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Controlled Continuous g-Frames in Hilbert C^*-Modules

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ABSTRACT. Frame Theory has a great revolution in recent years. This Theory have been extended from Hilbert spaces to Hilbert C^*-modules. The purpose of this paper is the introduction and the study of the concept of Controlled Continuous g-Frames in Hilbert C^*-Modules. Also we give some properties.

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1. Introduction and preliminaries

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaeffer [8] in 1952 to study some deep problems in nonharmonic Fourier series. After the fundamental paper [6] by Daubechies, Grossman and Meyer, frames theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames [12].

Hilbert C^*-module arose as generalization of the Hilbert space notion. The basic idea was to consider modules over C^*-algebras instead of linear spaces and to allow the inner product to take values in the C^*-algebras [14].

Theory of frames have been extended from Hilbert spaces to Hilbert C^*-modules [10], [19], [20], [21], [22], [23].

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Continuous frames defined by Ali, Antoine and Gazeau [1]. Gabardo and Han in [11] called these kinds frames or frames associated with measurable spaces. For more details, the reader can refer to [17], [18] and [16].

The Extended results of this more general framework are not a routine generalization, because there are essential differences between Hilbert C^*-module and Hilbert Space. For Exemple, any closed subspace in a Hilbert space has an orthogonal complement, but this fails in Hilbert C^*-module. Also, there is no explicit analogue of the Riesz representation theorem of continuous fonctionals in Hilbert C^*-module.

The aim of this paper is to extend results of Yahya Alizadeh, Mohammad Reza Abdollahpour [2], given in the Hilbert space to Hilbert C^*-module.

For this end, we introduce and we study the concept of Controlled Continuous g-Frames in Hilbert C^*-module. Also, some properties are given.

In the following we briefly recall the definitions and basic properties of C^*-algebra and Hilbert $\mathcal{A}$-modules. Our references for C^*-algebras are [7, 5]. For a C^*-algebra $\mathcal{A}$, if $a \in \mathcal{A}$ is positive we write $a \geq 0$ and $\mathcal{A}^+$ denotes the set of positive elements of $\mathcal{A}$.

Definition 1.1. *([5]). Let $\mathcal{A}$ be a unital C^*-algebra and U be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and U are compatible. U is a pre-Hilbert $\mathcal{A}$-module if U is equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle_{\mathcal{A}} : U \times U \to \mathcal{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words,*

(i) $\langle x, x \rangle_{\mathcal{A}} \geq 0$, for all $x \in U$, and $\langle x, x \rangle_{\mathcal{A}} = 0$ if and only if $x = 0$.
(ii) $\langle ax + y, z \rangle_{\mathcal{A}} = a\langle x, z \rangle_{\mathcal{A}} + \langle y, z \rangle_{\mathcal{A}}$, for all $a \in \mathcal{A}$ and $x, y, z \in U$.
(iii) $\langle x, y \rangle_{\mathcal{A}} = \langle y, x \rangle^*_{\mathcal{A}}$, for all $x, y \in U$.

For $x \in U$, we define $||x|| = ||\langle x, x \rangle_{\mathcal{A}}||^{\frac{1}{2}}$. If U is complete with $||.||$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$.
For every a in C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on U is defined by $|x| = \langle x, x \rangle^{\frac{1}{2}}_{\mathcal{A}}$, for all $x \in U$.

Let U and V be two Hilbert $\mathcal{A}$-modules, a map $T : U \to V$ is said to be adjointable if there exists a map $T^* : V \to U$ such that $\langle Tx, y \rangle_{\mathcal{A}} = \langle x, T^*y \rangle_{\mathcal{A}}$ for all $x \in U$ and $y \in V$.

We reserve the notation $End^*_{\mathcal{A}}(U, V)$ for the set of all adjointable operators from U to V and $End^*_{\mathcal{A}}(U, U)$ is abbreviated to $End^*_{\mathcal{A}}(U)$.

The following lemmas will be used to prove our mains results.

Lemma 1.1. *([15]). Let U be an Hilbert $\mathcal{A}$-module. If $T \in End^*_{\mathcal{A}}(U)$, then*

$$\langle Tx, Tx \rangle_{\mathcal{A}} \leq \|T\|^2 \langle x, x \rangle_{\mathcal{A}}, \qquad x \in U.$$

Lemma 1.2. *([3]). Let U and V be two Hilbert $\mathcal{A}$-modules and $T \in End^*_{\mathcal{A}}(U, V)$. Then the following statements are equivalent:*

(i) T is surjective.
(ii) T^* is bounded below with respect to norm, i.e., there is $m > 0$ such that $\|T^*x\| \geq m\|x\|$, for all $x \in V$.
(iii) T^* is bounded below with respect to the inner product, i.e., there is $m' > 0$ such that $\langle T^*x, T^*x \rangle_{\mathcal{A}} \geq m' \langle x, x \rangle_{\mathcal{A}}$, for all $x \in V$.

Lemma 1.3. *([24]). Let (Ω, μ) be a measure space, X and Y are two Banach spaces, $\lambda : X \longrightarrow Y$ be a bounded linear operator and $f : \Omega \longrightarrow X$ measurable function; then,*

$$\lambda\left(\int_{\Omega} f d\mu\right) = \int_{\Omega} (\lambda f) d\mu.$$

2. Controlled continuous g-Frames in Hilbert C^*-Modules

Let X be a Banach space, (Ω, μ) a measure space, and $f : \Omega \to X$ be a measurable function. Integral of Banach-valued function f has been defined by Bochner and others. Most properties of this integral are similar to those of the integral of real-valued functions (see [9, 24]). Since every C^*-algebra and Hilbert C^*-module are Banach space, we can use this integral and its properties.

