Probabilistic zero bounds of certain random polynomials

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Abstract

This paper introduces the notion of probabilistic zero bounds for random polynomials. It presents new results regarding the probabilistic bounds of random polynomials whose coefficients are independently and identically distributed as standard normal variates. Additionally, the paper provides a clear exposition of the developed methodology. To establish our results, we develop a novel approach utilizing the classical Cauchy's bounds for the zeros of a deterministic polynomial with complex coefficients. We also corroborate our analytical results with extensive simulations. The methodology developed in the paper can potentially be applied to a broad class of problems regarding bounds and the distribution of zeros in the theory of random polynomials.

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1. INTRODUCTION

The field of random polynomials, which originated from the seminal work of Bloch and Polya in 1932, has since developed into a thriving area of research [1; 2]. Their pioneering contributions have paved the way for foundational investigations by Littlewood and Offord [2; 3; 4]. In the 1940s, Kac devised a formula, now referred to as Kac-formula, which stands as one of the most significant outcomes in the field [5]. This formula furnishes an asymptotic expression for the expected number of real roots of high-degree random polynomials. The topic of the distribution of zeros of random polynomials has been extensively studied, with contributions coming from works like [3; 4; 6; 7; 8; 9], as well as the detailed exposition by Bharucha-Reid and Sambandham [2]. As per Kac's formula, the expected number of real zeros of a degree n polynomial with independent standard normal coefficients is

$$\begin{aligned}\mathbb{E}N_n(\mathbb{R}) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\sqrt{\frac{1}{(t^2-1)^2}-\frac{(n+1)^2t^{2n}}{(t^{2n+2}-1)^2}}\,dt \\ &= \frac{4}{\pi}\int_{0}^{1}\sqrt{\frac{1}{(1-t^2)^2}-\frac{(n+1)^2t^{2n}}{(1-t^{2n+2})^2}}\,dt\end{aligned}$$

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Kac [5] further showed that

$$\mathbb{E}N_n(\mathbb{R}) \sim \frac{2}{\pi}\log n.$$

But several researchers have sharpened Kac's original estimate, one can see [2] for a rigorous treatment of this estimate.

In their elegant paper Edelman and Kostlan [9] provide an elementary geometric derivation of the Kac integral formula, similar to the Buffon's needle, problem for the expected number of real zeros of a random polynomial with independent standard normally distributed coefficients. When the coefficients of the polynomial are independent and conform to a standard normal distribution, Edelman and Kostlan [10], proved the following:

Theorem (Edelman-Kostlan) *Let P_n be the Kac polynomial of degree n. Then as $n \to \infty$, the expected number of real zeros of the polynomial is given by:*

$$\mathbb{E}N_n(\mathbb{R}) = \frac{2}{\pi}\log(n) + 0.6257358072\ldots + \frac{2}{n\pi} + O\left(\frac{1}{n^2}\right). \tag{1}$$

While the distribution of roots for large-degree polynomials and the concentration phenomena have been extensively investigated in [2; 8; 9; 11; 12; 13], the probabilistic bounds for finite and low-degree polynomials remain largely unexplored. In the study of random polynomials, it is of significant interest to determine the likelihood that the roots of such polynomials are confined within a certain bound. More formally, consider the ensemble $\mathscr{P}_n$ of random polynomials of the form

$$P(z) = \sum_{k=0}^{n} a_k z^k, \tag{2}$$

where the coefficients a_k adhere to a given probability distribution. With respect to the uniform probability measure $d\mu_P$ on $\mathscr{P}_n$, we seek to ascertain lower bounds on the probability that a given positive constant c serves as a bound for the roots of a polynomial randomly selected from $\mathscr{P}_n$. That is, we aim to determine the probability p such that the roots of the polynomial $P(z)$ lie within the disc $|z| \leq c$ with at least probability p. Computing this exact probability is arguably a formidable problem and, to make a start, we propose and present a method to compute the bounds of this probability.

