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LINEAR PROGRAMMING WITH INTERVAL COEFFICIENTS

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Abstract: *Many operational research optimization models, such as linear programming, assume that the data used to build a model is accurate, but in the real world most of these assumptions are only approximately true. In linear programming models, uncertainty is associated with the model coefficients during the formulation stage. A small number of papers discuss these models of linear programming with interval coefficients (LPIC). Using the method presented in paper [1], capable of solving linear programming problems with numerical intervals as intervals, we give in this paper a practical example, to demonstrate the use and the practicality of interval coefficients in linear programming models.*

Keywords: linear programming, interval coefficients

1. Introduction

Many operational research optimization models, such as linear programming, assume that the data used to build a model is accurate, but in the real world most of these assumptions are only approximately true. In linear programming models, uncertainty is associated with the model coefficients during the formulation stage. In addition, in real-life models, it is more realistic to represent coefficients as numerical intervals than as exact real numbers.

A small number of papers discuss these models of linear programming with interval coefficients (LPIC).

The most important and relevant results on numerical intervals and linear programming are obtained by Shaocheng [3]. It introduces inequalities with maximum value ranges and inequalities with minimum value ranges to reduce an LPIC problem from a non-deterministic one to two deterministic LP problems. These two models can be solved with either method, such as the Simplex method, to get an optimal solution for the initial problem.

Shaocheng's basic idea highlights the results of the work [2]. Shaocheng deals only with non-negative variables, while in [2] the results are extended to non-positive variables or variables unrestricted in sign, and also with equality interval constraints.

The purpose of the paper [2] is to develop a method capable of solving linear programming problems with numerical intervals as intervals.

The problem is stated as follows ([2]) :

Given a linear programming problem with interval coefficients in either or both of the objective function and the constraints:

$$\min(\max) Z = \sum_{j=1}^{n} [\underline{c}_j, \overline{c}_j] x_j$$
$$\sum_{j=1}^{n} [\underline{a}_{ij}, \overline{a}_{ij}] x_j \geq (=, \leq) [\underline{b}_i, \overline{b}_i], \quad \text{for i=1,...,}m \qquad (1)$$

derive an algorithm which will compute the best and the worst optimum.

As mentioned earlier, the main idea is to transform the LPIC form from a non-

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deterministic problem into two deterministic linear programming problems. This is achieved by introducing inequalities with ranges of maximum and minimum values. This helps us to determine for any objective function with coefficients at intervals at least two different extreme versions, one will be the optimistic version (ie it should give the best optimum) and the other the pessimistic version (ie it should give the worst optimal).
They present a general algorithm for solving LPIC. Only minimum problems are presented, for maximum problems the problem is transformed into a minimum one, by multiplying the objective function by (-1).

Assuming $x_i \geq 0$ *and we have a minimization LPIC , then* $\sum_{j=1}^{n} \underline{c}_j x_j$ *is called the most favorable objectiv function, and* $\sum_{j=1}^{n} \overline{c}_j x_j$ *the least favorable objectiv function* . ([2])
For a maximization problem we use the opposite and for a non-positive variable we use a change of variable to convert it to a non-negative variable.
Suppose we are given the following LPIC problem:

$$\min Z = \sum_{j=1}^{n} [\underline{c}_j, \overline{c}_j] x_j$$

$$\sum_{j=1}^{n} [\underline{a}_{ij}, \overline{a}_{ij}] x_j \geq [\underline{b}_i, \overline{b}_i], \quad \text{i=1,...,m} \qquad (2)$$

$$x_j \geq 0 \ \forall j$$

The the best optimum and worst optimum are computed as follows:

(I) For the best optimum:

$$\min \underline{z} = \sum_{j=1}^{n} c'_j x_j,$$

$$\text{where} \quad c'_j = \begin{cases} \underline{c}_j, & \text{dacă } x_j \geq 0 \\ \overline{c}_j, & \text{dacă } x_j \leq 0 \end{cases}$$

$$\sum_{j=1}^{n} a'_{ij} x_j \geq \underline{b}_i, \forall i \qquad (3)$$

$$\text{where } a'_{ij} = \begin{cases} \overline{a}_{ij}, & \text{dacă } x_j \geq 0 \\ \underline{a}_{ij}, & \text{dacă } x_j \leq 0 \end{cases}$$

$$x_j \geq 0 \ \forall j$$

(II) For the worst optimum:

$$\min \overline{z} = \sum_{j=1}^{n} c''_j x_j,$$

$$\text{where } c''_j = \begin{cases} \overline{c}_j, & \text{dacă } x_j \geq 0 \\ \underline{c}_j, & \text{dacă } x_j \leq 0 \end{cases}$$

$$\sum_{j=1}^{n} a''_{ij} x_j \geq \overline{b}_i, \forall i \qquad (4)$$

$$\text{where } a''_{ij} = \begin{cases} \underline{a}_{ij}, & \text{dacă } x_j \geq 0 \\ \overline{a}_{ij}, & \text{dacă } x_j \leq 0 \end{cases}$$

