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Confidence Intervals of Gini Coefficient Under Unequal Probability Sampling

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We propose an estimator for the Gini coefficient, based on a ratio of means. We show how bootstrap and empirical likelihood can be combined to construct confidence intervals. Our simulation study shows the estimator proposed is usually less biased than customary estimators. The observed coverages of the empirical likelihood confidence interval proposed are also closer to the nominal value.

Key words: Bootstrap; empirical likelihood; inclusion probability; survey weight; sampling design.

1. Introduction

Gini's (1914) coefficient is a widely used indicator for measuring income inequality in a wide range of area of economics and finance (e.g., Koshevoy and Mosler 1997; Ogwang 2000; Gajdos and Weymark 2005). The Gini coefficient is defined as the ratio of the area that lies between the 45° line and the Lorenz's (1905) curve given by

$$\mathcal{L}(x) := \frac{1}{E(Y)} \int_0^x y \, dF_Y(y), \tag{1}$$

where $F_Y(\cdot)$ is the cumulative distribution function of a positive random variable Y, and $E(Y)$ is the expectation of Y. An excellent review of various formulations of the Gini coefficient can be found in Giorgi and Gigliarano (2017).

Surveys are usually used to estimate the Gini coefficient. However, sampled units are rarely selected independently with equal probability, because of sample selection, which involves stratification and unequal probabilities. Two customary estimators for unequal probability sampling can be found in the literature (e.g., Langel and Tillé 2013, for a review). They are defined by Equations (13) and (14) in Section 5. The proposed estimator is different and based on a ratio, which allows to express it as an empirical likelihood estimator. Single stage designs are considered in this article. The proposed approach can be extended for multi-stage designs by using Berger's (2018a) approach.

Variance estimation of the Gini coefficient has been widely studied in the literature (e.g., Nair 1936; Hoeffding 1948; Glasser 1962; Sendler 1979; Beach and Davidson 1983;

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Gastwirth and Gail 1985; Schezhtman and Yitzhaki 1987; Sandström et al. 1985, 1988; Nygård and Sandström 1989; Yitzhaki 1991; Shao 1994; Binder and Kovaćević 1995; Bishop et al. 1997; Karagiannis and Kovaćević 2000; Ogwang 2000; Giles 2004; Modarres and Gastwirth 2006; Davidson 2009). Yitzhaki (1991) and Qin et al. (2010) proposed a variance estimator under stratified random samples. Asymptotic variance under stratified and clustered survey data can be found in Bhattacharya (2007). Berger (2008) proposed a jackknife variance estimator under unequal probability sampling. Langel and Tillé (2013) provided a comprehensive literature review on variance estimation for the Gini coefficient.

Sandström et al. (1988) have developed a confidence interval for the Gini coefficient based on normal approximation. Mills and Zandvakili (1997) consider bootstrap methods to compute interval estimates for the Gini coefficient. Qin et al. (2010) proposed pseudoempirical likelihood confidence intervals for the Gini coefficient under simple random sampling, using bootstrap and empirical likelihood methods. Qin et al.'s (2010) approach requires estimating the distribution function, and is not designed for unequal probability sampling. Other empirical likelihood intervals with independent and identically distributed observations can be found in Peng (2011). Empirical likelihood confidence intervals are range preserving; that is, the lower bound and the upper bound cannot be outside the parameter space [0, 1] of the Gini coefficient. The bounds are driven by the distribution observed from the data, rather than an asymptotic distribution. Empirical likelihood also offers the possibility of using some auxiliary information, which may improve the estimation of the Gini coefficient (Berger and Torres 2016). A review of empirical likelihood under unequal probability sampling can be found in Berger (2018b). Note that the confidence intervals proposed do not require an effective sample size or a design effect, unlike the pseudoempirical likelihood approach (Wu and Rao 2006) for unequal probability sampling.