Method and Motivation: Given a positive number c, and a random polynomial with *iid* standard Gaussian coefficients, the question of determining the probability that c will act as the upper bound for the zeros may appear a very simple one, especially for

large n in the light of the Circular laws [11] which guarantee that for polynomials of high degree with independently and identically (*iid*) coefficients the zeros tend to cluster around the unit circle with high probability. However, we only need to recall that the convergence is probabilistic and the distribution of the zeros with the largest modulus is a different question. Moreover, the determination of that probability for finite n is an absolutely non-trivial problem. However, it is possible to establish probabilistic bounds for this occurrence. These probabilistic zero bounds are potentially of crucial significance for understanding the typical root locations of finite-degree random polynomials as well as providing asymptotic root bounds for polynomials of high degree. In this paper, we introduce new findings on the probabilistic bounds for the roots of random polynomials whose coefficients are distributed according to the standard normal distribution by applying Cauchy's classic root bounds for complex polynomials. To that end, we propose an approach that can be adapted to a broad range of problems about the probabilistic bounds for roots of random polynomials. More specifically, the method consists in starting with any known deterministic bound of a polynomial, which generally happens to be some function of the coefficients, say $f(a_0, a_1, \cdots, a_n)$. For a random polynomial, the quantity $f(a_0, a_1, \cdots, a_n)$ is a random variable. The essence of the method is to use probabilistic methods and inference to elicit information about the location of zeros of the polynomial. It is worth-noting that the proposed method can be adapted to a wide variety of problems in polynomial theory and despite its straightforward adaptability it has the potential to deliver significant results to some otherwise very difficult problems regarding the location of zeros of random polynomials.

The remainder of the article is structured as follows: In Section 2, we present prerequisites in the form of definitions and lemmas. The main results regarding the probabilistic bounds for roots of polynomials with coefficients distributed as standard normal variables are presented in Section 3. In Section 4, we discuss the numerical simulations of the obtained results. This is followed by the Conclusion in section 5 and finally the suggested Future work in section 6.

2. SOME PRELIMINARIES AND DEFINITIONS

In this section, we introduce the notion of probabilistic bound along with a formal definition and present another definition related to the transitivity of random variables as well as two lemmas for subsequent developments.

DEFINITION 1. *(Probabilistic Bound)*: For a class of $\mathscr{P}_n$ of random polynomials $P(z) = \sum_{0}^{n} a_n z^n$ with a prescribed distribution on its coefficients and degree n, let $d\mu_P$ be the uniform probability measure on $\mathscr{P}_n$ a positive number c is a probabilistic bound for the roots with probability p, if the roots of randomly picked polynomial in $\mathscr{P}_n$ with probability measure $d\mu_P$ lie within the disc $|z| \leq c$ with probability p.

LEMMA 1. *(Probabilistic Transitivity)*: *If $X(\omega)$ and $Y(\omega)$ are random variables such that, $X(\omega) \leq Y(\omega)$ and the distribution function of Y is $F(y)$, then for any positive number c :*

$$Y \leq c \implies \mathbb{P}(X \leq c) \geq F(c).$$

PROOF. Since $X(\omega) \leq Y(\omega) \leq c$,we have $\{Y(\omega) \leq c\} \subset \{X(\omega) \leq c\}$. Hence, it immediately follows that

$$\mathbb{P}(X(\omega) \leq c) \geq \mathbb{P}(Y(\omega) \leq c).$$

As $\mathbb{P}(Y(\omega) \leq c)$ is precisely equal to $F(c)$, the Lemma is established.

LEMMA 2. *(Cauchy's theorem for a monic polynomial)*

The polynomial $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ has all its roots in the interval

$$|z| \leq 1 + M \qquad ,where \qquad M = \max_{0 \leq j \leq n-1} |a_j|. \tag{3}$$

LEMMA 3. *(Cauchy's theorem for a general polynomial)*

The roots of the polynomial $P(z) := \sum_{i=1}^{n} a_n z^i$ lie in the disc

$$|z| \leq 1 + \max_{0 \leq i \leq n-1} \left\{ \left| \frac{a_i}{a_n} \right| \right\}. \tag{4}$$

For the proof of Lemma 2 and Lemma 3 above, one may consult [14]. For more details into the probabilistic preliminaries and distribution theory, one may consult any standard book on probability theory such as [15; 16]

3. MAIN RESULTS

In the following theorems, we present some results regarding the probabilistic bounds for roots of monic polynomials with coefficients distributed as standard normal variables.

