Fractional Simpson like type inequalities for differentiable s-convex functions

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Abstract

In this paper, based on new identity we establish some new Simpson like type inequalities for functions whose first derivatives are s-convex via Riemann-Liouville fractional integrals. The case where the derivatives are bounded as well as the case where the derivatives satisfy the Hölder condition are also discussed. The obtained results extend some known results and refine another one. Applications of the results are given at the end.

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1. INTRODUCTION

Let I be an interval of real numbers

Definition 1.1. [15] A function $f : I \to \mathbb{R}$ is said to be convex, if

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

The theory of convexity plays a central and attractive role in many fields of research. This theory provides us with a powerful tool for solving a large class of problems that arise in pure and applied mathematics. In recent years, the concept of convexity has been improved, generalized and extended in many directions. Among these generalizations, we note that introduced by Breckner called s-convexity and can be defined as follows

Definition 1.2. [6] A nonnegative function $f : I \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense for some fixed $s \in (0,1]$, if

$$f(tx+(1-t)y) \leq t^s f(x)+(1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

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The concept of convexity has a close relationship in the development of the theory of inequalities. The combination of these theories has attracted a lot of attention from researchers due to their nature and properties.

One of the most important and widely requested inequalities is that of Simpson which can be stated as follows

$$\left|\tfrac{1}{6}\left(f(a)+4f\left(\tfrac{a+b}{2}\right)+f(b)\right)-\tfrac{1}{b-a}\int_a^b f(u)\,du\right|\leq\tfrac{(b-a)^4}{2880}\left\|f^{(4)}\right\|_\infty, \tag{1}$$

where f is four-times continuously differentiable function on (a,b), and $\left\|f^{(4)}\right\|_\infty=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|$.

Concerning some papers related to the inequality (1) see [1; 2; 3; 4; 5; 7; 8; 9; 10; 13; 16; 18], and references therein.

Recently, Shuang et al. [17] discussed an inequality similar to (1), Among the obtained results for convex functions, we quote

$$\begin{aligned}&\left|\tfrac{1}{8}\left(f(a)+6f\left(\tfrac{a+b}{2}\right)+f(b)\right)-\tfrac{1}{b-a}\int_a^b f(t)\,dt\right|\\ \leq\ &\tfrac{b-a}{768}\left(19\left|f'(a)\right|+82\left|f'\left(\tfrac{a+b}{2}\right)\right|+19\left|f'(b)\right|\right),\end{aligned}$$

$$\begin{aligned}&\left|\tfrac{1}{8}\left(f(a)+6f\left(\tfrac{a+b}{2}\right)+f(b)\right)-\tfrac{1}{b-a}\int_a^b f(t)\,dt\right|\\ \leq\ &\tfrac{b-a}{16}\left(\tfrac{1+3^{p+1}}{4(p+1)}\right)^{\frac{1}{p}}\left(\left(\tfrac{\left|f'(a)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\tfrac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}\right)\end{aligned}$$

and

$$\begin{aligned}&\left|\tfrac{1}{8}\left(f(a)+6f\left(\tfrac{a+b}{2}\right)+f(b)\right)-\tfrac{1}{b-a}\int_a^b f(t)\,dt\right|\\ \leq\ &\tfrac{5(b-a)}{64}\left(\left(\tfrac{19\left|f'(a)\right|^q+41\left|f'\left(\frac{a+b}{2}\right)\right|^q}{60}\right)^{\frac{1}{q}}+\left(\tfrac{41\left|f'\left(\frac{a+b}{2}\right)\right|^q+19\left|f'(b)\right|^q}{60}\right)^{\frac{1}{q}}\right).\end{aligned}$$

In [14], Luo et al. gave the analogue weighted version of the result given by Shuang et al. [14]. They also discussed the cases where the first derivatives are bounded and Lipschitzian, of which we quote two of the established results: For functions whose first derivatives are bounded i.e. $m\leq f(x)\leq M$ for all $x\in[a,b]$, we have

$$\left|\tfrac{1}{8}\left(f(a)+6f\left(\tfrac{a+b}{2}\right)+f(b)\right)-\tfrac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\tfrac{9(b-a)(M-m)}{64}.$$

For functions whose first derivatives satisfies a Lipschitz condition i.e. $|f'(x)-f'(y)|\leq$

$L|x-y|$ for all $x,y\in[a,b]$, we have

$$\left|\frac{1}{8}\left(f(a)+6f\left(\frac{a+b}{2}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{41(b-a)^2L}{768}+\frac{b-a}{16}\left(f'(a)+f'(b)\right).$$

Very recently, Kirmaci [12], gave the following inequalities for s-convex and convex first derivatives

$$\begin{aligned}&\left|\frac{1}{8}\left(f(a)+6f\left(\frac{a+b}{2}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ \leq\ &\frac{(b-a)\left(3^{p+1}+1\right)^{\frac{1}{p}}}{8\times16^{\frac{1}{p}}}\left(\frac{1}{2^{\frac{s+1}{q}}}+\left(1-\frac{1}{2^{s+1}}\right)^{\frac{1}{q}}\right)\left(|f'(a)|+|f'(b)|\right)\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{1}{8}\left(f(a)+6f\left(\frac{a+b}{2}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ \leq\ &\frac{(b-a)\left(3^{p+1}+1\right)^{\frac{1}{p}}}{16\times8^{\frac{1}{p}}}\left(|f'(a)|+2\left|f'\left(\frac{a+b}{2}\right)\right|+|f'(b)|\right).\end{aligned}$$

In last decades the fractional calculus has attracted the attention of many researchers due to its has wide applications in pure and applied mathematics, especially the Riemann-Liouville operator which we recall the definition

Definition 1.3. [11] Let $f\in L^1[a,b]$. The Riemann-Liouville fractional integrals $I^{\alpha}_{a^+}f$ and $I^{\alpha}_{b^-}f$ of order $\alpha>0$ with $a\geq0$ are defined by

$$\begin{aligned}I^{\alpha}_{a^+}f(x)&=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt,\quad x>a,\\ I^{\alpha}_{b^-}f(x)&=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)dt,\quad b>x,\end{aligned}$$

respectively. Here $\Gamma(\alpha)=\int_0^{\infty}e^{-t}t^{\alpha-1}dt$, is the gamma function and $I^0_{a^+}f(x)=I^0_{b^-}f(x)=f(x)$.

Motivated by the results cited above, the aim of this study is to extend the results given in [17] for functions whose first derivatives are s-convex via Riemann-Liouville fractional integrals. We also discuss the cases where the derivatives are bounded and satisfy the Hölder condition. The results obtained are based on a new fractional identity and refine those of [12; 14]. We end the paper with a few applications.