In Section 2, we define the Gini coefficient. The proposed estimator is defined in Section 3. In Section 4, we show how bootstrap and empirical likelihood can be combined to construct confidence intervals. The empirical likelihood confidence intervals have the advantage of having bounds within the range of the Gini coefficient. Linearisation will not be required for empirical likelihood confidence intervals. Our simulation study in Section 5 shows that the proposed estimator can be more efficient than the customary estimator (e.g., Berger 2008; Langel and Tillé 2013). The coverages of the proposed empirical likelihood confidence interval are usually not significantly different from the nominal value.

2. The Gini Coefficient

Let $Y \geq 0$ denote a positive random variable with a distribution function $F_Y(y)$. The Gini coefficient is defined by

$$G_0 := \frac{2}{E(Y)} \int_0^\infty y F_Y(y) dF_Y(y) - 1 = 1 - \frac{1}{E(Y)} \int_0^\infty \{1 - F_Y(y)\}^2 dy. \tag{2}$$

Yitzhaki (1998) proposed an alternative expression of G_0 based on the minimum

$$Z := \min\{Y_a, Y_b\}$$

of two independent copies Y_a and Y_b of Y. Since $Z \geq 0$, we always have that $E(Z) = \int_0^\infty \{1 - F_Z(z)\} dz$, where $F_Z(z)$ denotes the cumulative distribution of Z.

Furthermore, since Z is the minimum of two random variables with the same distribution, we have that $F_Z(z) = 1 - \{1 - F_Y(z)\}^2$. This implies $E(Z) = \int_0^\infty \{1 - F_Y(z)\}^2 dz$. Thus, Equation (2) gives Yitzhaki's (1998) alternative expression (see also Peng 2011),

$$G_0 = 1 - \frac{E(Z)}{E(Y)}. \tag{3}$$

Let U be a finite population of N units, where N is a fixed quantity that is not necessarily known. Consider that we have N independent copies $\{Y_i : i \in U\}$ of Y. Let $\{y_i : i \in U\}$ be the realisation of these copies.

The empirical equivalent of $E(Z)$ is therefore

$$\bar{y}_U^* := \frac{1}{N(N-1)} \sum_{i \in U} \sum_{\substack{j \in U \\ j \neq i}} \min\{y_i, y_j\} = \frac{1}{N} \sum_{i \in U} y_i^*,$$

where

$$y_i^* := \frac{1}{N-1} \sum_{\substack{j \in U \\ j \neq i}} \min\{y_i, y_j\}.$$

Thus, the empirical version of Equation (3) is the finite population parameter

$$G_U := 1 - \frac{\bar{y}_U^*}{\bar{y}_U}, \tag{4}$$

where

$$\bar{y}_U := \frac{1}{N} \sum_{i \in U} y_i. \tag{5}$$

3. Estimation of the Gini Coefficient

Suppose that a sample S is randomly selected from U. We observe the values y_i for the sampled units $i \in S$. We shall use Neyman's (1938) design-based approach; that is, the sampling distribution is conditional on $\{y_i : i \in U\}$ and driven by the random selection of S. Thus, the values $\{y_i : i \in U\}$ and the parameter G_U will be treated as constants.

We consider that the population U is broken up into disjoint strata $U_1, \ldots, U_h, \ldots, U_H$ and $\cup_{h=1}^H U_h = U$. Within each stratum U_h, a sample of n_h units is selected with-replacement with unequal selection probabilities P_i, where $\sum_{i \in U_h} P_i = 1$. Let $\pi_i = n_h P_i$, when $i \in U_h$. Let S_h be the set of n_h labels for stratum U_h, where $S = \cup_{h=1}^H S_h$ and $n = \sum_{h=1}^H n_h$. We assume that we have a with-replacement or without-replacement sampling design with negligible sampling fractions, in order to justify the bootstrap approach. Fortunately, in practice, the Gini coefficient is estimated from social surveys, which are often based on negligible sampling fractions. The negligible sampling fraction is only needed for variance estimation and confidence intervals.

