Firstly, we prove that the operator F is bounded.

Let $u, v \in W^{1,p}(\Omega)$, by using the Hölder's inequality we have

$$\begin{aligned}\langle Fu, v\rangle &= \int_\Omega a(x,u,\nabla u)\nabla v\, dx \\ &\leq C\Big(\int_\Omega |a(x,u,\nabla u)|^{p'}dx\Big)^{1/p'}\Big(\int_\Omega |\nabla v|^p dx\Big)^{1/p}.\end{aligned}$$

On the other hand, from (A_2) we can easily show that $\|a(x,u,\nabla u)\|_{p'}$ is bounded for all u in $W^{1,p}(\Omega)$. Hence

$$\begin{aligned}\langle Fu, v\rangle &\leq C'\Big(\int_\Omega |\nabla v|^p dx\Big)^{1/p} \\ &\leq const\, \|v\|,\end{aligned}$$

as results the operator F is bounded.

Secondly, we prove that F is coercive. Let $v \in W^{1,p}(\Omega)$, we get from (A_1)

$$\begin{aligned}\frac{\langle Fv, v\rangle}{\|v\|} &= \frac{\int_\Omega a(x,v,\nabla v)\nabla v dx}{\|v\|} \\ &\geq \alpha\, \frac{\int_\Omega |\nabla v|^p dx + \int_\Omega |v|^p dx - \int_\Omega |v|^p dx}{\|v\|} \\ &\geq \alpha \|v\|^{p-1} - \alpha \frac{\|v\|_p}{\|v\|} \\ &\geq \alpha \|v\|^{p-1} - C. \quad (\text{ Due to } W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega))\end{aligned}$$

Therefore,

$$\frac{\langle Fv, v\rangle}{\|v\|} \to \infty \quad \text{as} \quad \|v\| \to \infty.$$

Hence F is coercive.

Now, we show that F is continuous, let $u_n \to u$ in $W^{1,p}(\Omega)$. Then $\nabla u_n \to \nabla u$ in $\left(L^p(\Omega)\right)^N$. Hence there exist a subsequence (u_k) of (u_n) and measurable functions h in $L^p(\Omega)$ and g in $(L^p(\Omega))^N$ such that

$$\begin{aligned} u_k(x) &\to u(x) \quad \text{and} \quad \nabla u_k(x) \to \nabla u(x), \\ |u_k(x)| &\le h(x) \quad \text{and} \quad |\nabla u_k(x)| \le |g(x)|, \end{aligned}$$

for $a.e. x \in \Omega$ and all $k \in \mathbb{N}$. Since a satisfies the Carathéodory condition, we obtain

$$a(x, u_k, \nabla u_k) \to a(x, u, \nabla u) \qquad \text{a.e. } x \in \Omega. \tag{3.1}$$

According to (A_2), we have

$$|a(x, u_k, \nabla u_k)| \le \beta(k(x) + |h(x)|^{p-1} + |g(x)|^{p-1}),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$. Since

$$\left(k(x) + |h(x)|^{p-1} + |g(x)|^{p-1}\right) \in L^{p'}(\Omega),$$

and by using (3.1) we have

$$\int_\Omega |a(x, u_k, \nabla u_k) - a(x, u, \nabla u)|^{p'} dx \longrightarrow 0.$$

The dominated convergence theorem imply that

$$a(x, u_k, \nabla u_k) \to a(x, u, \nabla u) \qquad \text{in} \quad \left(L^{p'}(\Omega)\right)^N.$$

Thus the entire sequence $a(x, u_n, \nabla u_n)$ converges to $a(x, u, \nabla u)$ in $\left(L^{p'}(\Omega)\right)^N$. Therefore, for all $v \in W^{1,p}(\Omega)$ we have $\langle Fu_n, v\rangle \to \langle Fu, v\rangle$, which implies that the operator F is continuous.

It remains to prove that the operator F is of type (S_+).