2. MAIN RESULTS

In order to prove our results, we need the following lemma

LEMMA 2.1. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^\circ$ with $a < b$, and $f' \in L^1[a,b]$, then the following equality holds:

$$\begin{aligned}
&\tfrac{1}{8}\left(f(a)+6f\left(\tfrac{a+b}{2}\right)+f(b)\right)-\tfrac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}}\left(I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right)\\
=\;&\tfrac{b-a}{4}\left(\int_0^1\left(t^{\alpha}-\tfrac{1}{4}\right)f'\left((1-t)a+t\tfrac{a+b}{2}\right)dt\right.\\
&\left.-\int_0^1\left((1-t)^{\alpha}-\tfrac{1}{4}\right)f'\left((1-t)\tfrac{a+b}{2}+tb\right)dt\right).
\end{aligned}$$

PROOF. Let

$$\begin{aligned}
I_1 &= \int_0^1\left(t^{\alpha}-\tfrac{1}{4}\right)f'\left((1-t)a+t\tfrac{a+b}{2}\right)dt\\
I_2 &= \int_0^1\left(\tfrac{1}{4}-(1-t)^{\alpha}\right)f'\left((1-t)\tfrac{a+b}{2}+tb\right)dt.
\end{aligned}$$

Integrating by parts I_1, we get

$$\begin{aligned}
I_1 =\;& \tfrac{2}{b-a}\left(t^{\alpha}-\tfrac{1}{4}\right)f\left((1-t)a+t\tfrac{a+b}{2}\right)\Big|_{t=0}^{t=1}\\
&-\tfrac{2\alpha}{b-a}\int_0^1 t^{\alpha-1}f\left((1-t)a+t\tfrac{a+b}{2}\right)dt\\
=\;& \tfrac{3}{2(b-a)}f\left(\tfrac{a+b}{2}\right)+\tfrac{1}{2(b-a)}f(a)-\tfrac{2\alpha}{b-a}\int_0^1 t^{\alpha-1}f\left((1-t)a+t\tfrac{a+b}{2}\right)dt\\
=\;& \tfrac{3}{2(b-a)}f\left(\tfrac{a+b}{2}\right)+\tfrac{1}{2(b-a)}f(a)-\alpha\left(\tfrac{2}{b-a}\right)^{\alpha+1}\int_a^{\frac{a+b}{2}}(u-a)^{\alpha-1}f(u)\,du\\
=\;& \tfrac{3}{2(b-a)}f\left(\tfrac{a+b}{2}\right)+\tfrac{1}{2(b-a)}f(a)-\tfrac{2^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a). \qquad (2)
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
I_2 =\;& \tfrac{2}{b-a}\left((1-t)^{\alpha}-\tfrac{1}{4}\right)f\left((1-t)\tfrac{a+b}{2}+tb\right)\Big|_{t=0}^{t=1}\\
&+\tfrac{2\alpha}{b-a}\int_0^1(1-t)^{\alpha-1}f\left((1-t)\tfrac{a+b}{2}+tb\right)dt\\
=\;& -\tfrac{1}{2(b-a)}f(b)-\tfrac{3}{2(b-a)}f\left(\tfrac{a+b}{2}\right)+\tfrac{2\alpha}{b-a}\int_0^1(1-t)^{\alpha-1}f\left((1-t)\tfrac{a+b}{2}+tb\right)dt\\
=\;& -\tfrac{1}{2(b-a)}f(b)-\tfrac{3}{2(b-a)}f\left(\tfrac{a+b}{2}\right)+\alpha\left(\tfrac{2}{b-a}\right)^{\alpha+1}\int_{\frac{a+b}{2}}^{b}(b-u)^{\alpha-1}f(u)\,du\\
=\;& -\tfrac{1}{2(b-a)}f(b)-\tfrac{3}{2(b-a)}f\left(\tfrac{a+b}{2}\right)+\tfrac{2^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b). \qquad (3)
\end{aligned}$$

Making the difference between (2) and (3), and then multiplying the resulting equality by $\frac{b-a}{4}$, we get the desired result.

Before giving our main results, let's recall some special functions that we will call in the sequel

Definition 2.2. [11] For any complex numbers and nonpositive integers x, y such that $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(y) > 0$. The beta function is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \tfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma(.)$ is the Euler gamma function. The incomplete beta function.

Definition 2.3. [11] For any complex numbers and nonpositive integers x, y such that $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(y) > 0$, we have

$$B_a(x,y) = \int_0^a t^{x-1}(1-t)^{y-1}\,dt, \;\; a < 1.$$

Definition 2.4. [11] The hypergeometric function is defined for $\mathrm{Re}c > \mathrm{Re}b > 0$ and $|z| < 1$, as follows

$${}_2F_1(a,b,c;z) = \tfrac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt,$$

where $c > b > 0, |z| < 1$ and $B(.,.)$ is the beta function.

THEOREM 2.5. Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) such that $f' \in L^1[a,b]$ with $0 \le a < b$. If $|f'|$ is s-convex in the second sense for some fixed $s \in (0,1]$, then we have

$$\begin{aligned} &\left| \tfrac{1}{8}\left(f(a) + 6f\left(\tfrac{a+b}{2}\right) + f(b)\right) - \tfrac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}} \left(I^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) + I^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b) \right) \right| \\ \le\; & \tfrac{b-a}{4}\left(\Theta_{s,\alpha}\,|f'(a)| \right. \\ & \left. + \left(\tfrac{\alpha}{(s+1)(\alpha+s+1)} \left(\tfrac{1}{4}\right)^{\frac{s+1}{\alpha}} + \tfrac{3s+3-\alpha}{2(s+1)(\alpha+s+1)} \right) \left| f'\left(\tfrac{a+b}{2}\right) \right| + \Theta_{s,\alpha}\,|f'(b)| \right), \end{aligned}$$

where

$$\begin{aligned} \Theta_{s,\alpha} = \; & \tfrac{1}{4(s+1)} \left(1 - 2\left(1 - \left(\tfrac{1}{4}\right)^{\frac{1}{\alpha}}\right)^{s+1} \right) \\ & + B_{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}(s+1, \alpha+1) - B_{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}(\alpha+1, s+1). \end{aligned} \tag{4}$$