$$x_j \geq 0 \ \forall j$$

Suppose $\underline{x} = (x'_1, x'_2, \ldots, x'_n)$ is the optimal solution for (I) with $\underline{z}$ the optimal value and $\overline{x} = (x''_1, x''_2, \ldots, x''_n)$ is the optimal solution for (II) with $\overline{z}$ the optimal value. Then the best optimum solution will be $\underline{z}$, at $(x'_1, x'_2, \ldots, x'_n)$ and with coefficient settings c'_j, a'_{ij} and $\underline{b}_i$, $\forall i$ and j and also the worst optimum solution $\overline{z}$ with $(x''_1, x''_2, \ldots, x''_n)$ and with coefficient settings c''_j, a''_{ij} $\forall i$ și j .
The general solution of LPIC will be $Z = [\underline{z}, \overline{z}]$ i.e. optimal value of the problem will be between $\underline{z}$ (the best case scenario) and $\overline{z}$ (the worst case scenario).

2. A practical example

We will give a practical example, an marketing example, to demonstrate the use and the practicality of interval coefficients in linear programming models.
The original example incorporates real numbers only. We modified some of the coefficients to be represented by interval numbers.
An advertising company wishes to plan an advertising campaign for three different media: television, radio and a magazine. The purpose of the advertising is to reach as many potential customers as possible. The following are results of a market study:

Table 1 Market study

Strategies	Television			
	Prime Day	Prime Time	Radio	Magazine
Cost of an advertising unit	Rs.[38,42]	Rs.[74,76]	Rs.[29,32]	Rs.[13,16]
Number of potential costumers reached per unit	[380,420]	[880,910]	[490,520]	[190,230]
Number of women customers reached per unit	[270,300]	[390,420]	[180,220]	[80,110]

The company wants to spend at most between Rs 790 and 810 thousands on advertising.
Also required that exposures of women to be at least between 1 million 980 thousands (we will note 1980, to be the same that data in the table) and 2 million 10 thousands, the cost of advertising on television to be at most between Rs 490 and 520 thousands, at least 3 advertising units be bought on prime day and 2 units during prime time and also at least 5 advertising units on radio and magazine. And the purpose is to maximize potential customers reach.

Observation 1: In the table above the numbers are orders of thousands.

Let X_i be the number of advertising units bought in prime day and time on television, radio and magazine, respectively, $i = \overline{1,4}$.

Now company problem to maximize potential customers reach can be formulated beginning with the objective function as follows:

Max Z=[380,420]x_1+[880,910] x_2
+[490,520] x_3+[190,230] x_4 (5)

For the constrains:

- at most between 790 and 810 thousands on advertising budget: [38,42]x_1+[74,76] x_2+[29,32] x_3 +[13,16] x_4 ≤ [790,810]
- at least between 1 million 980 thousands and 2 million 10 thousands number of women customers reached by the advertising campaign: [270,300] x_1+[390,420] x_2+[180,220] x_3+[80,110] x_4 ≥ [1980,2010]
- at most between 490 and 520 thousands on television advertising: [38,42] x_1+[74,76] x_2 ≤ [490,520]
- to limit the advertising unit bought on prime day and prime time and also on radio and magazine, these additional constrains are necessary: $x_1 \geq 3$, $x_2 \geq 2$, $x_3 \geq 5$, $x_4 \geq 5$

So, the problem has been formulated as an LPIC model. First we transform the LPIC model in two LP models and then we solve them separately to determine the best and the worst optimum solution.

We need to set up the problem in the standard form by converting the problem in to a minimization problem and change any ≤ inequalities into ≥ inequalities.

The new model is:

Min -Z = [-420,-380]x_1+[-910,-880] x_2
+[-520,-490] x_3+[-230,-190] x_4
[-42,-38] x_1 +[-76,-74] x_2+[-32,-29] x_3
+[-16,-13] x_4 ≥ [-810,-790]
[270,300] x_1+[390,420] x_2+[180,220] x_3+
[80,110] x_4 ≥ [1980,2010]
[-42,-38] x_1+[-76,-74] x_2 ≥ [-520,-490] (6)
$x_1 \geq 3$
$x_2 \geq 2$
$x_3 \geq 5$
$x_4 \geq 5$

We obtain:

- The best optimum :

$$\begin{aligned}\min\ -\underline{z} = &-420x_1-910x_2-520x_3-230x_4\\ &-38x_1-74x_2-29x_3-13x_4 \geq -810\\ &300x_1+420x_2+220x_3+110x_4 \geq 1980\\ &-38x_1-74x_2 \geq -520\\ &x_1 \geq 3\\ &x_2 \geq 2\\ &x_3 \geq 5\\ &x_4 \geq 5\end{aligned} \quad (7)$$

- The worst optimum :

$$\begin{aligned}\min\ -\overline{z} = &-380x_1-880x_2-490x_3-190x_4\\ &-42x_1-76x_2-32x_3-16x_4 \geq -790\\ &270x_1+390x_2+180x_3+80x_4 \geq 2010\\ &-42x_1-76x_2 \geq -490\\ &x_1 \geq 3\\ &x_2 \geq 2\\ &x_3 \geq 5\\ &x_4 \geq 5\end{aligned} \quad (8)$$