Let (Ω, μ) be a measure space, U and V be two Hilbert C^*-modules over a unital C^*-algebra and $\{V_w\}_{w\in\Omega}$ is a family of submodules of V. $End^*_{\mathcal{A}}(U, V_w)$ is the collection of all adjointable $\mathcal{A}$-linear maps from U into V_w.

We define, following:

$$l^2(\Omega, \{V_w\}_{\omega\in\Omega}) = \left\{x = \{x_w\}_{w\in\Omega} : x_w \in V_w, \left\|\int_{\Omega} |x_w|^2 d\mu(w)\right\| < \infty\right\}.$$

For any $x = \{x_w\}_{w\in\Omega}$ and $y = \{y_w\}_{w\in\Omega}$, the $\mathcal{A}$-valued inner product is defined by $\langle x, y\rangle = \int_{\Omega} \langle x_w, y_w\rangle_{\mathcal{A}} d\mu(w)$ and the norm is defined by $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. In this case the $l^2(\Omega, \{V_w\}_{\omega\in\Omega})$ is an Hilbert C^*-module (see [14]).

Let $GL^+(U)$ be the set of all positive bounded linear invertible operators on U with bounded inverse.

Definition 2.1. ([13]). *We call $\Lambda := \{\Lambda_w \in End^*_{\mathcal{A}}(U, V_w) : w \in \Omega\}$ a continuous g-frame for Hilbert C^*-module U with respect to $\{V_w : w \in \Omega\}$ if:*
(a) for any $x \in U$, the mapping $\tilde{x} : \Omega \to V_w$ defined by $\tilde{x}(w) = \Lambda_w x$ is measurable; (b) there is a pair of constants $0 < A, B$ such that, for any $x \in U$,

$$A\langle x, x\rangle_{\mathcal{A}} \leq \int_{\Omega} \langle \Lambda_w x, \Lambda_w x\rangle_{\mathcal{A}} d\mu(w) \leq B\langle x, x\rangle_{\mathcal{A}}, \quad x \in U. \tag{2.1}$$

The constants A and B are called continuous g-frame bounds. If $A = B$ we call this continuous g-frame a continuous tight g-frame, and if $A = B = 1$ it is called a continuous Parseval g-frame. If only the right-hand inequality of (2.1) is satisfied, we call $\Lambda := \{\Lambda_w w \in \Omega\}$ the continuous g-Bessel for U with respect to $\{V_w : w \in \Omega\}$ with Bessel bound B.
The continuous g-frame operator S on U is defined by :

$$Sx = \int_{\Omega} \Lambda^*_{\omega} \Lambda_{\omega} x d\mu(\omega), \quad x \in U.$$

The frame operator S is a bounded, positive, selfadjoint, and invertible (see [13]).

Theorem 2.1. ([13]). *Let $\Lambda = \{\Lambda_w \in End^*_{\mathcal{A}}(U, V_w) : w \in \Omega\}$. Λ is a continuous g-frame for U with respect to $\{V_w : w \in \Omega\}$ if and only if there exist a constants $0 < A, B$ such that for any $x \in U$:*

$$A\|x\|^2 \leq \left\| \int_\Omega \langle \Lambda_w x, \Lambda_w x \rangle_{\mathcal{A}} d\mu(w) \right\| \leq B\|x\|^2.$$

3. Main Results

Definition 3.1. *Let $\Lambda = \{\Lambda_w\}_{w\in\Omega}$ a family in $End^*_{\mathcal{A}}(U, V_w)$ for all $\omega \in \Omega$, and $C, C^{'} \in GL^+(U)$. We say that the family Λ is a $(C - C^{'})$-controlled continuous g-frame for Hilbert C^*-module U with respect to $\{V_w : w \in \Omega\}$ if it is a continuous g-Bessel family and there is a pair of constants $0 < A, B$ such that, for any $x \in U$,*

$$A\langle x, x\rangle_{\mathcal{A}} \leq \int_\Omega \langle \Lambda_w C x, \Lambda_w C^{'} x\rangle_{\mathcal{A}} d\mu(w) \leq B\langle x, x\rangle_{\mathcal{A}}, \quad x \in U. \tag{3.1}$$

A and B are called the $(C - C^{'})$-controlled continuous g-frames bounds.
If $C^{'} = I$ then we call $\Lambda = \{\Lambda_w\}_{w\in\Omega}$ a C-controlled continuous g-frames for U with respect to $\{V_w : w \in \Omega\}$.