The estimator proposed for Equation (4) is

$$\hat{G}_\pi := 1 - \frac{\bar{y}_\pi^*}{\bar{y}_\pi}, \tag{6}$$

where $\bar{y}_{\pi}^{*}$ and $\bar{y}_{\pi}$ denote Hájek's (1971) estimators given by

$$\bar{y}_{\pi} := \frac{1}{\hat{N}} \sum_{i \in S} \frac{y_i}{\pi_i},$$

$$\bar{y}_{\pi}^{*} := \frac{1}{\hat{N}} \sum_{i \in S} \frac{\hat{y}_i^{*}}{\pi_i},$$

$$\hat{y}_i^{*} := \frac{1}{\hat{N} - \pi_i^{-1}} \sum_{\substack{j \in S \\ j \neq i}} \frac{1}{\pi_j} \min\{y_i, y_j\},$$

$$\hat{N} := \sum_{i \in S} \pi_i^{-1}.$$

The advantage of Equation (6) is the fact that it does not involve the estimation $F_Y(y)$. Note that Equation (6) reduces to Yitzhaki's (1998) under simple random sampling with a single stratum (see also Peng 2011; Giorgi and Gigliarano 2017).

Rescaled bootstrap (Rao et al. 1992; Rust and Rao 1996) can be used for variance estimation. This method is based on bootstrap weights (Rust and Rao 1996), given by

$$w_i^{(b)} := \frac{r_i n}{\pi_i (n-1)} \tag{7}$$

where r_i is the number of times i-th unit is selected by bootstrap. The variance between the bootstrap replicates can be used as a variance estimate. A bootstrap confidence interval based on the bootstrap quantiles can be derived (the "*histogram approach*").

The theory of bootstrap is well established, and little needs to be added. However, empirical likelihood is a new emerging topic, and little has been done on empirical likelihood confidence intervals for Gini, under unequal probability sampling. Peng's (2011) approach assumed an independent and identically distributed setting. Qin et al.'s (2010) method is based on simple random sampling. In Section 4, we show how an empirical likelihood confidence interval can be constructed with unequal probability sampling, in conjunction with bootstrap.

4. Empirical Likelihood Confidence Intervals

In this section, we show how Berger and Torres's (2016) approach can be combined with bootstrap. Empirical likelihood is based on estimating equations. It can be shown that Equation (6) is the solution to

$$\sum_{i \in S} \frac{1}{\pi_i} e(y_i, \hat{y}_i^{*}, G) = 0. \tag{8}$$

where

$$e(y_i, \hat{y}_i^{*}, G) := y_i(G-1) + \hat{y}_i^{*}. \tag{9}$$

By substituting Equation (9) within Equation (8), we obtain $\sum_{i \in S} \pi_i^{-1} y_i (G-1) + \sum_{i \in S} \pi_i^{-1} \hat{y}_i^{*} = (G-1)\hat{N}\bar{y}_{\pi} + \hat{N}\bar{y}_{\pi}^{*} = 0$. The solution to the last equation is indeed Equation (6).

Berger and Torres's (2012, 2014, 2016) "*empirical log-likelihood function*" is defined by

$$\ell_{\max}(G) := \max_{p_i : i \in S} \left\{ \sum_{i \in S} \log(p_i) : p_i > 0,\ \sum_{i \in S} \frac{p_i}{\pi_i} e(y_i, \hat{y}_i^*, G) = 0,\ \sum_{i \in S} p_i \boldsymbol{\delta}_i = \frac{\vec{\boldsymbol{n}}}{n} \right\}, \quad (10)$$

where G denotes a value within the parameter space, $\boldsymbol{\delta}_i$ is the vector of stratification variables defined by

$$\boldsymbol{\delta}_i := (\delta_{i1}, \ldots, \delta_{ih}, \ldots, \delta_{iH})^\top$$

and $\vec{\boldsymbol{n}}$ is the strata allocation given by

$$\vec{\boldsymbol{n}} := \sum_{i \in S} \tilde{\boldsymbol{\delta}}_i = (n_1, \ldots, n_h, \ldots, n_H)^\top.$$

Within Equation (10), we have two types of constraints. The constraint involving G is a moment condition which contains the standard sampling weights π_i^{-1}. We also have a stratification constraint $\sum_{i \in S} p_i \boldsymbol{\delta}_i = \vec{\boldsymbol{n}} n^{-1}$, which is not motivated by moment conditions. Equation (10) reduces to Owen's (1988) empirical log-likelihood function when we have a single stratum and $\pi_i = n/N, \forall i \in U$. The advantage of Equation (10) is that it can be used as a standard likelihood function for design-based inference. Note that Equation (10) differs from Peng's (2011) approach, even with a single stratum and $\pi_i = n/N$, because Peng's (2011) approach is based on splitting the sample randomly into two sub-samples of same size.