Let $(u_n)_n$ be a sequence in $W^{1,p}(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in } W^{1,p}(\Omega) \\ \limsup\limits_{n\to\infty} \langle Fu_n, u_n - u\rangle \le 0. \end{cases} \tag{3.2}$$

We will show that $u_n \to u$ in $W^{1,p}(\Omega)$.

The first part of (3.2) yields $M_1 > 0$ such that

$$\|\nabla u_n\|_p \le M_1 \qquad \text{for all } n \ge 1.$$

In view of (A_3) and (3.2), we get

$$\lim_{n\to\infty} \langle Fu_n,\ u_n - u\rangle = \lim_{n\to\infty} \langle Fu_n - Fu,\ u_n - u\rangle = 0. \tag{3.3}$$

Hence

$$D_n \to 0 \quad \text{in } L^1(\Omega),$$

where $D_n = \left[a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)\right](\nabla u_n - \nabla u)$.

By using $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$, we can find a subsequence still denoted by u_n such that

$$\begin{cases} u_n \to u & \text{a.e. in } \Omega, \\ D_n \to 0 & \text{a.e. in } \Omega. \end{cases}$$

Then, there exists a subset B of Ω, of zero measure, such that for $x \in \Omega \backslash B$, $|u(x)| < \infty$, $|\nabla u(x)| < \infty$, $|k(x)| < \infty$, $u_n(x) \to u(x)$, $D_n(x) \to 0$.

If we put $\xi_n = \nabla u_n$, $\xi = \nabla u$, we have

$$\begin{aligned} D_n(x) &= \big[a(x,u_n,\xi_n) - a(x,u_n,\xi)\big].(\xi_n - \xi) \\ &= a(x,u_n,\xi_n)\xi_n + a(x,u_n,\xi)\xi - a(x,u_n,\xi_n)\xi - a(x,u_n,\xi)\xi_n \\ &\geq \alpha|\xi_n|^p + \alpha|\xi|^p - \beta\big(k(x) + |u_n|^{p-1} + |\xi_n|^{p-1}\big)|\xi| \\ &\quad - \beta\big(k(x) + |u_n|^{p-1} + |\xi|^{p-1}\big)|\xi_n| \\ &\geq \alpha|\xi_n|^p - C_x[1 + |\xi_n|^{p-1} + |\xi_n|], \end{aligned}$$

where C_x is a constant which depends on x, but does not depend on n. Since $u_n(x) \to u(x)$ we have $|u_n(x)| \leq M_x$, where M_x is some positive constant. Then by a standard argument $|\xi_n|$ is bounded uniformly with respect to n, we deduce that

$$D_n(x) \geq |\xi_n|^p \Big(\alpha - \frac{C_x}{|\xi_n|^p} - \frac{C_x}{|\xi_n|} - \frac{C_x}{|\xi_n|^{p-1}}\Big).$$

If $|\xi_n| \to \infty$ (for a subsequence), then $D_n(x) \to \infty$ which gives a contradiction.

Let now ξ^* be a cluster point of ξ_n. We have $|\xi^*| < \infty$ and by using the continuity of a we have

$$\big[a(x,u_n,\xi^*) - a(x,u_n,\xi)\big](\xi^* - \xi) = 0.$$

According to (A_3), we get $\xi^* = \xi$, which implies that

$$\nabla u_n(x) \longrightarrow \nabla u(x) \qquad \text{a.e. in } \Omega.$$

On the other hand, since the sequence $\big(a(x,u_n,\nabla u_n)\big)$ is bounded in $\big(L^{p'}(\Omega)\big)^N$, and $a(x,u_n,\nabla u_n)$ converge to $a(x,u,\nabla u)$ a.e. in Ω, in view of Lemma 3.1, we have

$$a(x,u_n,\nabla u_n) \rightharpoonup a(x,u,\nabla u) \quad \text{in } (L^{p'}(\Omega))^N \quad \text{a.e. in } \Omega. \tag{3.4}$$

We set $\overline{y}_n = a(x,u_n,\nabla u_n)\nabla u_n$ and $\overline{y} = a(x,u,\nabla u)\nabla u$. We can write

$$\overline{y}_n \to \overline{y} \quad \text{in } L^1(\Omega).$$

From (A_1), we have

$$\alpha\,|\nabla u_n|^p \leq a(x,u_n,\nabla u_n)\nabla u_n\,.$$

Let $z_n = |\nabla u_n|^p$, $z = |\nabla u|^p$, $y_n = \dfrac{\overline{y}_n}{\alpha}$, and $y = \dfrac{\overline{y}}{\alpha}$.