THEOREM 1. *Consider the ensemble $\mathscr{P}_n$ of random polynomials $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ of degree n, and let $d\mu_P$ be the uniform probability measure on $\mathscr{P}_n$, where $a_i \sim N(0,1)$ are independently distributed standard normal random variables. Then, a real number $c > 1$ is a probabilistic bound for the roots of the polynomial $P(z)$, with a probability at least p, if*

$$c > \sqrt{2}\,\mathrm{erf}^{-1}\left(p^{1/n}\right) + 1, \tag{5}$$

where erf^{-1} *is the inverse error function.*

PROOF. Consider a polynomial $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ chosen with measure $d\mu_P$ from the ensemble $\mathscr{P}_n$, where $a_i \sim N(0,1)$ are independently distributed standard normal random variables. By Lemma 2, the polynomial $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ has all its roots in the disc $|z| \leq 1 + M$, where $M = \max\limits_{0\leq j\leq n-1} |a_j|$. Denote by $F_X(x)$ the distribution of $a_k \sim \mathbb{N}(0,1)$, and the distribution of $Y = |a_k|$ by $F_Y(y)$. Then by definition of the cumulative distribution function, we have

$$\begin{aligned} F_Y(y) &= P[Y \leq y] \\ &= P[-y \leq a_k \leq y] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-y}^{y} e^{-y^2/2} dy \\ &= F_X(y) - F_X(-y). \end{aligned}$$

We recall from distribution theory:
If $X \sim N(0,1)$, then the CDF $F_X(x)$ in terms of the error function is given by

$$F_X(x) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right), \tag{6}$$

where the error function is given by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2/2} dt.$$

From equation **(6)**, it follows that the distribution of $Y = |X|$ is given by :

$$F_Y(y) = \frac{1}{2}[(1+erf(\frac{y}{\sqrt{2}})) - (1+erf(\frac{-y}{\sqrt{2}}))] \tag{7}$$

$$= erf(\frac{y}{\sqrt{2}}). \tag{8}$$

Hence, from equation **(8)** it follows that the distribution of $V = 1 + \max_{0\leq j\leq n-1} |a_j|$ is given by

$$(F_Y(y-1))^n. \tag{9}$$

Using the distribution of $\max_{0\leq j\leq n-1} |a_j|$, the expression (5) yields:

$$\left(erf\left(\frac{y-1}{\sqrt{2}}\right)\right)^n. \tag{10}$$

Therefore, if p is the probability that the roots of $P(z)$ are all less than a number c, then

$$p \geq erf\left(\frac{c-1}{\sqrt{2}}\right)^n. \tag{11}$$

Hence, c is a bound with a probability of at least p if

$$erf\left(\frac{c-1}{\sqrt{2}}\right) \leq p^{\frac{1}{n}}. \tag{12}$$

Noting the inverse of the error function is a decreasing function, inequality (**12**) can be recast as

$$c > \sqrt{2}\,\mathrm{erf}^{-1}\left(p^{1/n}\right) + 1.$$

This completes the proof of Theorem 1.

REMARK. Setting $p = 0.99$ in inequality (5), we see that 6.32 will be a bound of the roots of $P(z)$ of degree 100000, with at least 99 percent of probability.

THEOREM 2. *Consider the ensemble $\mathscr{P}_n$ of random polynomials $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ of degree n, where the coefficients $a_i \sim N(0,1)$ for $i = 0,1,\ldots,n-1,n$ are independently distributed standard normal random variables. Further, let $d\mu_P$ be the uniform probability measure on $\mathscr{P}_n$. The roots of a randomly selected polynomial $P(z)$ from this ensemble lie within the disc $|z| \leq c$ with a probability at most*

$$\mathrm{erf}\left(\frac{c^n}{\sqrt{2}}\right). \tag{13}$$

PROOF. For the polynomial $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ chosen from the ensemble $\mathscr{P}_n$ with uniform measure $d\mu P_n$, let $\xi_1, \xi_2, \ldots, \xi_n$ be the n roots of $P(z)$. Then, from Vieta's formula,

$$|\xi_1 \cdot \xi_2 \cdots \xi_n| = |a_0|$$

If $\xi_{\max} = \max_{1 \le i \le n-1} |\xi_i|$, we have

$$\xi_{\max}^n \geq |\xi_1 \cdot \xi_2 \cdots \xi_n| = |a_0|,$$

which leads to

$$\xi_{\max} \geq |a_0|^{1/n}.$$

Now, for the random variable $W = |a_0|^{1/n}$, we have the distribution function $F_W(w)$ of W, is given by

$$F_W(w) = P\left[|a_0|^{1/n} \leq w\right] = P\left[|a_0| \leq w^n\right]. \tag{14}$$