The "*maximum empirical likelihood estimator*" $\hat{G}_{EL}$ is defined as the quantity that maximises $\ell_{\max}(G)$. Berger and Torres (2016) show that this implies that $\hat{G}_{EL}$ is the solution to Equation (8). Thus, $\hat{G}_{EL} = \hat{G}_\pi$.

The empirical likelihood approach can also be used for confidence intervals based upon Equation (6). Consider the "*empirical log-likelihood ratio statistic*"

$$\hat{r}(G) := 2\left\{\ell_{\max}(\hat{G}) - \ell_{\max}(G)\right\}. \quad (11)$$

Berger and Torres (2016) showed that the empirical log-likelihood ratio statistic converges to an ancillary quadratic form, when $G = G_0$. Unfortunately, this quadratic form will not necessarily converge to a χ^2-distribution, because the $\hat{y}_i^*$ are estimated. In other words, this quadratic form is an ancillary statistic with an unknown distribution. We shall approximate this distribution using bootstrap.

In order to compute a α-level confidence interval, we would need to know the upper α-quantile of the distribution of $\hat{r}(G_0)$. This quantile can be approximated by the bootstrap distribution. Consider the rescaled bootstrap sampling weights given by Equation (7). Let $\hat{r}(G)^b$ be the b-th bootstrap value of Equation (11) based on bootstrap sampling weights given by Equation (7), with $G = \hat{G}_\pi$. The α-level bootstrap confidence interval is

$$\left[\min\{G : \hat{r}(G) \leq r_\alpha\};\ \max\{G : \hat{r}(G) \leq r_\alpha\}\right], \quad (12)$$

where r_α is the α-quantile of $\{\hat{r}(\hat{G}_\pi)^1, \ldots, \hat{r}(\hat{G}_\pi)^b, \ldots, \hat{r}(\hat{G}_\pi)^B\}$. Note that $\hat{r}(G)$ is a convex non-symmetric function with a minimum at $G = \hat{G}_\pi$. This interval can be found by using any root search method, such as Brent (1973) and Dekker's (1969) method, since the

bounds are the two roots of $\hat{r}(G) - r_\alpha = 0$. This can be achieved numerically by calculating $\hat{r}(G)$ for several values of G.

The empirical likelihood confidence intervals cannot be disjoint because $\hat{r}(G)$ is always convex, because of the strict concavity of the function $\sum_{i \in S} \log(p_i)$ within Equation (10).

5. Simulation Studies

Two customary estimators can be found in the literature (e.g., Berger 2008; Langel and Tillé 2013). They are given by

$$\hat{G}_\pi^{(1)} := \frac{2}{\hat{N}\bar{y}_\pi} \sum_{i \in S} \frac{y_i}{\pi_i} \hat{F}_\pi(y_i) - 1, \tag{13}$$

$$\hat{G}_\pi^{(2)} := \frac{1}{2\hat{N}^2\bar{y}_\pi} \sum_{i \in S} \sum_{j \in S} \frac{1}{\pi_i \pi_j} |y_i - y_j|, \tag{14}$$

where

$$\hat{F}_\pi(y_i) := \frac{1}{\hat{N}} \sum_{i \in S} \frac{1}{\pi_i} I\{y_i < y\}.$$

In this section, we compare via simulation the proposed estimator $\hat{G}_\pi$ in Equation (6) with Equations (13) and (14). We also compare their variance estimators and coverages of their 95% confidence intervals. Our simulation study will show that the proposed estimator in Equation (6) can be less biased than Equations (13) and (14). The observed coverages of the empirical likelihood confidence interval are also closer to the nominal value.