PROOF. From Lemma 2.1, properties of modulus, and s-convexity in the second

sense of $|f'|$, we have

$$\begin{aligned}
&\left|\tfrac{1}{8}\left(f(a)+6f\left(\tfrac{a+b}{2}\right)+f(b)\right)-\tfrac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}}\left(I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right)\right| \\
\leq\ & \tfrac{b-a}{4}\left(\int_0^1\left|t^{\alpha}-\tfrac{1}{4}\right|\left|f'\left((1-t)a+t\tfrac{a+b}{2}\right)\right|dt\right. \\
&\left.+\int_0^1\left|(1-t)^{\alpha}-\tfrac{1}{4}\right|\left|f'\left((1-t)\tfrac{a+b}{2}+tb\right)\right|dt\right) \\
=\ & \tfrac{b-a}{4}\left(\int_0^{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}\left(\tfrac{1}{4}-t^{\alpha}\right)\left|f'\left((1-t)a+t\tfrac{a+b}{2}\right)\right|dt\right. \\
&+\int_{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}^1\left(t^{\alpha}-\tfrac{1}{4}\right)\left|f'\left((1-t)a+t\tfrac{a+b}{2}\right)\right|dt \\
&+\int_0^{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}\left((1-t)^{\alpha}-\tfrac{1}{4}\right)\left|f'\left((1-t)\tfrac{a+b}{2}+tb\right)\right|dt \\
&\left.+\int_{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}^1\left(\tfrac{1}{4}-(1-t)^{\alpha}\right)\left|f'\left((1-t)\tfrac{a+b}{2}+tb\right)\right|dt\right) \\
\leq\ & \tfrac{b-a}{4}\left(\int_0^{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}\left(\tfrac{1}{4}-t^{\alpha}\right)\left((1-t)^{s}|f'(a)|+t^{s}\left|f'\left(\tfrac{a+b}{2}\right)\right|\right)dt\right. \\
&+\int_{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}^1\left(t^{\alpha}-\tfrac{1}{4}\right)\left((1-t)^{s}|f'(a)|+t^{s}\left|f'\left(\tfrac{a+b}{2}\right)\right|\right)dt \\
&+\int_0^{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}\left((1-t)^{\alpha}-\tfrac{1}{4}\right)\left((1-t)^{s}\left|f'\left(\tfrac{a+b}{2}\right)\right|+t^{s}|f'(b)|\right)dt \\
&\left.+\int_{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}^1\left(\tfrac{1}{4}-(1-t)^{\alpha}\right)\left((1-t)^{s}\left|f'\left(\tfrac{a+b}{2}\right)\right|+t^{s}|f'(b)|\right)dt\right) \\
=\ & \tfrac{b-a}{4}\left(|f'(a)|\left(\int_0^{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}\left(\tfrac{1}{4}-t^{\alpha}\right)(1-t)^{s}dt+\int_{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}^1\left(t^{\alpha}-\tfrac{1}{4}\right)(1-t)^{s}dt\right)\right. \\
&\left.+\left|f'\left(\tfrac{a+b}{2}\right)\right|\left(\int_0^{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}\left(\tfrac{1}{4}-t^{\alpha}\right)t^{s}dt+\int_0^{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}\left((1-t)^{\alpha}-\tfrac{1}{4}\right)(1-t)^{s}dt\right.\right.
\end{aligned}$$

$$+\int_{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}^{1}\left(t^{\alpha}-\tfrac{1}{4}\right)t^{s}dt+\int_{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}^{1}\left(\tfrac{1}{4}-(1-t)^{\alpha}\right)(1-t)^{s}dt\Bigg)$$

$$+\left|f'(b)\right|\left(\int_{0}^{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}\left((1-t)^{\alpha}-\tfrac{1}{4}\right)t^{s}dt+\int_{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}^{1}\left(\tfrac{1}{4}-(1-t)^{\alpha}\right)t^{s}dt\right)\Bigg)$$

$$=\frac{b-a}{4}\left(\Theta_{s,\alpha}\left|f'(a)\right|\right.$$

$$\left.+\left(\frac{\alpha}{(s+1)(\alpha+s+1)}\left(\frac{1}{4}\right)^{\frac{s+1}{\alpha}}+\frac{3s+3-\alpha}{2(s+1)(\alpha+s+1)}\right)\left|f'\left(\frac{a+b}{2}\right)\right|+\Theta_{s,\alpha}\left|f'(b)\right|\right),$$

where we have used

$$\int_{0}^{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}\left(\tfrac{1}{4}-t^{\alpha}\right)(1-t)^{s}dt=\int_{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}^{1}\left(\tfrac{1}{4}-(1-t)^{\alpha}\right)t^{s}dt \tag{5}$$

$$=\frac{1}{4(s+1)}\left(1-\left(1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}\right)^{s+1}\right)-B_{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}(\alpha+1,s+1),$$

$$\int_{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}^{1}\left(t^{\alpha}-\tfrac{1}{4}\right)(1-t)^{s}dt=\int_{0}^{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}\left((1-t)^{\alpha}-\tfrac{1}{4}\right)t^{s}dt \tag{6}$$

$$=B_{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}(s+1,\alpha+1)-\frac{1}{4(s+1)}\left(1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}\right)^{s+1},$$

$$\int_{0}^{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}\left(\tfrac{1}{4}-t^{\alpha}\right)t^{s}dt=\int_{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}^{1}\left(\tfrac{1}{4}-(1-t)^{\alpha}\right)(1-t)^{s}dt \tag{7}$$

$$=\frac{\alpha}{4(s+1)(\alpha+s+1)}\left(\frac{1}{4}\right)^{\frac{s+1}{\alpha}},$$

$$\int_{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}^{1}\left(t^{\alpha}-\tfrac{1}{4}\right)t^{s}dt=\int_{0}^{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}\left((1-t)^{\alpha}-\tfrac{1}{4}\right)(1-t)^{s}dt \tag{8}$$

$$=\frac{\alpha}{4(s+1)(\alpha+s+1)}\left(\frac{1}{4}\right)^{\frac{s+1}{\alpha}}+\frac{3s+3-\alpha}{4(s+1)(\alpha+s+1)},$$

and $\Theta_{s,\alpha}$ is defined as in (4). The proof is completed.

COROLLARY 2.6. In Theorem 2.5, if we take $s=1$, then we get

$$\left|\frac{1}{8}\left(f(a)+6f\left(\frac{a+b}{2}\right)+f(b)\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}}\left(I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right)\right|$$

$$\leq \frac{b-a}{4}\left(\Theta_{1,\alpha}\left|f'(a)\right|+\left(\frac{\alpha}{2(\alpha+2)}\left(\frac{1}{4}\right)^{\frac{2}{\alpha}}+\frac{6-\alpha}{4(\alpha+2)}\right)\left|f'\left(\frac{a+b}{2}\right)\right|+\Theta_{1,\alpha}\left|f'(b)\right|\right),$$

where

$$\Theta_{1,\alpha}=\frac{(\alpha+1)(\alpha+2)+8}{8(\alpha+1)(\alpha+2)}-\frac{1}{4}\left(1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}\right)^2+2\left(\frac{(\alpha+1)-(\alpha+2)4^{\frac{1}{\alpha}}}{(\alpha+1)(\alpha+2)}\right)\left(\frac{1}{4}\right)^{\frac{\alpha+2}{\alpha}}. \tag{9}$$

COROLLARY 2.7. In Theorem 2.5, if we take $\alpha=1$, then we get

$$\begin{aligned}&\left|\frac{1}{8}\left(f(a)+6f\left(\frac{a+b}{2}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ \leq\ &\frac{b-a}{4(s+1)(s+2)}\left(\left(\frac{s-2}{4}+2\left(\frac{3}{4}\right)^{s+2}\right)\left|f'(a)\right|+\left(\left(\frac{1}{4}\right)^{s+1}+\frac{3s+2}{2}\right)\left|f'\left(\frac{a+b}{2}\right)\right|\right.\\ &\left.+\left(\frac{s-2}{4}+2\left(\frac{3}{4}\right)^{s+2}\right)\left|f'(b)\right|\right).\end{aligned}$$

Remark 2.8. Corollary 2.7 will be reduced to Corollary 3.1.3 from [17], if we take $s=1$.