Using the expression for commutative distribution function(cdf) of the modulus of a standard normal variate from (9), we obtain :

$$F_W(w) = \operatorname{erf}\left(\frac{w^n}{\sqrt{2}}\right). \tag{15}$$

Considering $\xi_{\max} > w > c$, we have

$$P\left[\xi_{\max} > c\right] \geq P[w > c],$$

or equivalently,

$$1 - P\left[\xi_{\max} \leq c\right] \geq 1 - F_w(c)$$

or,

$$P\left[\xi_{\max} \leq c\right] \leq F_w(c). \tag{16}$$

Now, using equation (15), the inequality (16) yields

$$P\left[\xi_{\max} \leq c\right] \leq \operatorname{erf}\left(\frac{c^n}{\sqrt{2}}\right).$$

Therefore, it follows that the probability of all roots of $P(z)$ lying in $|z| \leq c$ is at most $\operatorname{erf}\left(\frac{c^n}{\sqrt{2}}\right)$.

This completes the proof of the Theorem 2.

REMARK. It is worth noting the Theorem 1 gives us the lower bounds of the probability that all the roots of $P(z)$ lie within $|z| \leq c$, while the Theorem 2 provides us the upper bounds of the same probability, thus giving us a better idea about the probabilistic bounds for roots of monic polynomial with coefficients distributed as standard normal variables.

In the next theorem, we obtain the probabilistic bounds for roots of general polynomial with coefficients distributed as standard normal variables.

THEOREM 3. *Consider the ensemble $\mathscr{Q}_n$ of random polynomials $Q(z) = \sum_{i=1}^{n} a_i z^i$ of degree n, where the coefficients $a_i \sim \mathbb{N}(0,1)$ for $i = 1,\ldots,n$ are independently distributed standard normal random variables, and let $d\mu_Q$ be the uniform probability measure on $\mathscr{Q}_n$. A positive real number $c > 1$ serves as a probabilistic bound for the roots of a randomly selected polynomial $Q(z)$ from this ensemble with a probability of at least p, if*

$$c \geq 1 + \tan\left(\frac{\pi}{2} p^{1/n}\right). \tag{17}$$

Before proceeding with the proof of the theorem we establish the following lemma:
Lemma A: Distribution of Modulus of the ratio of two standard normal variables: *Suppose X_1 and X_2 are two independently and identically distributed standard random variables in the interval and*

$$W = \left|\frac{X_1}{X_2}\right|,$$

then Y follows a CDF given by:

$$F_W(w) = \frac{2}{\pi} tan^{-1}(w) \tag{18}$$

Proof of Lemma A: In order to derive the cdf in equation **(18)**, let $X_1, X_2 \sim \mathbb{N}(0,1)$ and

$$W = \left|\frac{X_1}{X_2}\right|$$

The random variable, $Y = X_1/X_2$ being the ratio of two standard normal variates follows a standard Cauchy distribution with cdf

$$F_Y(y) = \frac{1}{\pi} tan^{-1}(y) + \frac{1}{2} \tag{19}$$

Hence the distribution of $W = |Y|$ can be computed as :

$$\begin{aligned} F_W(w) &= P(W \leq w) \\ &= P(|Y| \leq w) \\ &= P(-w \leq Y \leq w) \\ &= F_Y(w) - F_Y(-w) \\ &= \frac{1}{\pi}\left[\tan^{-1}(w) - \tan^{-1}(-w)\right] \\ &= \frac{2}{\pi}\tan^{-1}(w) \end{aligned}$$

which establishes lemma **A**.