Example 3.1. *Let $U = \left\{ A = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \end{pmatrix} / a, b \in \mathbb{C} \right\}$,*
and $\mathcal{A} = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} / x, y \in \mathbb{C} \right\}$ which is a C^*-algebra.
We define the inner product :

$$\begin{array}{ccc} U \times U & \to & \mathcal{A} \\ (A, B) & \mapsto & A(\overline{B})^t \end{array}$$

This inner product makes U a C^*-module on $\mathcal{A}$.
Now, we consider a measure space $(\Omega = [0, 1], d\mu)$, whose $d\mu$ is a Lebesgue measure restraint on the interval $[0, 1]$.
Let $\{\Lambda_w\}_{w\in\Omega}$ be a sequence of operators defined by:

$$\Lambda_w(A) = \begin{pmatrix} wa & 0 & 0 \\ 0 & 0 & w^\alpha b \end{pmatrix},$$

whose $A \in U$ and $\alpha > 1$.
These operators are continuous because they are bounded:

$$\|\Lambda_w\| \leq \max(w, w^\alpha).$$

We assume that V_w are equal to $\mathcal{A}$ for all $\omega \in \Omega$.
We have :

$$\int_\Omega \; < \; \Lambda_w(A), \Lambda_w(A) >_{\mathcal{A}} d\mu(w)$$

$$= \begin{pmatrix} \int_0^1 w^2 d\mu(w)\,|a|^2 & 0 \\ 0 & \int_0^1 w^{2\alpha} d\mu(w)\,|b|^2 \end{pmatrix}$$
$$= \begin{pmatrix} 1/3\,|a|^2 & 0 \\ 0 & (\frac{1}{2\alpha+1})\,|b|^2 \end{pmatrix}.$$

It's clear that,

$$\int_\Omega < \Lambda_w(A), \Lambda_w(A) >_{\mathcal{A}} d\mu(w) \leq \frac{1}{3}\,\|A\|^2,$$

because

$$\|A\|^2 =< A, A >_{\mathcal{A}},$$

which shows that $\{\Lambda_\omega\}_{\omega\in\Omega}$ *is a g-Bessel family.*
We define an operator C by :

$$C : U \to U$$
$$C(A) = \begin{pmatrix} \frac{a}{2} & 0 & 0 \\ 0 & 0 & \sqrt{(2\alpha+1)}\frac{b}{2} \end{pmatrix}.$$

We define an operator $C^{'}$ *by :*

$$C^{'} : U \to U$$
$$C^{'}(A) = \begin{pmatrix} 2a & 0 & 0 \\ 0 & 0 & \sqrt{(2\alpha+1)}2b \end{pmatrix}.$$

It is easy to show that C and $C^{'}$ *are a continuous and invertible operator.*
Furthermore,

$$\int_\Omega < \Lambda_w C(A), \Lambda_w C^{'}(A) >_{\mathcal{A}} d\mu(w)$$
$$= \begin{pmatrix} \int_0^1 w^2 d\mu(w)\,|a|^2 & 0 \\ 0 & \int_0^1 (2\alpha+1) w^{2\alpha} d\mu(w)\,|b|^2 \end{pmatrix}$$
$$= \begin{pmatrix} 1/3\,|a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix}.$$

Finally, we have:

$$\frac{1}{3}\,\|A\|^2 \leq \int_\Omega < \Lambda_w C(A), \Lambda_w C^{'}(A) >_{\mathcal{A}} d\mu(w) \leq \|A\|^2.$$

Which shows that $\{\Lambda_\omega\}_{\omega\in\Omega}$ *is a* $(C - C^{'})$*-controlled continuous g-frame for U with respect to* $\{V_\omega, \omega \in \Omega\}$.

Let $\Lambda = \{\Lambda_w \in End^*_{\mathcal{A}}(U, V_w) : w \in \Omega\}$ be a $(C - C^{'})$-controlled continuous g-frames for U with respect to $\{V_w : w \in \Omega\}$ with bounds A and B.The bounded linear operator:

$$T_{CC'} : l^2(\Omega, \{V_w\}_{w\in\Omega}) \to U,$$

given by:

$$T_{CC'}(\{y_w\}_{w\in\Omega}) = \int_\Omega (CC')^{\frac{1}{2}}\Lambda_\omega^* y_\omega d\mu(w), \qquad \{y_w\}_{w\in\Omega} \in l^2(\Omega, \{V_w\}_{w\in\Omega}).$$

The operator $T_{CC'}$ is called the $(C - C')$ synthesis operator of Λ.
The operator:

$$T_{CC'}^* : U \to l^2(\Omega, \{V_w\}_{w\in\Omega}),$$

given by:

$$T_{CC'}^*(x) = \{\Lambda_\omega (C'C)^{\frac{1}{2}}x\}_{\omega\in\Omega}, \qquad x \in U, \tag{3.2}$$

is called the $(C - C')$ analysis operator for Λ.
Consider now $S_{CC'} := T_{CC'}T_{CC'}^*$ as a mapping on U. By construction $S_{CC'}$ is self adjoint. Using [4] and [14] , $S_{CC'}$ is $\mathcal{A}$-linear and bounded. Thus, it is adjointable. Since $\langle S_{CC'}x, x\rangle \geq 0$, for any $x \in U$, it result, again from [4] and [14], that $S_{CC'}$ is positive.

Remark 3.1. *If C and C' commute and each of them commute with $\Lambda_\omega^*\Lambda_\omega$, then*

$$S_{CC'}x = \int_\Omega C'\Lambda_w^*\Lambda_w Cx d\mu(w),$$

where $S_{CC'}$ is positive in $End_\mathcal{A}^*(U)$ which is C^*-algebra (see [4], [14]). Therefore $S_{CC'}$ is invertible. Indeed, by lemma 1.3, for all $x \in U$ we have:

$$\langle S_{CC'}x, x\rangle_\mathcal{A} = \langle \int_\Omega C'\Lambda_w^*\Lambda_w Cx d\mu(w), x\rangle_\mathcal{A} = \int_\Omega \langle \Lambda_w Cx, \Lambda_w C'x\rangle_\mathcal{A} d\mu(w) \tag{3.3}$$

By (3.1) and (3.3) we obtain

$$A.I \leq S_{CC'} \leq B.I.$$

Then

$$0 \leq I - B^{-1}S_{CC'} \leq \frac{B-A}{B}.I,$$

and consequently:

$$\|I - B^{-1}S_{CC'}\| = \sup_{x\in U, \|x\|=1} \|\langle (I - B^{-1}S_{CC'})x, x\rangle_\mathcal{A}\| \leq \frac{B-A}{B} < 1,$$

which shows that $S_{CC'}$ is invertible.