Thanks to Fatou's lemma,

$$\int_\Omega 2\,y\,dx \leq \liminf_{n\to\infty} \int_\Omega y + y_n - |z_n - z|dx,$$

i.e., $0 \leq -\limsup_{n\to\infty} \int_\Omega |z_n - z|\, dx$. Then

$$0 \leq \liminf_{n\to\infty} \int_\Omega |z_n - z|\, dx \leq \limsup_{n\to} \int_\Omega |z_n - z|\, dx \leq 0,$$

this implies

$$\nabla u_n \longrightarrow \nabla u \quad \text{in} \quad \big(L^p(\Omega)\big)^N. \tag{3.5}$$

From the compact embedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$, we have from the assumptions that $\|u_n\|_p \to \|u\|_p$ as $n \to \infty$, whence $\|u_n\| \longrightarrow \|u\|$ (see (3.5)). Since $W^{1,p}(\Omega)$ satisfies the Kadec-Klee property [28, Remark 2.47(a),(c)], the facts that $u_n \rightharpoonup u$ and $\|u_n\| \longrightarrow \|u\|$ ensure that $u_n \longrightarrow u$ in $W^{1,p}(\Omega)$ as $n \to \infty$, which completes the proof.

Lemma 3.3. *Suppose that the hypothese (H_1) holds. Then the operator $S : W^{1,p}(\Omega) \to \big(W^{1,p}(\Omega)\big)^*$ defined by*

$$\langle Su,\ v\rangle = -\int_\Omega \big(b(x)|u|^{p-2}u + \lambda H(x,u,\nabla u)\big)v\, dx, \qquad \forall u,\ v \in W^{1,p}(\Omega)$$

is compact.

Proof. We split the proof into three steps.

Step 1

Let $\phi : W^{1,p}(\Omega) \to L^{p'}(\Omega)$ be the operator setting by

$$\phi\, u(x) := -b(x)|u(x)|^{p-2}u(x) \quad \text{for} \quad u \in W^{1,p}(\Omega) \quad \text{and} \quad x \in \Omega.$$

It is obvious that ϕ is continuous. Next we show that ϕ is bounded.

For every $u \in W^{1,p}(\Omega)$, by using the compact embedding $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$, we have

$$\begin{aligned}
\|\phi\, u\|_{p'}^{p'} &= \int_\Omega |-b(x)|u|^{p-2}u|^{p'} dx \\
&\leq \|b^{p'}\|_\infty \int_\Omega |u|^{(p-1)p'} dx \\
&\leq \|b^{p'}\|_\infty \int_\Omega |u|^p dx \\
&\leq C\, \|u\|^p.
\end{aligned}$$

This implies that ϕ is bounded on $W^{1,p}(\Omega)$.

Step 2

We show that the operator ψ defined from $W^{1,p}(\Omega)$ into $L^{p'}(\Omega)$ by

$$\psi u(x) := -\lambda H(x,u,\nabla u) \quad \text{for} \quad u \in W^{1,p}(\Omega) \quad \text{and} \quad x \in \Omega$$

is bounded and continuous. Let $u \in W^{1,p}(\Omega)$, by using the growth condition (H_1) we obtain

$$\begin{aligned}
\|\psi u\|_{p'}^{p'} &\leq \int_\Omega |\lambda H(x,u,\nabla u)|^{p'} dx \\
&\leq (\varrho\lambda)^{p'} \int_\Omega \left(|e(x)|^{p'} + |u|^{(p-1)p'} + |\nabla u|^{(p-1)p'}\right) dx \\
&\leq (\varrho\lambda)^{p'} \int_\Omega |e(x)|^{p'} dx + (\varrho\lambda)^{p'} \int_\Omega |u|^p + |\nabla u|^p \, dx \\
&\leq (\varrho\lambda)^{p'} \|e\|_{p'}^{p'} + (\varrho\lambda)^{p'} \|u\|^p \\
&\leq C_m \|u\|^p,
\end{aligned} \tag{3.6}$$