We generated $N = 10{,}000$ population values y_i from different distributions as in Davidson (2009), Qin et al. (2010) and Peng (2011), namely the χ^2, exponential, lognormal, Pareto and Weibull distributions. The different values of G_0 defined by Equation (2) are given in Table 1. We selected 2,000 randomized systematic samples of size $n = 200$ and 500. The inclusion probabilities π_i are generated from a linear model with y_i as covariate, in order to obtain a correlation of 0.7 between π_i and y_i. We chose this correlation to highlight the effect of the design. We use $B = 1{,}000$ replicates for the bootstrap procedures.

In Table 1, we have the observed relative bias (RB) and mean squared error (MSE) given by

$$RB(\hat{G}) := \frac{\hat{E}(\hat{G}) - G_0}{G_0} \times 100\%,$$

$$MSE(\hat{G}) := \hat{E}\{(\hat{G} - G_0)^2\}$$

for $\hat{G} = \hat{G}_\pi, \hat{G}_\pi^{(1)}$ and $\hat{G}_\pi^{(2)}$. Here, $\hat{E}(\cdot)$ denotes the means over the 2,000 observed values. The RB of $\hat{G}_\pi$ is slightly smaller than with $\hat{G}_\pi^{(2)}$. The RB of $\hat{G}_\pi^{(1)}$ tends to be the smallest for large values of G_0. However, $\hat{G}_\pi^{(1)}$ has the largest RB when G_0 is small. The MSE of $\hat{G}_\pi$ and $\hat{G}_\pi^{(2)}$ are similar. The MSE of $\hat{G}_\pi^{(1)}$ is slightly larger when $n = 200$. With $n = 500$, all the MSE are similar. From Table 1, we conclude that $\hat{G}_\pi$ tends to have the smallest bias with a MSE comparable to one observed for $\hat{G}_\pi^{(2)}$.

Table 1. Relative bias (%) and mean squared error (MSE) of $\hat{G}_\pi$, $\hat{G}_\pi^{(1)}$ and $\hat{G}_\pi^{(2)}$ for several distributions. G_0 is given by (3). The rows are sorted according to G_0.

			Relative bias (%)			MSE × 10,000		
n	Distributions	G_0	$\hat{G}_\pi$	$\hat{G}_\pi^{(1)}$	$\hat{G}_\pi^{(2)}$	$\hat{G}_\pi$	$\hat{G}_\pi^{(1)}$	$\hat{G}_\pi^{(2)}$
200	Pareto($\alpha = 10$, $\beta = 1$)	0.05	−0.4	8.5	−0.9	0.2	0.4	0.2
	Weibull($\alpha = 10$, $\beta = 1$)	0.07	−0.7	6.3	−1.2	0.2	0.3	0.2
	Pareto($\alpha = 5$, $\beta = 1$)	0.11	1.0	5.1	0.6	0.8	1.1	0.8
	Weibull($\alpha = 5$, $\beta = 1$)	0.13	−0.6	2.8	−1.0	0.6	0.7	0.6
	$\Gamma(\alpha = 10, \beta = 1)$	0.18	−0.9	2.0	−1.3	1.2	1.5	1.2
	$\Gamma(\alpha = 5, \beta = 1)$	0.25	−2.0	0.2	−2.4	2.4	2.6	2.5
	LogN($\mu = 0$, $\sigma = 0.5$)	0.28	−0.6	0.9	−1.0	1.8	1.8	1.8
	Exp($\lambda = 1$)	0.50	−0.9	−0.3	−1.4	6.2	6.0	6.4
	χ^2_1	0.64	−1.9	−1.4	−2.3	8.8	8.1	9.7
	$\Gamma(\alpha = 0.2, \beta = 1)$	0.80	0.0	0.1	−0.3	3.5	3.5	3.6
500	Pareto($\alpha = 10$, $\beta = 1$)	0.05	−0.2	3.4	−0.3	0.1	0.1	0.1
	Weibull($\alpha = 10$, $\beta = 1$)	0.07	−0.7	2.1	−0.9	0.1	0.1	0.1
	Pareto($\alpha = 5$, $\beta = 1$)	0.11	0.9	2.6	0.8	0.3	0.4	0.3
	Weibull($\alpha = 5$, $\beta = 1$)	0.13	−0.5	0.9	−0.7	0.2	0.2	0.2
	$\Gamma(\alpha = 10, \beta = 1)$	0.18	−0.5	0.7	−0.7	0.5	0.5	0.5
	$\Gamma(\alpha = 5, \beta = 1)$	0.25	−1.7	−0.9	−1.9	1.1	1.0	1.1
	LogN($\mu = 0$, $\sigma = 0.5$)	0.28	−0.4	0.2	−0.6	0.7	0.7	0.7
	Exp($\lambda = 1$)	0.50	−0.9	−0.7	−1.1	2.5	2.4	2.6
	χ^2_1	0.64	−1.7	−1.5	−1.9	4.6	4.3	4.9
	$\Gamma(\alpha = 0.2, \beta = 1)$	0.80	−0.1	0.0	−0.2	1.4	1.4	1.4