Theorem 3.1. . *Let $\Lambda = \{\Lambda_w \in End_\mathcal{A}^*(U, V_w) : w \in \Omega\}$ be $(C - C')$-controlled continuous g-bessel family for U and C, C' be in $GL^+(U)$ which commute and each of them commute with $\Lambda_\omega^*\Lambda_\omega$. Then Λ is a $(C - C')$-controlled continuous g-frames for U with respect to $\{V_w : w \in \Omega\}$ if and only if there exist a positive constants A and B such that :*

$$A\|x\|^2 \leq \| \int_\Omega \langle \Lambda_w Cx, \Lambda_w C'x\rangle_\mathcal{A} d\mu(w)\| \leq B\|x\|^2, \qquad x \in U. \tag{3.4}$$

Proof. Suppose that (3.4) holds, we have:

$$\langle S_{CC'}^{\frac{1}{2}}x, S_{CC'}^{\frac{1}{2}}x\rangle_{\mathcal{A}} = \langle S_{CC'}x, x\rangle_{\mathcal{A}} = \int_{\Omega}\langle \Lambda_w Cx, \Lambda_w C^{'}x\rangle_{\mathcal{A}} d\mu(w). \tag{3.5}$$

Using (3.4), we get for all $x \in U$:

$$\sqrt{A}\|x\| \leq \|S_{CC'}^{\frac{1}{2}}x\| \leq \sqrt{B}\|x\|,$$

by lemma 1.2, there exist $m, M > 0$ such that :

$$m\langle x, x\rangle_{\mathcal{A}} \leq \langle S_{CC'}^{\frac{1}{2}}x, S_{CC'}^{\frac{1}{2}}x\rangle_{\mathcal{A}} \leq M\langle x, x\rangle_{\mathcal{A}}.$$

Therefore $\{\Lambda_w : w \in \Omega\}$ is a $(C - C^{'})$-controlled continuous g-frames for U with respect to $\{V_w : w \in \Omega\}$.
Conversely, Let $\{\Lambda_w\}_{w\in\Omega}$ be a $(C - C^{'})$-controlled continuous g-frames for U with respect to $\{V_w : w \in \Omega\}$ with bounds A and B.
Hence, we have

$$A\langle x, x\rangle_{\mathcal{A}} \leq \int_{\Omega}\langle \Lambda_w Cx, \Lambda_w C^{'}x\rangle_{\mathcal{A}} d\mu(w) \leq B\langle x, x\rangle_{\mathcal{A}}, \qquad x \in U. \tag{3.6}$$

Since $0 \leq \langle x, x\rangle$, for all $x \in U$, then we can take the norme in the left, middle and right termes of the above inequality (3.6).
Thus we have:

$$\|A\langle x, x\rangle_{\mathcal{A}}\| \leq \left\|\int_{\Omega}\langle \Lambda_w Cx, \Lambda_w C^{'}x\rangle_{\mathcal{A}} d\mu(w)\right\| \leq \|B\langle x, x\rangle_{\mathcal{A}}\|, \qquad x \in U.$$

So,

$$A\|x\|^2 \leq \left\|\int_{\Omega}\langle \Lambda_w Cx, \Lambda_w C^{'}x\rangle_{\mathcal{A}} d\mu(w)\right\| \leq B\|x\|^2, \qquad x \in U.$$

Theorem 3.2. *Let $\Lambda = \{\Lambda_w\}_{w\in\Omega}$ be a family in $End^{*}_{\mathcal{A}}(U, V_w)$ for all $\omega \in \Omega$, and $C \in GL^{+}(U)$ wich commute with $\Lambda^{*}_{\omega}\Lambda_{\omega}$. Then Λ is a continuous g-frame for U with respect to $\{V_w : w \in \Omega\}$ if and only if Λ is a $(C - C)$-controlled continuous g-frames for U with respect to $\{V_w : w \in \Omega\}$.*

Proof. Let $\{\Lambda_w : w \in \Omega\}$ be a $(C - C)$-controlled continuous g-frames with bounds A and B, then :

$$A\|x\|^2 \leq \left\|\int_{\Omega}\langle \Lambda_w Cx, \Lambda_w Cx\rangle_{\mathcal{A}} d\mu(w)\right\| \leq B\|x\|^2, \qquad x \in U.$$

For all $x \in U$, we have:

$$\begin{aligned} A\|x\|^2 = A\|CC^{-1}x\|^2 &\leq A\|C\|^2\|C^{-1}x\|^2 \\ &\leq \|C\|^2\left\|\int_{\Omega}\langle \Lambda_w CC^{-1}x, \Lambda_w CC^{-1}x\rangle_{\mathcal{A}} d\mu(w)\right\| \\ &= \|C\|^2\left\|\int_{\Omega}\langle \Lambda_w x, \Lambda_w x\rangle_{\mathcal{A}} d\mu(w)\right\|. \end{aligned}$$

Hence,

$$A\|C\|^{-2}\|x\|^2 \leq \left\| \int_\Omega \langle \Lambda_w x, \Lambda_w x\rangle_{\mathcal{A}} d\mu(w) \right\|. \tag{3.7}$$