THEOREM 2.9. Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) such that $f'\in L^1[a,b]$ with $0\leq a<b$. If $|f'|^q$ is s-convex in the second sense for some fixed $s\in(0,1]$ where $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have

$$\begin{aligned}&\left|\frac{1}{8}\left(f(a)+6f\left(\frac{a+b}{2}\right)+f(b)\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}}\left(I^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right)\right|\\ \leq\ &\frac{b-a}{4}\left(\frac{1}{4^{p+\frac{1}{\alpha}}\alpha}B\left(\frac{1}{\alpha},p+1\right)+\frac{3^{p+1}}{4^{p+1}\alpha(p+1)}\cdot{}_2F_1\left(1-\frac{1}{\alpha},1,p+2;\frac{3}{4}\right)\right)^{\frac{1}{p}}\\ &\times\left(\frac{\left|f'(a)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}\Bigg).\end{aligned}$$

PROOF. From Lemma 2.1, properties of modulus, Hölder's inequality, and s-convexity in the second sense of $|f'|^q$, we have

$$\begin{aligned}&\left|\frac{1}{8}\left(f(a)+6f\left(\frac{a+b}{2}\right)+f(b)\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}}\left(I^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right)\right|\\ \leq\ &\frac{b-a}{4}\left(\left(\int_0^1\left|t^{\alpha}-\frac{1}{4}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left((1-t)a+t\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_0^1\left|(1-t)^{\alpha}-\frac{1}{4}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left((1-t)\frac{a+b}{2}+tb\right)\right|^q dt\right)^{\frac{1}{q}}\right)\\ \leq\ &\frac{b-a}{4}\left(\left(\int_0^{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}\left(\frac{1}{4}-t^{\alpha}\right)^p dt+\int_{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}^1\left(t^{\alpha}-\frac{1}{4}\right)^p dt\right)^{\frac{1}{p}}\right.\end{aligned}$$

$$\times\left(\int_0^1\left|f'\left((1-t)a+t\tfrac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}\left((1-t)^\alpha-\tfrac{1}{4}\right)^p dt+\int_{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}^1\left(\tfrac{1}{4}-(1-t)^\alpha\right)^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\left|f'\left((1-t)\tfrac{a+b}{2}+tb\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg)$$

$$\leq \frac{b-a}{4}\left(\int_0^{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}\left(\tfrac{1}{4}-t^\alpha\right)^p dt+\int_{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}^1\left(t^\alpha-\tfrac{1}{4}\right)^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\left(\int_0^1\left((1-t)^s\left|f'(a)\right|^q+t^s\left|f'\left(\tfrac{a+b}{2}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left((1-t)^s\left|f'\left(\tfrac{a+b}{2}\right)\right|^q+t^s\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right)$$

$$= \frac{b-a}{4}\left(\frac{1}{4^{p+\frac{1}{\alpha}}\alpha}B\left(\tfrac{1}{\alpha},p+1\right)+\frac{3^{p+1}}{4^{p+1}\alpha(p+1)}\cdot{}_2F_1\left(1-\tfrac{1}{\alpha},1,p+2;\tfrac{3}{4}\right)\right)^{\frac{1}{p}}$$

$$\times\left(\frac{\left|f'(a)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}\Bigg),$$

where we have used the fact that

$$\int_0^{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}\left(\tfrac{1}{4}-t^\alpha\right)^p dt=\frac{1}{\alpha}\int_0^{\frac{1}{4}}\left(\tfrac{1}{4}-w\right)^p w^{\frac{1}{\alpha}-1}dw=\frac{1}{4^p\alpha}\int_0^{\frac{1}{4}}(1-4w)^p w^{\frac{1}{\alpha}-1}dw$$

$$=\frac{1}{4^{p+\frac{1}{\alpha}}\alpha}\int_0^1 u^{\frac{1}{\alpha}-1}(1-u)^p du=\frac{1}{4^{p+\frac{1}{\alpha}}\alpha}B\left(\tfrac{1}{\alpha},p+1\right),$$

and

$$\int_{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}^1\left(t^\alpha-\tfrac{1}{4}\right)^p dt=\frac{1}{\alpha}\int_{\frac{1}{4}}^1\left(w-\tfrac{1}{4}\right)^p w^{\frac{1}{\alpha}-1}dw=\left(\tfrac{3}{4}\right)^p\frac{1}{\alpha}\int_0^{\frac{3}{4}}\left(1-\tfrac{4}{3}u\right)^p(1-u)^{\frac{1}{\alpha}-1}du$$

$$=\left(\tfrac{3}{4}\right)^{p+1}\frac{1}{\alpha}\int_0^1(1-z)^p\left(1-\tfrac{3}{4}z\right)^{\frac{1}{\alpha}-1}dz$$

$$=\frac{3^{p+1}}{4^{p+1}\alpha(p+1)}\cdot{}_2F_1\left(1-\tfrac{1}{\alpha},1,p+2;\tfrac{3}{4}\right).$$

The proof is completed.

COROLLARY 2.10. In Theorem 2.9, if we take $s=1$, then we get

$$\left|\frac{1}{8}\left(f(a)+6f\left(\tfrac{a+b}{2}\right)+f(b)\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^\alpha}\left(I^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+I^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right)\right|$$

$$\leq \frac{b-a}{4}\left(\frac{1}{4^{p+\frac{1}{\alpha}}\alpha}B\left(\frac{1}{\alpha},p+1\right)+\frac{3^{p+1}}{4^{p+1}\alpha(p+1)}\cdot{}_2F_1\left(1-\frac{1}{\alpha},1,p+2;\frac{3}{4}\right)\right)^{\frac{1}{p}} \times\left(\frac{\left|f'(a)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}\Bigg).$$

COROLLARY 2.11. In Theorem 2.9, if we take $\alpha=1$, then we get

$$\left|\frac{1}{8}\left(f(a)+6f\left(\frac{a+b}{2}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{b-a}{16}\left(\frac{1+3^{p+1}}{4(p+1)}\right)^{\frac{1}{p}}\left(\left(\frac{\left|f'(a)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}\right).$$

Remark 2.12. Corollary 2.11 will be reduced to Corollary 3.2.1 from [17], if we take $s=1$.