For both models (The best and the worst), LINDO (a linear programming solver) was used to obtain the solutions.

```
min -420x1-910x2-520x3-230x4
subject to
Budget) -38x1-74x2-29x3-13x4>= -810
Women)  300x1+420x2+220x3+110x4>= 1980
TV)     -38x1-74x2>= -520
         x1>=3
         x2>=2
         x3>=5
         x4>=5
end
```

Figure 1: LINDO Best Optimum

```
LP OPTIMUM FOUND AT STEP      1

       OBJECTIVE FUNCTION VALUE

       1)     -12890.69

 VARIABLE        VALUE          REDUCED COST
       X1         3.000000          0.000000
       X2         2.000000          0.000000
       X3        16.655172          0.000000
       X4         5.000000          0.000000


      ROW   SLACK OR SURPLUS     DUAL PRICES
   BUDGET)         0.000000       -17.931034
    WOMEN)      3974.137939         0.000000
       TV)       258.000000         0.000000
        5)         0.000000      -261.379303
        6)         0.000000      -416.896545
        7)        11.655172         0.000000
        8)         0.000000        -3.103448

NO. ITERATIONS=       1

RANGES IN WHICH THE BASIS IS UNCHANGED:

                         OBJ COEFFICIENT RANGES
VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                   COEF          INCREASE         DECREASE
      X1     -420.000000         INFINITY       261.379303
      X2     -910.000000         INFINITY       416.896545
      X3     -520.000000         6.923077         INFINITY
      X4     -230.000000         INFINITY         3.103448


                        RIGHTHAND SIDE RANGES
     ROW         CURRENT        ALLOWABLE        ALLOWABLE
                   RHS          INCREASE         DECREASE
  BUDGET     -810.000000       338.000000         INFINITY
   WOMEN     1980.000000      3974.137939         INFINITY
      TV     -520.000000       258.000000         INFINITY
       5        3.000000         6.789474         3.000000
       6        2.000000         3.486486         2.000000
       7        5.000000        11.655172         INFINITY
       8        5.000000        26.000000         5.000000
```

Figure 2: LINDO solution for Best Optimum

```
min -380x1-880x2-490x3-190x4
subject to
Budget) -42x1-76x2-32x3-16x4>= -790
Women)  270x1+390x2+180x3+80x4>= 2010
TV)     -42x1-76x2>= -490
         x1>=3
         x2>=2
         x3>=5
         x4>=5
end
```

Figure 3: LINDO Worst Optimum

```
LP OPTIMUM FOUND AT STEP      0

       OBJECTIVE FUNCTION VALUE

       1)     -10465.00

 VARIABLE        VALUE          REDUCED COST
       X1         3.000000          0.000000
       X2         2.000000          0.000000
       X3        13.500000          0.000000
       X4         5.000000          0.000000


     ROW   SLACK OR SURPLUS     DUAL PRICES
  BUDGET)         0.000000       -15.312500
   WOMEN)      2410.000000         0.000000
      TV)       212.000000         0.000000
       5)         0.000000      -263.125000
       6)         0.000000      -283.750000
       7)         8.500000         0.000000
       8)         0.000000       -55.000000

NO. ITERATIONS=       0

RANGES IN WHICH THE BASIS IS UNCHANGED:

                        OBJ COEFFICIENT RANGES
VARIABLE         CURRENT        ALLOWABLE         ALLOWABLE
                   COEF          INCREASE          DECREASE
      X1      -380.000000        INFINITY        263.125000
      X2      -880.000000        INFINITY        283.750000
      X3      -490.000000      110.000000          INFINITY
      X4      -190.000000        INFINITY         55.000000

                          RIGHTHAND SIDE RANGES
         ROW         CURRENT        ALLOWABLE         ALLOWABLE
                       RHS          INCREASE          DECREASE
      BUDGET     -790.000000       272.000000          INFINITY
       WOMEN     2010.000000      2410.000000          INFINITY
          TV     -490.000000       212.000000          INFINITY
           5        3.000000         5.047619          3.000000
           6        2.000000         2.789474          2.000000
           7        5.000000         8.500000          INFINITY
           8        5.000000        17.000000          5.000000
```

Figure 4: LINDO solution for Worst Optimum

From the output in Figure 2 we can observe that the best optimum is $-\underline{z}$=-12890 for the minimal problem, so for our maximum problem $\underline{z}$=12890 (i.e 12 million 890 thousands), and the coefficients used to get this solution are as indicated in Figure 2.

Similar for the worst optimum, Figure 4 shows that the worst optimum for this LPIC problem is $\overline{z}$=10465 (i.e. 10 million 465 thousands) for the maximum problem with the fixed coefficients used to get this solution.

So, by using this method to solve this LPIC model we can conclude that optimum solution Z of this problem lies between the best optimum $\underline{z}$=12890 and the worst optimum $\overline{z}$=10465.

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