Moreover, we have :

$$\left\| \int_\Omega \langle \Lambda_w x, \Lambda_w x\rangle_{\mathcal{A}} d\mu(w) \right\| = \left\| \int_\Omega \langle \Lambda_w CC^{-1}x, \Lambda_w CC^{-1}x\rangle_{\mathcal{A}} d\mu(w) \right\|.$$

Then,

$$\left\| \int_\Omega \langle \Lambda_w x, \Lambda_w x\rangle_{\mathcal{A}} d\mu(w) \right\| \leq B\|C^{-1}x\|^2 \leq B\|C^{-1}\|^2\|x\|^2. \tag{3.8}$$

From (3.7) and (3.8) and theorem 2.1, we conclude that $\{\Lambda_w\}_{w\in\Omega}$ is a continuous g-frame with bounds $A\|C\|^{-2}$ and $B\|C^{-1}\|^2$.
Conversely, let $\{\Lambda_w, w \in \Omega\}$ be a continuous g-frame with bounds E and F.
For all $x \in U$, we have :

$$E\langle x, x\rangle_{\mathcal{A}} \leq \int_\Omega \langle \Lambda_w x, \Lambda_w x\rangle_{\mathcal{A}} d\mu(w) \leq F\langle x, x\rangle_{\mathcal{A}}.$$

From right last inequality and lemma 1.1, we have :

$$\int_\Omega \langle \Lambda_w Cx, \Lambda_w Cx\rangle_{\mathcal{A}} d\mu(w) \leq F\langle Cx, Cx\rangle_{\mathcal{A}} \leq F\|C\|^2\langle x, x\rangle_{\mathcal{A}}. \tag{3.9}$$

Moreover, we have:

$$E\langle x, x\rangle_{\mathcal{A}} = E\langle C^{-1}Cx, C^{-1}Cx\rangle_{\mathcal{A}} \leq E\|C^{-1}\|^2\langle Cx, Cx\rangle_{\mathcal{A}}.$$

Then,

$$E\langle x, x\rangle_{\mathcal{A}} \leq \|C^{-1}\|^2 \int_\Omega \langle \Lambda_w Cx, \Lambda_w Cx\rangle_{\mathcal{A}} d\mu(w) \tag{3.10}$$

From (3.9) and (3.10), we have:

$$E\|C^{-1}\|^{-2}\langle x, x\rangle_{\mathcal{A}} \leq \int_\Omega \langle \Lambda_w Cx, \Lambda_w Cx\rangle_{\mathcal{A}} d\mu(w) \leq F\|C\|^2\langle x, x\rangle_{\mathcal{A}}.$$

Which allows us to conclude that Λ is a $(C-C)$-controlled continuous g-frames with bounds $E\|C^{-1}\|^{-2}$ and $F\|C\|^2$.

Proposition 3.1. *Let $\{\Lambda_w, w \in \Omega\}$ be a continuous g-frame for U with respect to $\{V_w : w \in \Omega\}$ and $C, C^{'}$ be in $GL^+(U)$ which commute and each of them commute with $\Lambda_\omega^*\Lambda_\omega$. Then $\{\Lambda_w, w \in \Omega\}$ is $(C - C^{'})$-controlled continuous g-frames.*

Proof. Let $\{\Lambda_w, w \in \Omega\}$ be a continuous g-frame with bounds A and B and S the g-frame operator associated.
By Theorem (2.1) we have:

$$A\|x\|^2 \leq \left\| \int_\Omega \langle \Lambda_w x, \Lambda_w x\rangle_{\mathcal{A}} d\mu(w) \right\| \leq B\|x\|^2, \qquad x \in U.$$

Hence,

$$A\|x\|^2 \leq \|\langle Sx, x\rangle_{\mathcal{A}}\| \leq B\|x\|^2, \qquad x \in U. \tag{3.11}$$

and

$$\left\|\int_{\Omega}\langle\Lambda_w Cx,\Lambda_w C^{'}x\rangle_{\mathcal{A}}d\mu(w)\right\|\leq\|C\|\|C^{'}\|\left\|\int_{\Omega}\langle\Lambda_w x,\Lambda_w x\rangle_{\mathcal{A}}d\mu(w)\right\|=\|C\|\|C^{'}\|\|\langle Sx,x\rangle_{\mathcal{A}}\| \tag{3.12}$$

From (3.11) and (3.12), we have :

$$\left\|\int_{\Omega}\langle\Lambda_w Cx,\Lambda_w C^{'}x\rangle_{\mathcal{A}}d\mu(w)\right\|\leq B\|C\|\|C^{'}\|\|x\|^2,\qquad x\in U. \tag{3.13}$$

Moreover, we have :

$$\int_{\Omega}\langle\Lambda_w Cx,\Lambda_w C^{'}x\rangle_{\mathcal{A}}d\mu(w)=\int_{\Omega}\langle\Lambda_w (CC^{'})^{\frac{1}{2}}x,\Lambda_w (CC^{'})^{\frac{1}{2}}x\rangle_{\mathcal{A}}d\mu(w). \tag{3.14}$$

Then :

$$A\langle(CC^{'})^{\frac{1}{2}}x,(CC^{'})^{\frac{1}{2}}x\rangle_{\mathcal{A}}\leq\int_{\Omega}\langle\Lambda_w Cx,\Lambda_w C^{'}x\rangle_{\mathcal{A}}d\mu(w)\leq B\langle(CC^{'})^{\frac{1}{2}}x,(CC^{'})^{\frac{1}{2}}x\rangle_{\mathcal{A}}. \tag{3.15}$$