THEOREM 2.13. Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) such that $f'\in L^1[a,b]$ with $0\leq a<b$. If $|f'|^q$ is s-convex in the second sense for some fixed $s\in(0,1]$ where $q\geq 1$, then we have

$$\left|\frac{1}{8}\left(f(a)+6f\left(\frac{a+b}{2}\right)+f(b)\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^\alpha}\left(I^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+I^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right)\right| \leq \frac{b-a}{4}\left(\frac{\left(2-4^{\frac{1}{\alpha}}\right)\alpha+3\times 4^{\frac{1}{\alpha}}}{4^{\frac{\alpha+1}{\alpha}}(\alpha+1)}\right)^{1-\frac{1}{q}} \times\left(\left(\Theta_{s,\alpha}\left|f'(a)\right|^q+\frac{\left(2-4^{\frac{s+1}{\alpha}}\right)\alpha+(3s+3)4^{\frac{s+1}{\alpha}}}{4^{\frac{s+\alpha+1}{\alpha}}(s+1)(\alpha+s+1)}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} +\left(\frac{\left(2-4^{\frac{s+1}{\alpha}}\right)\alpha+(3s+3)4^{\frac{s+1}{\alpha}}}{4^{\frac{s+\alpha+1}{\alpha}}(s+1)(\alpha+s+1)}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\Theta_{s,\alpha}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\Bigg),$$

where $\Theta_{s,\alpha}$ is defined as in (4).

PROOF. From Lemma 2.1, properties of modulus, power mean inequality, and s-convexity in the second sense of $|f'|^q$, we have

$$\left|\frac{1}{8}\left(f(a)+6f\left(\frac{a+b}{2}\right)+f(b)\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^\alpha}\left(I^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+I^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right)\right| \leq \frac{b-a}{4}\left(\int_0^1\left|t^\alpha-\frac{1}{4}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|t^\alpha-\frac{1}{4}\right|\left|f'\left((1-t)a+t\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}} +\left(\int_0^1\left|(1-t)^\alpha-\frac{1}{4}\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\left|(1-t)^{\alpha}-\tfrac{1}{4}\right|\left|f'\left((1-t)\tfrac{a+b}{2}+tb\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg)$$

$$\leq \frac{b-a}{4}\Bigg(\left(\int_0^{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}\left(\tfrac{1}{4}-t^{\alpha}\right)dt+\int_{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}^1\left(t^{\alpha}-\tfrac{1}{4}\right)dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\left|t^{\alpha}-\tfrac{1}{4}\right|\left|f'\left((1-t)a+t\tfrac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}\left((1-t)^{\alpha}-\tfrac{1}{4}\right)dt+\int_{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}^1\left(\tfrac{1}{4}-(1-t)^{\alpha}\right)dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\left|(1-t)^{\alpha}-\tfrac{1}{4}\right|\left|f'\left((1-t)\tfrac{a+b}{2}+tb\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg)$$

$$\leq \frac{b-a}{4}\left(\int_0^{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}\left(\tfrac{1}{4}-t^{\alpha}\right)dt+\int_{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}^1\left(t^{\alpha}-\tfrac{1}{4}\right)dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\left(\int_0^1\left|t^{\alpha}-\tfrac{1}{4}\right|\left((1-t)^s\left|f'(a)\right|^q+t^s\left|f'\left(\tfrac{a+b}{2}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left|(1-t)^{\alpha}-\tfrac{1}{4}\right|\left((1-t)^s\left|f'\left(\tfrac{a+b}{2}\right)\right|^q+t^s\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right)$$

$$= \frac{b-a}{4}\left(\frac{\left(2-4\frac{1}{\alpha}\right)\alpha+3\times 4\frac{1}{\alpha}}{4^{\frac{\alpha+1}{\alpha}}(\alpha+1)}\right)^{1-\frac{1}{q}}$$

$$\times\left(\left(\left|f'(a)\right|^q\left(\int_0^{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}\left(\tfrac{1}{4}-t^{\alpha}\right)(1-t)^s dt+\int_{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}^1\left(t^{\alpha}-\tfrac{1}{4}\right)(1-t)^s dt\right)\right.\right.$$

$$\left.\left.+\left|f'\left(\tfrac{a+b}{2}\right)\right|^q\left(\int_0^{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}\left(\tfrac{1}{4}-t^{\alpha}\right)t^s dt+\int_{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}^1\left(t^{\alpha}-\tfrac{1}{4}\right)t^s dt\right)\right)^{\frac{1}{q}}\right.$$

$$+\left(\left(\int_0^{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}\left((1-t)^{\alpha}-\tfrac{1}{4}\right)(1-t)^s dt\right.\right.$$

$$\left.\left.+\int_{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}^1\left(\tfrac{1}{4}-(1-t)^{\alpha}\right)(1-t)^s dt\right)\left|f'\left(\tfrac{a+b}{2}\right)\right|^q\right)$$

$$+\left(\int_0^{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}\left((1-t)^{\alpha}-\tfrac{1}{4}\right)t^{s}dt+\int_{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}^{1}\left(\tfrac{1}{4}-(1-t)^{\alpha}\right)t^{s}dt\right)\left|f'(b)\right|^{q}\Bigg)^{\frac{1}{q}}\Bigg)$$

$$=\frac{b-a}{4}\left(\frac{\left(2-4^{\frac{1}{\alpha}}\right)\alpha+3\times4^{\frac{1}{\alpha}}}{4^{\frac{\alpha+1}{\alpha}}(\alpha+1)}\right)^{1-\frac{1}{q}}$$

$$\times\left(\left(\Theta_{s,\alpha}\left|f'(a)\right|^{q}+\frac{\left(2-4^{\frac{s+1}{\alpha}}\right)\alpha+(3s+3)4^{\frac{s+1}{\alpha}}}{4^{\frac{s+\alpha+1}{\alpha}}(s+1)(\alpha+s+1)}\left|f'\left(\tfrac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\left(2-4^{\frac{s+1}{\alpha}}\right)\alpha+(3s+3)4^{\frac{s+1}{\alpha}}}{4^{\frac{s+\alpha+1}{\alpha}}(s+1)(\alpha+s+1)}\left|f'\left(\tfrac{a+b}{2}\right)\right|^{q}+\Theta_{s,\alpha}\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right),$$

where we have used (4)-(8). The proof is achieved.