Proof of Theorem: Coming back to the polynomial $Q(z) = \sum_{i=1}^{n} a_n z^i$, with $a_i \sim N(0,1)$. We note, by Lemma **A** the bound Y of the roots of $Q(z)$ is given by

$$Y = 1 + \max_{0 \leq i \leq n-1} \left\{ \left| \frac{a_i}{a_n} \right| \right\}.$$

Hence, in the light of lemma **A** the distribution of Y is given by

$$F_Y(y) = \left[\frac{2}{\pi}\tan^{-1}(y-1) \right]^n.$$

Thus in view of Lemma 1, a number c is a bound with a probability of at least

$$p \geq \left[\frac{2}{\pi}\tan^{-1}(y-1) \right]^n.$$

Equivalently,

$$\frac{2}{\pi}\tan^{-1}(c-1) \geq p^{1/n},$$

or

$$\tan^{-1}(c-1) \geq \frac{\pi}{2}\left(p^{1/n}\right).$$

Which leads to

$$c \geq 1 + \tan\left[\frac{\pi}{2}\left(p^{1/n}\right)\right].$$

This completes the proof of Theorem 3.

REMARK. Substituting $n = 5$, and $p = 0.99$ in inequality (13), we find 13.91 acts a bound for the roots of polynomial $Q(z)$ with a probability of at least 0.99.

4. DISCUSSION OF NUMERICAL SIMULATION RESULTS

A numerical study of the formula in Theorem 1 reveals that for a monic random polynomial of degree 5 with coefficients as independently and identically distributed standard random variables, $c = 2$ is bound with a probability of at least 14 percent. While the actual probability of 91 percent makes that a poor estimate, we however also observe from a comparison of graphs in Fig 1 and Fig 2 that the lower bound on the probability of c to bound the roots of $P(z)$ goes on improving very rapidly as we go on increasing the value of c. For $c = 3$, the lower bound of probability from Theorem 1 is 62 percent, while the actual probability computed from simulation is 98 percent, which is a sharp improvement. Taking $c = 6.32$, we find from Theorem 1 that $c = 6.32$ is a bound of the roots with a probability of at least 99 percent. Numerical simulations also reveal that the average of lower and upper bounds given by Theorem 1 and Theorem 3, is a much better estimate for the actual probability computed from simulations. It is also interesting to mention that the lower bounds of probability by Theorem 2, give us much better estimates for the actual probability in comparison to those given by Theorem 1 for $c = 2$. The next two figures, Figure 3 and Figure 4, depict the probability of a number being of bound of $Q(z)$ from Theorem 3 and Monte Carlo simulations respectively, while Figures 5 and 6 show distribution of the root with maximum modulus for random polynomials $P(z)$ and $Q(z)$. The distribution of the zeros of a random non-monic polynomial with *iid* normal coefficients with degree 1000 has been depicted in Figure 7. Finally, Figure 8 shows the expected modulus of the largest root for polynomials of degrees 1 through 20, represented by the red line with circle markers. The expected Cauchy bound for these polynomials is represented by the blue line with square markers. The expected modulus of the largest root tends to increase with the degree of the polynomial. This trend suggests that higher-degree polynomials are more likely to have roots with larger moduli. The expected Cauchy bound, which is a theoretical upper bound on the modulus of all roots, also increases with the degree of the polynomial. However, it appears to grow at a faster rate than the actual expected modulus of the largest root, which indicates that while the Cauchy bound is a useful estimate, it is not tight. Though the Cauchy bound serves as a ready bound for the zeros of a polynomial, for more precise estimations or sharper bounds, one would likely need to rely on numerical simulations or more refined theoretical bounds tailored to the specific properties of the coefficients of the polynomials.

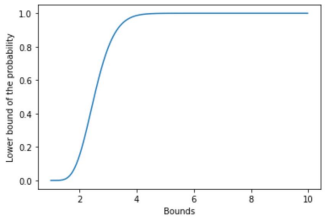

Fig. 1. Simulation of probabilistic bounds for roots of $P(z)$ for $n = 5$ by Theorem 1

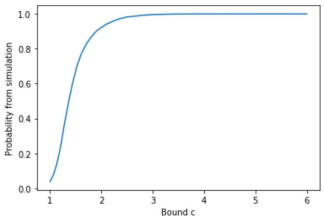

Fig. 2. Probabilistic bounds for roots of $P(z)$ for $n = 5$ by Monte Carlo simulation