In Table 2, we have the observed coverages of the 95% confidence intervals. For $\hat{G}_\pi$, we consider two confidence intervals: The "*bootstrap confidence interval*" based on the 2.5% and 97.5% quantiles of the bootstrap (column "*Boot*"), and the empirical likelihood confidence intervals in Equation (12) (column "*EL*"). The usual confidence intervals based on linearised variance estimates are used for $\hat{G}_\pi^{(1)}$ and $\hat{G}_\pi^{(2)}$. The quantity G_0 is the target parameter on which the confidence intervals are based. The relative bias of the variance estimator

$$RB\{\hat{V}(\hat{G})\} := \frac{\hat{E}\{\hat{V}(\hat{G})\} - V(\hat{G})}{V(\hat{G})} \times 100\%$$

is given in the last three columns, where $V(\hat{G})$ denotes the observed variance. The bootstrap variance is used for $\hat{G}_\pi$. For $\hat{G}_\pi^{(1)}$ and $\hat{G}_\pi^{(2)}$, we use the linearisation variance estimates (e.g., Berger 2008; Langel and Tillé 2013) based on Hartley and Rao's (1962) variance estimator.

The observed coverages of the empirical likelihood approach are usually not significantly different from 95%, when the other coverages are different from 95%. The low coverages of $\hat{G}_\pi^{(1)}$ and $\hat{G}_\pi^{(2)}$ can be explained by lack of normality. With small values of G_0, the lower bounds of $\hat{G}_\pi^{(1)}$ and $\hat{G}_\pi^{(2)}$ can be negative. This could also explain the low coverage of $\hat{G}_\pi^{(1)}$. When the coverage of the empirical likelihood approach is significantly

Table 2. Observed coverages (%) of 95% confidence intervals of $\hat{G}_\pi$ (bootstrap and empirical likelihood), $\hat{G}_\pi^{(1)}$ and $\hat{G}_\pi^{(2)}$. Relative bias RB$\{\hat{V}(\hat{G})\}$ *(%), of the bootstrap variance estimator of $\hat{G}_\pi$ and the linearised variance of $\hat{G}_\pi^{(1)}$ and $\hat{G}_\pi^{(2)}$. Several distributions are considered. The rows are sorted according to G_0 (see Table 1 for the values of G_0).*