COROLLARY 2.14. In Theorem 2.13, if we take $s=1$, then we get

$$\left|\tfrac{1}{8}\left(f(a)+6f\left(\tfrac{a+b}{2}\right)+f(b)\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}}\left(I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right)\right|$$

$$\leq\frac{b-a}{4}\left(\frac{\left(2-4^{\frac{1}{\alpha}}\right)\alpha+3\times4^{\frac{1}{\alpha}}}{4^{\frac{\alpha+1}{\alpha}}(\alpha+1)}\right)^{1-\frac{1}{q}}$$

$$\times\left(\left(\Theta_{1,\alpha}\left|f'(a)\right|^{q}+\frac{\left(2-4^{\frac{2}{\alpha}}\right)\alpha+6\times4^{\frac{2}{\alpha}}}{4^{\frac{2+3\alpha}{\alpha}}(\alpha+2)}\left|f'\left(\tfrac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\left(2-4^{\frac{2}{\alpha}}\right)\alpha+6\times4^{\frac{2}{\alpha}}}{4^{\frac{2+3\alpha}{\alpha}}(\alpha+2)}\left|f'\left(\tfrac{a+b}{2}\right)\right|^{q}+\Theta_{1,\alpha}\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right),$$

where $\Theta_{1,\alpha}$ is defined in (9).

COROLLARY 2.15. In Theorem 2.13, if we take $\alpha=1$, then we get

$$\left|\tfrac{1}{8}\left(f(a)+6f\left(\tfrac{a+b}{2}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

$$\leq\frac{b-a}{4[(s+1)(s+2)]^{\frac{1}{q}}}\left(\frac{5}{16}\right)^{1-\frac{1}{q}}$$

$$\times\left(\left(\left(\tfrac{s-2}{4}+2\left(\tfrac{3}{4}\right)^{s+2}\right)\left|f'(a)\right|^{q}+\frac{2+(3s+2)4^{s+1}}{4^{s+2}}\left|f'\left(\tfrac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{2+(3s+2)4^{s+1}}{4^{s+2}}\left|f'\left(\tfrac{a+b}{2}\right)\right|^{q}+\left(\tfrac{s-2}{4}+2\left(\tfrac{3}{4}\right)^{s+2}\right)\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right).$$

Remark 2.16. Corollary 2.15 will be reduced to Corollary 3.1.2 from [17], if we take $s=1$.

3. FURTHER RESULTS

THEOREM 3.1. Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) such that $f'\in L^1[a,b]$ with $0\le a<b$. If there exist constants $-\infty<m<M<+\infty$ such that $m\le f'(x)\le M$ for all $x\in[a,b]$, then we have

$$\begin{aligned}&\left|\tfrac{1}{8}\left(f(a)+6f\left(\tfrac{a+b}{2}\right)+f(b)\right)-\tfrac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}}\left(I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right)\right|\\ \le\ &\tfrac{(b-a)(M-m)}{4(\alpha+1)}\left(\tfrac{\alpha}{2}\left(\tfrac{1}{4}\right)^{\frac{1}{\alpha}}+\tfrac{3-\alpha}{4}\right).\end{aligned}$$

PROOF. From Lemma 2.1, we have

$$\begin{aligned}&\tfrac{1}{8}\left(f(a)+6f\left(\tfrac{a+b}{2}\right)+f(b)\right)-\tfrac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}}\left(I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right)\\ =\ &\tfrac{b-a}{4}\left(\int_0^1\left(t^{\alpha}-\tfrac{1}{4}\right)f'\left((1-t)a+t\tfrac{a+b}{2}\right)dt\right.\\ &\left.-\int_0^1\left((1-t)^{\alpha}-\tfrac{1}{4}\right)f'\left((1-t)\tfrac{a+b}{2}+tb\right)dt\right)\\ =\ &\tfrac{b-a}{4}\left(\int_0^1\left(t^{\alpha}-\tfrac{1}{4}\right)\left(f'\left((1-t)a+t\tfrac{a+b}{2}\right)-\tfrac{m+M}{2}+\tfrac{m+M}{2}\right)dt\right.\\ &\left.-\int_0^1\left((1-t)^{\alpha}-\tfrac{1}{4}\right)\left(f'\left((1-t)\tfrac{a+b}{2}+tb\right)-\tfrac{m+M}{2}+\tfrac{m+M}{2}\right)dt\right)\\ =\ &\tfrac{b-a}{4}\left(\int_0^1\left(t^{\alpha}-\tfrac{1}{4}\right)\left(f'\left((1-t)a+t\tfrac{a+b}{2}\right)-\tfrac{m+M}{2}\right)dt\right.\\ &-\int_0^1\left((1-t)^{\alpha}-\tfrac{1}{4}\right)\left(f'\left((1-t)\tfrac{a+b}{2}+tb\right)-\tfrac{m+M}{2}\right)dt\\ &\left.+\tfrac{m+M}{2}\left(\int_0^1\left(t^{\alpha}-\tfrac{1}{4}\right)dt-\int_0^1\left((1-t)^{\alpha}-\tfrac{1}{4}\right)dt\right)\right)\\ =\ &\tfrac{b-a}{4}\left(\int_0^1\left(t^{\alpha}-\tfrac{1}{4}\right)\left(f'\left((1-t)a+t\tfrac{a+b}{2}\right)-\tfrac{m+M}{2}\right)dt\right.\\ &\left.-\int_0^1\left((1-t)^{\alpha}-\tfrac{1}{4}\right)\left(f'\left((1-t)\tfrac{a+b}{2}+tb\right)-\tfrac{m+M}{2}\right)dt\right),\end{aligned}\tag{10}$$

where we have taken into consideration that

$$\int_0^1\left(t^{\alpha}-\tfrac{1}{4}\right)dt-\int_0^1\left((1-t)^{\alpha}-\tfrac{1}{4}\right)dt=\left(\tfrac{1}{\alpha+1}-\tfrac{1}{4}\right)-\left(\tfrac{1}{\alpha+1}-\tfrac{1}{4}\right)=0.$$

Applying the absolute value to both sides of (10), we get

$$\begin{aligned}&\left|\tfrac{1}{8}\left(f(a)+6f\left(\tfrac{a+b}{2}\right)+f(b)\right)-\tfrac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}}\left(I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right)\right|\\ =\ &\tfrac{b-a}{4}\left(\int_0^1\left|t^{\alpha}-\tfrac{1}{4}\right|\left|f'\left((1-t)a+t\tfrac{a+b}{2}\right)-\tfrac{m+M}{2}\right|dt\right.\end{aligned}$$

$$+\int_0^1 \left|(1-t)^{\alpha}-\tfrac{1}{4}\right|\left|f'\left((1-t)\tfrac{a+b}{2}+tb\right)-\tfrac{m+M}{2}\right|dt\Bigg). \tag{11}$$

Since $m \le f'(x) \le M$ for all $x \in [a,b]$, we have

$$\left|f'\left((1-t)a+t\tfrac{a+b}{2}\right)-\tfrac{m+M}{2}\right| \le \tfrac{M-m}{2}, \tag{12}$$

and

$$\left|f'\left((1-t)\tfrac{a+b}{2}+tb\right)-\tfrac{m+M}{2}\right| \le \tfrac{M-m}{2}. \tag{13}$$