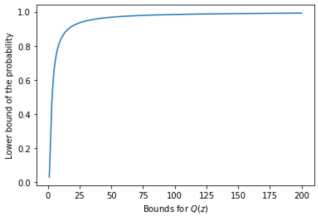

Fig. 3. Simulation of probabilistic bounds for roots of $Q(z)$ for $n = 5$ by Theorem 3

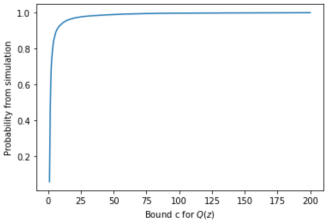

Fig. 4. Probabilistic bounds for roots of $Q(z)$ for $n = 5$ by Monte Carlo simulation

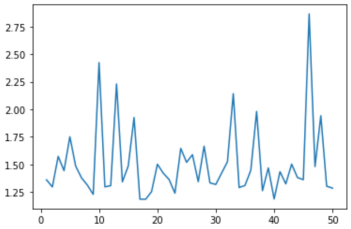

Fig. 5. Distribution of the root with maximum modulus for $P(z)$

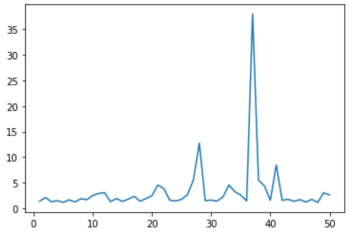

Fig. 6. Distribution of the root with maximum modulus for $Q(z)$

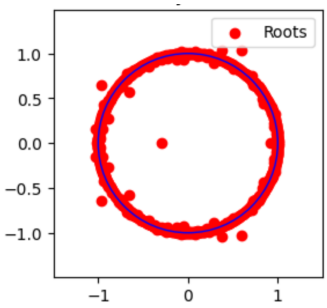

Fig. 7. Distribution of the roots with respect to unit circle for $Q(z)$

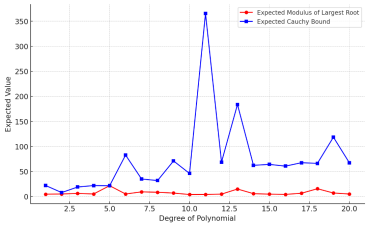

Fig. 8. Distribution of the expected maximum modulus and Cauchy bound for $Q(z)$

5. CONCLUSION

In this paper, we introduce the notion of probabilistic zero bounds and, through a novel yet simple method, illustrate its application by deriving some interesting theorems utilizing the classical bounds for zeros of a polynomial due to Cauchy. Our investigation into the probabilistic bounds of roots of random polynomials with independent and identically distributed standard normal coefficients has yielded concrete results that contribute to the existing body of knowledge on the distribution of zeros in random polynomials, exemplifying how classical mathematical concepts, such as Cauchy's theorem, can be leveraged in a probabilistic framework to yield insights into the behavior of complex random polynomials. The expected number of zeros within the unit disk is derived, adding to our understanding of the distribution of zeros for polynomials with random coefficients through an elegant symmetry argument. The methodological approach adopted in this paper—utilizing Cauchy's theorem and probabilistic logic to infer the locations of zeros—is noteworthy for its simplicity and effectiveness. This approach could potentially be generalized to a wider class of problems in polynomial theory, offering a robust toolset for tackling questions about root locations in random polynomial settings.

6. FUTURE WORK

The future work stemming from this paper's methodology opens several promising avenues for exploration. As a method paper that innovates upon the classical bounds for the zeros of polynomials, we envision extending the probabilistic zero-bounds approach to other well-known bounds beyond those of Cauchy. Several classical bounds, such as the ones established by Kalantari [18], Dehmer [19], Datt and Govil

[17] and plethora of other bounds offer different perspectives on the locations of polynomial zeros. Applying the probabilistic framework developed in this paper to these bounds could yield new insights into the distribution of zeros for random polynomials with various distributions of coefficients.

Conflict of Interests: The authors have no conflict of interest to declare.

Data Availability: The authors declare that they have not used any third-party or copyrighted data in their simulations and all the programs used can be made available on request.

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