By lemma 1.2, there exist $m^{'}>0$ such that :

$$Am^{'}\langle x,x\rangle_{\mathcal{A}}\leq A\langle(CC^{'})^{\frac{1}{2}}x,(CC^{'})^{\frac{1}{2}}x\rangle_{\mathcal{A}}. \tag{3.16}$$

From (3.15), (3.16) and (3.13), we have:

$$Am^{'}\|x\|^2\leq\left\|\int_{\Omega}\langle\Lambda_w Cx,\Lambda_w C^{'}x\rangle_{\mathcal{A}}d\mu(w)\right\|\leq B\|C\|\|C^{'}\|\|x\|^2,\qquad x\in U. \tag{3.17}$$

From the theorem 3.1 allows us to conclude that $\{\Lambda_w\}_{w\in\Omega}$ is $(C-C^{'})$-controlled continuous g-frames with bounds $Am^{'}$ and $B\|C\|\|C^{'}\|$.

Theorem 3.3. *Let $\Lambda=\{\Lambda_w\in End^{*}_{\mathcal{A}}(U,V_\omega),w\in\Omega\}$ and $C,C^{'}$ be in $GL^{+}(U)$ which commute and each of them commute with $\Lambda^{*}_{\omega}\Lambda_\omega$. Then the following statements are equivalent :*
(1) the family $\Lambda=\{\Lambda_w,w\in\Omega\}$ is a $(C-C^{'})$-controlled continuous g-Bessel family for U with respect $\{V_\omega\}_{\omega\in\Omega}$ with bound B.
(2) The operator $T_{CC^{'}}:l^2(\Omega,\{V_w\}_{w\in\Omega})\to U$ given by:

$$T_{CC^{'}}(\{y_w\}_{w\in\Omega})=\int_{\Omega}(CC^{'})^{\frac{1}{2}}\Lambda^{*}_{\omega}y_\omega d\mu(w),\qquad\{y_w\}_{w\in\Omega}\in l^2(\Omega,\{V_w\}_{w\in\Omega}),$$

is well defined and bounded with $\|T_{CC^{'}}\|\leq\sqrt{B}$.

Proof. $(1)\Longrightarrow(2)$
Let $\{\Lambda_w,w\in\Omega\}$ be a $(C-C^{'})$-controlled continuous g-Bessel family for U with respect $\{V_\omega\}_{\omega\in\Omega}$ with bound B.
From Theorem 3.1 we have :

$$\left\|\int_{\Omega}\langle\Lambda_w Cx,\Lambda_w C^{'}x\rangle_{\mathcal{A}}d\mu(w)\right\|\leq B\|x\|^2,\qquad x\in U. \tag{3.18}$$

From ([14], page 7) and [13], we have :

$$\|T_{CC^{'}}(\{y_w\}_{w\in\Omega})\|^2=\sup_{x\in U,\|x\|=1}\|\langle T_{CC^{'}}(\{y_w\}_{w\in\Omega}),x\rangle_{\mathcal{A}}\|^2,\qquad\{y_w\}_{w\in\Omega}\in l^2(\Omega,\{V_w\}_{w\in\Omega}).$$

By Lemma 1.3 and the Cauchy-Schwarz formula, we have :

$$\begin{aligned}
\left\|T_{CC'}(\{y_w\}_{w\in\Omega})\right\|^2 &= \sup_{x\in U,\|x\|=1}\left\|\langle\int_\Omega (CC')^{\frac{1}{2}}\Lambda_\omega^* y_\omega d\mu(w), x\rangle_{\mathcal{A}}\right\|^2\\
&= \sup_{x\in U,\|x\|=1}\left\|\int_\Omega\langle(CC')^{\frac{1}{2}}\Lambda_\omega^* y_\omega, x\rangle_{\mathcal{A}} d\mu(w)\right\|^2\\
&= \sup_{x\in U,\|x\|=1}\left\|\int_\Omega\langle y_\omega, \Lambda_\omega(CC')^{\frac{1}{2}}x\rangle_{\mathcal{A}} d\mu(w)\right\|^2\\
&\leq \sup_{x\in U,\|x\|=1}\left\|\int_\Omega\langle y_\omega, y_\omega\rangle_{\mathcal{A}} d\mu(w)\right\|\left\|\int_\Omega\langle\Lambda_\omega(CC')^{\frac{1}{2}}x, \Lambda_\omega(CC')^{\frac{1}{2}}x\rangle_{\mathcal{A}} d\mu(w)\right\|\\
&= \sup_{x\in U,\|x\|=1}\left\|\int_\Omega\langle y_\omega, y_\omega\rangle_{\mathcal{A}} d\mu(w)\right\|\left\|\int_\Omega\langle\Lambda_\omega Cx, \Lambda_\omega C'x\rangle_{\mathcal{A}} d\mu(w)\right\|\\
&\leq \sup_{x\in U,\|x\|=1}\left\|\int_\Omega\langle y_\omega, y_\omega\rangle_{\mathcal{A}} d\mu(w)\right\| B\|x\|^2 = B\|\{y_\omega\}_{\omega\in\Omega}\|^2.
\end{aligned}$$

Then we have

$$\|T_{CC'}(\{y_w\}_{w\in\Omega})\|^2 \leq B\|\{y_\omega\}_{\omega\in\Omega}\|^2 \Longrightarrow \|T_{CC'}\| \leq \sqrt{B}.$$