Using (12) and (13) in (11), we get

$$\begin{aligned}
&\left|\tfrac{1}{8}\left(f(a)+6f\left(\tfrac{a+b}{2}\right)+f(b)\right)-\tfrac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}}\left(I^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right)\right| \\
= &\quad \tfrac{(b-a)(M-m)}{8}\left(\int_0^1\left|t^{\alpha}-\tfrac{1}{4}\right|dt+\int_0^1\left|(1-t)^{\alpha}-\tfrac{1}{4}\right|dt\right) \\
= &\quad \tfrac{(b-a)(M-m)}{4}\left(\int_0^1\left|t^{\alpha}-\tfrac{1}{4}\right|dt\right) \\
= &\quad \tfrac{(b-a)(M-m)}{4(\alpha+1)}\left(\tfrac{\alpha}{2}\left(\tfrac{1}{4}\right)^{\frac{1}{\alpha}}+\tfrac{3-\alpha}{4}\right).
\end{aligned}$$

The proof is completed.

COROLLARY 3.2. In Theorem 3.1, if we take $\alpha = 1$, then we get

$$\left|\tfrac{1}{8}\left(f(a)+6f\left(\tfrac{a+b}{2}\right)+f(b)\right)-\tfrac{1}{b-a}\int_a^b f(t)\,dt\right| \le \tfrac{5(b-a)(M-m)}{64}.$$

Remark 3.3. The result of Corollary 2.11 is finer than the result of Corollary 3.1 of [14].

THEOREM 3.4. Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) such that $f' \in L^1[a,b]$ with $0 \le a < b$. If f' is r-L-Hölderian function on $[a,b]$ (i.e. there exist $L>0$ and $0<r\le 1$ such that $|f'(x)-f'(y)| \le L|x-y|^r$), then we have

$$\begin{aligned}
&\left|\tfrac{1}{8}\left(f(a)+6f\left(\tfrac{a+b}{2}\right)+f(b)\right)-\tfrac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}}\left(I^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right)\right| \\
\le &\ \tfrac{L(b-a)^{r+1}}{2^{r+2}}\Bigg(\tfrac{-\alpha^2+(6-r)\alpha+3r+7}{4(\alpha+1)(\alpha+r+1)}+\tfrac{\alpha}{2(r+1)(\alpha+r+1)}\left(\tfrac{1}{4}\right)^{\frac{r+1}{\alpha}}-\tfrac{1}{2(r+1)}\left(1-\left(\tfrac{1}{4}\right)^{\frac{1}{\alpha}}\right)^{r+1} \\
&\qquad +B_{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}(r+1,\alpha+1)-B_{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}(\alpha+1,r+1)\Bigg).
\end{aligned}$$

PROOF. From Lemma 2.1, we have

$$\tfrac{1}{8}\left(f(a)+6f\left(\tfrac{a+b}{2}\right)+f(b)\right)-\tfrac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}}\left(I^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right)$$

$$
\begin{aligned}
&= \frac{b-a}{4}\left(\int_0^1 \left(t^\alpha-\tfrac{1}{4}\right) f'\left((1-t)a+t\tfrac{a+b}{2}\right)dt\right.\\
&\quad \left.-\int_0^1 \left((1-t)^\alpha-\tfrac{1}{4}\right) f'\left((1-t)\tfrac{a+b}{2}+tb\right)dt\right)\\
&= \frac{b-a}{4}\left(\int_0^1 \left(t^\alpha-\tfrac{1}{4}\right)\left(f'\left((1-t)a+t\tfrac{a+b}{2}\right)-f'(a)+f'(a)\right)dt\right.\\
&\quad \left.-\int_0^1 \left((1-t)^\alpha-\tfrac{1}{4}\right)\left(f'\left((1-t)\tfrac{a+b}{2}+tb\right)-f'\left(\tfrac{a+b}{2}\right)+f'\left(\tfrac{a+b}{2}\right)\right)dt\right)\\
&= \frac{b-a}{4}\left(\int_0^1 \left(t^\alpha-\tfrac{1}{4}\right)\left(f'\left((1-t)a+t\tfrac{a+b}{2}\right)-f'(a)\right)dt\right.\\
&\quad -\int_0^1 \left((1-t)^\alpha-\tfrac{1}{4}\right)\left(f'\left((1-t)\tfrac{a+b}{2}+tb\right)-f'\left(\tfrac{a+b}{2}\right)\right)dt\\
&\quad \left.+f'(a)\int_0^1 \left(t^\alpha-\tfrac{1}{4}\right)dt-f'\left(\tfrac{a+b}{2}\right)\int_0^1 \left((1-t)^\alpha-\tfrac{1}{4}\right)dt\right)\\
&= \frac{b-a}{4}\left(\int_0^1 \left(t^\alpha-\tfrac{1}{4}\right)\left(f'\left((1-t)a+t\tfrac{a+b}{2}\right)-f'(a)\right)dt\right.\\
&\quad -\int_0^1 \left((1-t)^\alpha-\tfrac{1}{4}\right)\left(f'\left((1-t)\tfrac{a+b}{2}+tb\right)-f'\left(\tfrac{a+b}{2}\right)\right)dt\\
&\quad \left.+\tfrac{3-\alpha}{4(\alpha+1)}\left(f'(a)-f'\left(\tfrac{a+b}{2}\right)\right)\right). \qquad (14)
\end{aligned}
$$

Applying the absolute value in both sides of (14), and by using the fact that f' is r-L-Hölderian on $[a,b]$, we obtain