where $C_m = \max\left((\varrho\lambda)^{p'}\|e\|_{p'}^{p'}, (\varrho\lambda)^{p'}\right)$. (Due to $e(x)$ is a positive function in $L^{p'}(\Omega)$)

Therefore ψ is bounded on $W^{1,p}(\Omega)$.

Next, we show that ψ is continuous, let $u_n \to u$ in $W^{1,p}(\Omega)$, then $u_n \to u$ in $L^p(\Omega)$ and $\nabla u_n \to \nabla u$ in $\left(L^p(\Omega)\right)^N$. Thus there exist a subsequence still denoted by (u_n) and measurable functions φ in $L^p(\Omega)$ and σ in $\left(L^p(\Omega)\right)^N$ such that

$$\begin{aligned}
u_n(x) \to u(x) \quad &\text{and} \quad \nabla u_n(x) \to \nabla u(x), \\
|u_n(x)| \leq \varphi(x) \quad &\text{and} \quad |\nabla u_n(x)| \leq |\sigma(x)|,
\end{aligned}$$

for a.e. $x \in \Omega$ and all $n \in \mathbb{N}$. Since H satisfies the Carathéodory condition, we obtain

$$H(x, u_n(x), \nabla u_n(x)) \to H(x, u(x), \nabla u(x)) \quad \text{a.e. } x \in \Omega. \tag{3.7}$$

Thanks to (H_1) we obtain

$$|H(x, u_n(x), \nabla u_n(x))| \leq \varrho\left(e(x) + |\varphi(x)|^{p-1} + |\sigma(x)|^{p-1}\right)$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.

Since

$$e(x) + |\varphi(x)|^{p-1} + |\sigma(x)|^{p-1} \in L^{p'}(\Omega),$$

and from (3.7), we get

$$\int_\Omega |H(x, u_k(x), \nabla u_k(x)) - H(x, u(x), \nabla u(x))|^{p'} dx \longrightarrow 0,$$

by using the dominated convergence theorem we have

$$\psi u_k \to \psi u \quad \text{in} \quad L^{p'}(\Omega).$$

Thus the entire sequence (ψu_n) converges to ψu in $L^{p'}(\Omega)$ and then ψ is continuous.

Step 3

As the embedding $I : W^{1,p}(\Omega) \to L^p(\Omega)$ is compact, it is known that the adjoint operator $I^* : L^{p'}(\Omega) \to \left(W^{1,p}(\Omega)\right)^*$ is also compact. Therefore, the compositions $I^* o \phi$ and $I^* o \psi$ from $W^{1,p}(\Omega)$ into $\left(W^{1,p}(\Omega)\right)^*$ are compact. We deduce that $S = I^* o \phi + I^* o \psi$ is compact. which completes the present proof.

4. Main results

In this section, we study the existence of weak solutions for the nonlinear problem (1.1) based on the recent Berkovits topological degree introduce in section 2.

Theorem 4.1. *Assume that the assumptions* $(A_1)-(A_2)$ *and* (H_1) *hold. Then, the problem* (1.1) *has a weak solution* u *in* $W^{1,p}(\Omega)$.

Proof. Let F, S be two operators from $W^{1,p}(\Omega)$ into its dual as defined in Lemmas 3.2 and 3.3 respectively. Then $u \in W^{1,p}(\Omega)$ is a weak solution of the problem (1.1) if and only if

$$Fu = -Su \tag{4.1}$$

According to Lemma 3.3 the operator S is bounded, continuous and quasimonotone. On the other hand, in light of the properties of the operator F given in Lemma 3.2 and since the operator F is strictly monotone. Moreover, note by using the Minty-Browder Theorem (see [33], Theorem 26 A), the inverse operator $G := F^{-1}:\big(W^{1,p}(\Omega)\big)^* \to W^{1,p}(\Omega)$ is bounded, continuous and satisfies condition (S_+).