		Coverages (%)						
		$\hat{G}_\pi$				RB$\{\hat{V}(\hat{G})\}$ (%)		
n	Distributions	Boot	(11)	$\hat{G}_\pi^{(1)}$	$\hat{G}_\pi^{(2)}$	$\hat{G}_\pi$	$\hat{G}_\pi^{(1)}$	$\hat{G}_\pi^{(2)}$
200	Pareto($\alpha = 10$, $\beta = 1$)	94.2	94.6	83.2†	94.0†	2.6	1.6	1.6
	Weibull($\alpha = 10$, $\beta = 1$)	93.5†	94.6	85.6†	93.4†	4.8	3.7	4.0
	Pareto($\alpha = 5$, $\beta = 1$)	94.4	95.1	92.6†	94.3	4.4	2.1	2.2
	Weibull($\alpha = 5$, $\beta = 1$)	93.5†	94.5	93.3†	93.3†	3.9	2.7	3.2
	$\Gamma(\alpha = 10, \beta = 1)$	91.1†	92.9†	94.6	90.6†	−7.6	−17.5	−5.5
	$\Gamma(\alpha = 5, \beta = 1)$	89.4†	92.7†	93.4†	89.1†	−10.9	−2.6	−9.9
	LogN($\mu = 0$, $\sigma = 0.5$)	93.7†	95.4	94.8	93.7†	−1.0	−1.7	−2.0
	Exp($\lambda = 1$)	90.7†	92.7†	92.7†	90.4†	−13.7	−14.9	−14.4
	χ^2_1	89.1†	90.4†	90.8†	89.3†	−19.7	−20.8	−18.8
	$\Gamma(\alpha = 0.2, \beta = 1)$	94.7	95.4	94.4	94.6	−5.7	0.0	−2.8
500	Pareto($\alpha = 10$, $\beta = 1$)	95.1	95.2	90.1†	94.3	4.9	−0.9	−0.9
	Weibull($\alpha = 10$, $\beta = 1$)	93.5†	94.6	85.6†	93.4†	4.8	3.7	4.0
	Pareto($\alpha = 5$, $\beta = 1$)	95.1	94.8	93.1†	94.3	4.2	−3.6	−3.9
	Weibull($\alpha = 5$, $\beta = 1$)	93.2†	95.1	94.4	93.0†	0.6	−3.7	−3.5
	$\Gamma(\alpha = 10, \beta = 1)$	93.4†	94.7	94.4	93.2†	1.5	−2.7	−0.5
	$\Gamma(\alpha = 5, \beta = 1)$	88.6†	92.9†	92.3†	88.3†	−1.2	2.2	−1.9
	LogN($\mu = 0$, $\sigma = 0.5$)	94.4†	95.4	94.6	94.2	−1.0	−1.7	−2.0
	Exp($\lambda = 1$)	93.1†	93.8†	93.7†	92.8†	−6.0	−7.7	−8.2
	χ^2_1	87.7†	87.5†	89.0†	87.6†	3.0	−1.6	0.6
	$\Gamma(\alpha = 0.2, \beta = 1)$	95.5	96.0	94.6	95.2	8.6	0.6	1.6

† Coverage rates significantly different from 95%: p-value $\leq$ 0.05.

different from 95%, the other coverages are also significantly different (distributions $\Gamma(\alpha = 5, \beta = 1)$, $Exp(\lambda = 1)$ and χ^2_1). The distribution $\Gamma(\alpha = 10, \beta = 1)$ is an exception, because $\hat{G}_\pi^{(1)}$ has the best coverage, but with a biased variance estimator. We have observed one sample of size $n = 200$ with a negative lower bound for the confidence interval of Equation (13). This occurs with the data generated from a χ^2-distribution.

The RB of the variance of $\hat{G}_\pi$ can be large with $n = 200$, because they are based on bootstrap. However, with $n = 500$, all the RB are similar, and $\hat{G}_\pi$ may have the smallest RB. When $n = 200$, we have larger RB for large values of G_0 (distributions $Exp(\lambda = 1)$ and χ^2_1 and $\Gamma(\alpha = 0.2, \beta = 1)$).

In Table 3, we have the observed average length of the 95% confidence intervals, as well as the observed "*coefficient of variation*" (CV) of the lengths. The average length is very similar and in line with the coverages observed in Table 2, because confidence intervals with large coverage tend to be larger on average.

A small CV implies more stable confidence intervals, but this does not imply observed coverages closer to 95%. The CV of the bootstrap confidence intervals tends to be the smallest, but with observed coverage significantly different from 95%. For the Pareto and Weibull distribution, the CV of Equation (11) is slightly larger than the other confidence intervals, which have coverages usually different from 95%. This effect is more

Table 3. Observed Average Length of 95% confidence intervals and observed coeffcient of the lengths variation (CV) in percent. The rows are sorted according to G_0 (see Table 1 for the values of G_0).