We conclude that the operator $T_{CC'}$ is well defined and bounded.
$(2) \Longrightarrow (1)$
If (2) holds, then for any $x \in U$, we have:

$$\begin{aligned}
\int_\Omega\langle\Lambda_w Cx, \Lambda_w C'x\rangle_{\mathcal{A}} d\mu(w) &= \int_\Omega\langle C'\Lambda_w^*\Lambda_w Cx, x\rangle_{\mathcal{A}} d\mu(w)\\
&= \int_\Omega\langle(CC')^{\frac{1}{2}}\Lambda_w^*\Lambda_w(CC')^{\frac{1}{2}}x, x\rangle_{\mathcal{A}} d\mu(w)
\end{aligned}$$

Put: $y_w = \Lambda_w(CC')^{\frac{1}{2}}x$, $\omega \in \Omega$, then:

$$\begin{aligned}
\int_\Omega\langle\Lambda_w Cx, \Lambda_w C'x\rangle_{\mathcal{A}} d\mu(w) &= \langle T_{CC'}(\{y_w\}_{w\in\Omega}), x\rangle_{\mathcal{A}}\\
&\leq \|T_{CC'}\|\|(\{y_w\}_{w\in\Omega})\|\|x\|\\
&\leq \|T_{CC'}\|\left(\int_\Omega\|\Lambda_w(CC')^{\frac{1}{2}}x\|^2 d\mu(w)\right)^{\frac{1}{2}}\|x\|\\
&\leq \|T_{CC'}\|\|x\|\left(\int_\Omega\langle\Lambda_w Cx, \Lambda_w C'x\rangle_{\mathcal{A}} d\mu(w)\right)^{\frac{1}{2}}.
\end{aligned}$$

So,

$$\left(\int_{\Omega}\langle\Lambda_w Cx,\Lambda_w C^{'}x\rangle_{\mathcal{A}}d\mu(w)\right)^{\frac{1}{2}}\leq\|T_{CC'}\|\|x\|.$$

Therefore:

$$\int_{\Omega}\langle\Lambda_w Cx,\Lambda_w C^{'}x\rangle_{\mathcal{A}}d\mu(w)\leq\|T_{CC'}\|^2\|x\|^2.$$

As $\|T_{CC'}\|\leq\sqrt{B}$, we have :

$$\int_{\Omega}\langle\Lambda_w Cx,\Lambda_w C^{'}x\rangle_{\mathcal{A}}d\mu(w)\leq B\|x\|^2,$$

which end the proof.

Proposition 3.2. *Let $\Lambda=\{\Lambda_w\}_{\omega\in\Omega}$ and $\Gamma=\{\Gamma_w\}_{\omega\in\Omega}$ two $(C-C^{'})$-controlled continue g-Bessel family for U with respect to $\{V_\omega\}_{\omega\in\Omega}$ with bounds E_1 and E_2 respectively where $C,C^{'}$ be in $GL^{+}(U)$ which commute and each of them commute with $\Lambda_\omega^*\Lambda_\omega$. Then the operator $L_{CC'}:U\longrightarrow U$ defined by:*

$$L_{CC'}(x)=\int_{\Omega}C^{'}\Gamma_w^*\Lambda_\omega Cxd\mu(w)\qquad x\in U,\tag{3.19}$$

is well defined and bounded with $\|L_{CC'}\|\leq\sqrt{E_1E_2}$. Also its adjoint operator is given by:

$$L_{CC'}^*(g)=\int_{\Omega}C^{'}\Lambda_w^*\Gamma_\omega Cxd\mu(w).\tag{3.20}$$

Proof. For any $x\in U$, we have :

$$\begin{aligned}\left\|\int_{\Omega}C^{'}\Gamma_w^*\Lambda_\omega Cxd\mu(w)\right\|^2&=\sup_{y\in U,\|y\|=1}\left\|\langle\int_{\Omega}C^{'}\Gamma_w^*\Lambda_\omega Cxd\mu(w),y\rangle_{\mathcal{A}}\right\|^2\\&=\sup_{y\in U,\|y\|=1}\left\|\int_{\Omega}\langle\Lambda_\omega Cx,\Gamma_w C^{'}y\rangle_{\mathcal{A}}d\mu(w)\right\|^2\\&\leq\sup_{y\in U,\|y\|=1}\left\|\int_{\Psi}\langle\Lambda_\omega Cx,\Lambda_\omega Cx\rangle_{\mathcal{A}}d\mu(w)\right\|\left\|\int_{\Omega}\langle\Gamma_w C^{'}y,\Gamma_w C^{'}y\rangle_{\mathcal{A}}d\mu(w)\right\|\\&\leq\left\|\int_{\Omega}\langle\Lambda_\omega Cx,\Lambda_\omega Cx\rangle_{\mathcal{A}}d\mu(w)\right\|E_2\\&\leq E_1E_2\|x\|^2,\end{aligned}$$

then $\int_{\Omega}C^{'}\Gamma_w^*\Lambda_\omega Cxd\mu(w)$ converge in U and,

$$\|L_{CC'}\|=\left\|\int_{\Omega}C^{'}\Gamma_w^*\Lambda_\omega Cxd\mu(w)\right\|\leq\sqrt{E_1E_2}.$$