Hence, equation (4.1) is equivalent to the abstract Hammerstein equation

$$u = Gv \quad \text{and} \quad v + SoGv = 0 \tag{4.2}$$

To solve equations (4.2), we will using the Berkovits topological degree introduced in section 2. For that, we first proove that the set

$$B := \big\{\, v \in \big(W^{1,p}(\Omega)\big)^* \setminus \quad v + tSoGv = 0 \quad \text{for some} \quad t \in [0,1] \big\}$$

is bounded. Indeed, let $v \in B$ and take $u := Gv$.

Thanks to (A_1), (H_1), (3.6), the Young's inequality and the compact embedding $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$, we have

$$\begin{aligned}
\|Gv\|^p &= \int_\Omega |u|^p\,dx + \int_\Omega |\nabla u|^p\,dx \\
&\le \int_\Omega |u|^p\,dx + \frac{1}{\alpha}\int_\Omega a(x,u,\nabla u)\nabla u dx \\
&\le \int_\Omega |u|^p\,dx + \frac{1}{\alpha}\langle Fu,u\rangle = \int_\Omega |u|^p\,dx + \frac{1}{\alpha}\langle v, Gv\rangle \\
&\le \int_\Omega |u|^p\,dx + \frac{t}{\alpha}|\langle S\circ Gv, Gv\rangle| \\
&\le \int_\Omega |u|^p\,dx + \frac{t}{\alpha}\int_\Omega b(x)|u|^p dx + \frac{t}{\alpha}\int_\Omega |H(x,u,\nabla u)|u dx \\
&\le \int_\Omega |u|^p\,dx + \frac{t\|b\|_\infty}{\alpha}\int_\Omega |u|^p dx + \frac{t}{\alpha}C_{p'}\int_\Omega |H(x,u,\nabla u)|^{p'} dx + \frac{t}{\alpha}C_p\int_\Omega |u|^p dx \\
&\le C_1\int_\Omega |u|^p dx + C_2\|u\|^{p/p'} \\
&\le Cst\big(\|Gv\|^p + \|Gv\|^{p-1}\big)
\end{aligned}$$

This implies that $\{Gv \setminus v \in B\}$ is bounded.

Since the operator S is bounded, it is obvious from (4.2) that the set B is bounded in $\left(W^{1,p}(\Omega)\right)^*$. Therefore, we can choose a positive constant R such that

$$\|v\|_{\left(W^{1,p}(\Omega)\right)^*} < R \quad \text{for all} \quad v \in B.$$

It follows that

$$v + tSoGv \neq 0 \quad \text{for all} \quad v \in \partial B_R(0) \quad \text{and all} \quad t \in [0,1].$$

By Lemma 2.1 we get

$$I + SoG \in \mathcal{F}_T(\overline{B_R(0)}) \quad \text{and} \quad I = FoG \in \mathcal{F}_T(\overline{B_R(0)}).$$

Since the operators I, S and G are bounded, $I + SoG$ is also bounded. We conclude that

$$I + SoG \in \mathcal{F}_{T,B}(\overline{B_R(0)}) \quad \text{and} \quad I \in \mathcal{F}_{T,B}(\overline{B_R(0)}).$$

Consider an affine homotopy $\Lambda : [0,1] \times \overline{B_R(0)} \to W^{-1,p'}(\Omega)$ given by

$$\Lambda(t,v) := v + tSoGv \quad \text{for} \quad (t,v) \in [0,1] \times \overline{B_R(0)}.$$

Applying Theorem 2.1, we have

$$d(I + SoG, B_R(0), 0) = d(I, B_R(0), 0) = 1$$

then, there exists a point $v \in B_R(0)$ such that

$$v + SoGv = 0.$$

which says that $u = Gv$ is a weak solution of (1.1). This completes the proof.

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