		Average Lengths				CV(Lengths) %			
		$\hat{G}_\pi$				$\hat{G}_\pi$			
n	Distributions	Boot	(11)	$\hat{G}_\pi^{(1)}$	$\hat{G}_\pi^{(2)}$	Boot	(11)	$\hat{G}_\pi^{(1)}$	$\hat{G}_\pi^{(2)}$
200	Pareto($\alpha = 10, \beta = 1$)	0.017	0.018	0.017	0.017	16.4	18.1	16.2	16.2
	Weibull($\alpha = 10, \beta = 1$)	0.016	0.016	0.016	0.016	13.5	13.3	13.3	13.3
	Pareto($\alpha = 5, \beta = 1$)	0.036	0.037	0.035	0.035	14.6	15.1	14.2	14.2
	Weibull($\alpha = 5, \beta = 1$)	0.030	0.030	0.029	0.030	11.8	11.7	11.5	11.5
	$\Gamma(\alpha = 10, \beta = 1)$	0.039	0.040	0.041	0.039	23.1	38.4	66.9	28.5
	$\Gamma(\alpha = 5, \beta = 1)$	0.054	0.056	0.057	0.054	23.7	37.8	43.8	28.8
	LogN($\mu = 0, \sigma = 0.5$)	0.052	0.053	0.052	0.052	11.9	12.7	15.9	13.3
	Exp($\lambda = 1$)	0.089	0.092	0.089	0.090	19.3	30.9	26.2	23.7
	χ^2_1	0.089	0.095	0.091	0.091	27.4	38.5	40.1	33.8
	$\Gamma(\alpha = 0.2, \beta = 1)$	0.076	0.077	0.074	0.076	9.0	9.5	8.1	9.0
500	Pareto($\alpha = 10, \beta = 1$)	0.011	0.011	0.011	0.011	11.0	11.1	10.4	10.4
	Weibull($\alpha = 10, \beta = 1$)	0.010	0.010	0.010	0.010	8.6	8.6	8.3	8.3
	Pareto($\alpha = 5, \beta = 1$)	0.023	0.023	0.022	0.022	9.3	9.4	8.6	8.6
	Weibull($\alpha = 5, \beta = 1$)	0.019	0.019	0.019	0.019	8.2	8.1	7.6	7.6
	$\Gamma(\alpha = 10, \beta = 1)$	0.025	0.026	0.026	0.025	20.9	47.5	79.4	23.9
	$\Gamma(\alpha = 5, \beta = 1)$	0.036	0.037	0.036	0.035	22.0	37.6	33.9	26.8
	LogN($\mu = 0, \sigma = 0.5$)	0.033	0.033	0.032	0.032	9.9	9.9	12.0	10.9
	Exp($\lambda = 1$)	0.058	0.060	0.058	0.058	20.1	34.9	27.5	23.9
	χ^2_1	0.060	0.064	0.060	0.061	32.3	52.4	49.5	39.1
	$\Gamma(\alpha = 0.2, \beta = 1)$	0.048	0.048	0.046	0.046	5.7	6.0	5.1	5.1

pronounced with $n = 200$. With the Gamma and χ^2-distributions, we have a small CV with bootstrap and $\hat{G}_{\pi}^{(2)}$, but with very low coverages.

6. Discussion

Our simulation study shows the proposed estimator is usually less biased than the customary estimators. The observed coverages of the empirical likelihood confidence interval are also closer to the nominal value. We considered single stage design. However, the proposed approach can be extended for multi-stage designs with unit nonresponse, using Berger's (2018a) approach combined with bootstrap. Auxiliary information has not been considered for simplicity. Calibration weights can be used within Equation (6). The proposed empirical likelihood approach can also take into account some auxiliary information, by adding additional constraints within Equation (10) (see Berger and Torres 2016; Berger 2018a,b, for more details). These additional constraints imply that $\hat{G}_{EL}$ will be different but usually close to $\hat{G}_{\pi}$, because $\hat{G}_{EL}$ is based on calibrated weights.

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