Moreover, we see that:

$$\langle L_{CC'}x,y\rangle_{\mathcal{A}}=\langle\int_{\Omega}C^{'}\Gamma_w^*\Lambda_\omega Cxd\mu(w),y\rangle_{\mathcal{A}}$$

$$= \int_{\Omega} \langle C^{'}\Gamma_w^*\Lambda_\omega Cx, y\rangle_{\mathcal{A}} d\mu(w)$$
$$= \int_{\Omega} \langle x, C\Lambda_w^*\Gamma_\omega C^{'}y\rangle_{\mathcal{A}} d\mu(w)$$
$$= \langle x, \int_{\Omega} C\Lambda_w^*\Gamma_\omega C^{'}y d\mu(w)\rangle_{\mathcal{A}}.$$

Which give:

$$L_{CC'}^*(g) = \int_{\Omega} C^{'}\Lambda_w^*\Gamma_\omega Cx d\mu(w).$$

Corollary 3.1. *Let $\Lambda = \{\Lambda_w\}_{w\in\Omega} \subset End_{\mathcal{A}}^*(U, V_w)$ be a $(C - C^{'})$-controlled continue g-frames for U with respect to $\{V_\omega\}_{\omega\in\Omega}$ and $\Gamma = \{\Gamma_w\}_{w\in\Omega} \subset End_{\mathcal{A}}^*(U, V_w)$ be a $(C - C^{'})$-controlled continue g-Bessel family for U with respect to $\{V_\omega\}_{\omega\in\Omega}$, and $C, C^{'}$ be in $GL^+(U)$ which commute and each of them commute with $\Lambda_\omega^*\Lambda_\omega$ and commute with $\Gamma_w^*\Gamma_w$. If the operator $L_{CC'}$ is surjective then $\Gamma = \{\Gamma_w\}_{w\in\Omega}$ is also a $(C - C^{'})$-controlled continue g-frames for U with respect to $\{V_\omega\}_{\omega\in\Omega}$.*

Proof. Let $\Lambda = \{\Lambda_w \in End_{\mathcal{A}}^*(U, V_w) : w \in \Omega\}$ be a $(C - C^{'})$-controlled continuous g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$.
By theorem 3.3, the operator $T_{CC'} : l^2(\Omega, \{V_w\}_{w\in\Omega}) \to U$ given by

$$T_{CC'}(\{y_w\}_{w\in\Omega}) = \int_{w\in\Omega} (CC^{'})^{\frac{1}{2}}\Lambda_\omega^* y_\omega d\mu(w), \qquad \{y_w\}_{w\in\Omega} \in l^2(\Omega, \{V_w\}_{w\in\Omega})$$

is well defined and bounded.
By (3.2) its adjoint operator $T_{CC'}^* : U \to l^2(\Omega, \{V_w\}_{w\in\Omega})$ is given by :

$$T_{CC'}^*(x) = \{\Lambda_\omega (C^{'}C)^{\frac{1}{2}}x\}_{\omega\in\Omega}. \tag{3.21}$$

Since $\Gamma = \{\Gamma_w\}_{w\in\Omega}$ is also a $(C - C^{'})$-controlled continuous g-Bessel family for U with respect to $\{V_\omega\}_{\omega\in\Omega}$.
Again, by theorem 3.3, the operator $K_{CC'} : l^2(\Omega, \{V_w\}_{w\in\Omega}) \to U$ is given by :

$$K_{CC'}(\{y_w\}_{w\in\Omega}) = \int_{w\in\Omega} (CC^{'})^{\frac{1}{2}}\Gamma_\omega^* y_\omega d\mu(w), \qquad \{y_w\}_{w\in\Omega} \in l^2(\Omega, \{V_w\}_{w\in\Omega})$$

is well defined bounded operator. Its adjoint operator is given by

$$K_{CC'}^*(x) = \{\Gamma_\omega (C^{'}C)^{\frac{1}{2}}x\}_{\omega\in\Omega}, \qquad x \in U.$$

Hence, for any $x \in U$, the operator defined in (3.19) can be written as :

$$L_{CC'}(x) = \int_{\Omega} C^{'}\Gamma_w^*\Lambda_\omega Cx d\mu(w) = K_{CC'}T_{CC'}^*x.$$

Since $L_{CC'}$ is surjective then for any $y \in U$, there exists $x \in U$ such that:

$$y = L_{CC'}x = K_{CC'}T_{CC'}^*x.$$

This implies that $K_{CC'}$ is surjective.
From lemma 1.2, we have $K^*_{CC'}$ is bounded below, then there exists $m > 0$ such that:

$$\langle K^*_{CC'}x, K^*_{CC'}x\rangle_{\mathcal{A}} \geq m\langle x, x\rangle_{\mathcal{A}}, \qquad x \in U.$$

Hence,

$$\langle K_{CC'}K^*_{CC'}x, x\rangle_{\mathcal{A}} \geq m\langle x, x\rangle_{\mathcal{A}}, \qquad x \in U.$$

So,

$$\langle \int_{\Omega} (CC^{'})^{\frac{1}{2}}\Gamma^*_w\Gamma_\omega(CC^{'})^{\frac{1}{2}}xd\mu(\omega), x\rangle_{\mathcal{A}} \geq m\langle x, x\rangle_{\mathcal{A}}, \qquad x \in U,$$

therefore

$$\int_{\Omega} \langle \Gamma_w Cx, \Gamma_w C^{'}x\rangle_{\mathcal{A}} d\mu(\omega) \geq m\langle x, x\rangle_{\mathcal{A}}, \qquad x \in U.$$

Which shows the desired result.

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