$$
\begin{aligned}
&\left|\tfrac{1}{8}\left(f(a)+6f\left(\tfrac{a+b}{2}\right)+f(b)\right)-\tfrac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^\alpha}\left(I^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+I^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right)\right|\\
&\le \frac{b-a}{4}\left(\int_0^1 \left|t^\alpha-\tfrac{1}{4}\right|\left|f'\left((1-t)a+t\tfrac{a+b}{2}\right)-f'(a)\right|dt\right.\\
&\quad +\int_0^1 \left|(1-t)^\alpha-\tfrac{1}{4}\right|\left|f'\left((1-t)\tfrac{a+b}{2}+tb\right)-f'\left(\tfrac{a+b}{2}\right)\right|dt\\
&\quad \left.+\tfrac{3-\alpha}{4(\alpha+1)}\left(\left|f'(a)-f'\left(\tfrac{a+b}{2}\right)\right|\right)\right)\\
&\le \frac{b-a}{4}L\left(\int_0^1 \left|t^\alpha-\tfrac{1}{4}\right|\left|(1-t)a+t\tfrac{a+b}{2}-a\right|^r dt\right.\\
&\quad \left.+\int_0^1 \left|(1-t)^\alpha-\tfrac{1}{4}\right|\left|(1-t)\tfrac{a+b}{2}+tb-\tfrac{a+b}{2}\right|^r dt+\tfrac{3-\alpha}{4(\alpha+1)}\left(\left|a-\tfrac{a+b}{2}\right|^r\right)\right)\\
&= \frac{L}{2}\left(\tfrac{b-a}{2}\right)^{r+1}\left(\int_0^1 \left|t^\alpha-\tfrac{1}{4}\right|t^r dt+\int_0^1 \left|(1-t)^\alpha-\tfrac{1}{4}\right|t^r dt+\tfrac{3-\alpha}{4(\alpha+1)}\right)\\
&= \frac{L(b-a)^{r+1}}{2^{r+2}}\left(\tfrac{-\alpha^2+(6-r)\alpha+3r+7}{4(\alpha+1)(\alpha+r+1)}+\tfrac{\alpha}{2(r+1)(\alpha+r+1)}\left(\tfrac{1}{4}\right)^{\frac{r+1}{\alpha}}-\tfrac{1}{2(r+1)}\left(1-\left(\tfrac{1}{4}\right)^{\frac{1}{\alpha}}\right)^{r+1}\right.\\
&\quad \left.B_{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}(r+1,\alpha+1)-B_{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}(\alpha+1,r+1)\right),
\end{aligned}
$$

where we have used

$$\int_0^1 \left|t^\alpha - \tfrac{1}{4}\right| t^r dt = \tfrac{\alpha}{2(r+1)(\alpha+r+1)} \left(\tfrac{1}{4}\right)^{\frac{r+1}{\alpha}} + \tfrac{3r+3-\alpha}{4(r+1)(\alpha+r+1)}$$

and

$$\int_0^1 \left|(1-t)^\alpha - \tfrac{1}{4}\right| t^r dt = B_{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}(r+1,\alpha+1) - B_{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}(\alpha+1,r+1) + \tfrac{1}{4(r+1)} - \tfrac{1}{2(r+1)}\left(1-\left(\tfrac{1}{4}\right)^{\frac{1}{\alpha}}\right)^{r+1}.$$

The proof is completed.

Remark 3.5. By a simple calculation of definite integrals, we have

1/ $B_{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}(2,\alpha+1) = \tfrac{1}{(\alpha+1)(\alpha+2)} - \tfrac{1}{4(\alpha+1)}\left(\tfrac{1}{4}\right)^{\frac{1}{\alpha}} + \tfrac{1}{4(\alpha+2)}\left(\tfrac{1}{4}\right)^{\frac{2}{\alpha}}$.

2/ $B_{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}(\alpha+1,2) = \tfrac{1}{4(\alpha+1)}\left(\tfrac{1}{4}\right)^{\frac{1}{\alpha}} - \tfrac{1}{4(\alpha+2)}\left(\tfrac{1}{4}\right)^{\frac{2}{\alpha}}$.

3/ $B_{\frac{3}{4}}(r+1,2) = \tfrac{r+5}{4(r+1)(r+2)}\left(\tfrac{3}{4}\right)^{r+1}$.

4/ $B_{\frac{1}{4}}(2,r+1) = \tfrac{1}{(r+1)(r+2)} - \tfrac{r+5}{4(r+1)(r+2)}\left(\tfrac{3}{4}\right)^{r+1}$.

5/ $B_{\frac{3}{4}}(2,2) = \tfrac{9}{64}$.

6/ $B_{\frac{1}{4}}(2,2) = \tfrac{5}{192}$.

COROLLARY 3.6. Under the assumptions of Theorem 3.4, if f' is a Lipschitzian function, then we have

$$\left|\tfrac{1}{8}\left(f(a)+6f\left(\tfrac{a+b}{2}\right)+f(b)\right) - \tfrac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^\alpha}\left(I^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a) + I^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right)\right| \leq \tfrac{L(b-a)^2}{8}\left(\tfrac{6+\alpha-\alpha^2}{2(\alpha+1)(\alpha+2)} + \tfrac{\alpha}{2(\alpha+1)}\left(\tfrac{1}{4}\right)^{\frac{1}{\alpha}}\right).$$

COROLLARY 3.7. In Theorem 3.4, if we take $\alpha = 1$, then we get

$$\left|\tfrac{1}{8}\left(f(a)+6f\left(\tfrac{a+b}{2}\right)+f(b)\right) - \tfrac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \tfrac{L(b-a)^{r+1}}{2^{r+2}}\left(\tfrac{r^2+7r+2}{4(r+1)(r+2)} + \tfrac{2\left(1+3^{r+2}\right)}{4^{r+2}(r+1)(r+2)}\right).$$

COROLLARY 3.8. In Corollary 3.7, if we take $r = 1$, then we get

$$\left|\tfrac{1}{8}\left(f(a)+6f\left(\tfrac{a+b}{2}\right)+f(b)\right) - \tfrac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \tfrac{9L(b-a)^2}{128}.$$

4. APPLICATIONS

Simpson like quadrature formula

Let Υ be the partition of the points $a = x_0 < x_1 < ... < x_n = b$ of the interval $[a,b]$, and consider the quadrature formula

$$\int_a^b f(u)\,du = \lambda(f,\Upsilon) + R(f,\Upsilon),$$

where

$$\lambda(f,\Upsilon) = \sum_{i=0}^{n-1} \tfrac{x_{i+1}-x_i}{8}\left(f(x_i) + 6f\left(\tfrac{x_i+x_{i+1}}{2}\right) + f(x_{i+1})\right)$$

and $R(f,\Upsilon)$ denotes the associated approximation error.

PROPOSITION 4.1. Let $n \in \mathbb{N}$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $0 \le a < b$ and $f' \in L^1[a,b]$. If $|f'|$ is s-convex function in the second sense for some fixed $s \in (0,1]$, we have

$$|R(f,\Upsilon)| \le \sum_{i=0}^{n-1} \tfrac{(x_{i+1}-x_i)^2}{4(s+1)(s+2)}\left(\left(\tfrac{s-2}{4} + 2\left(\tfrac{3}{4}\right)^{s+2}\right)|f'(x_i)| + \left(\left(\tfrac{1}{4}\right)^{s+1} + \tfrac{3s+2}{2}\right)\left|f'\left(\tfrac{x_i+x_{i+1}}{2}\right)\right| + \left(\tfrac{s-2}{4} + 2\left(\tfrac{3}{4}\right)^{s+2}\right)|f'(x_{i+1})|\right).$$

PROOF. Applying Theorem 2.9 on the subintervals $[x_i, x_{i+1}]$ $(i = 0,1,...,n-1)$ of the partition Υ, we get

$$\left|\tfrac{1}{8}\left(f(x_i) + 6f\left(\tfrac{x_i+x_{i+1}}{2}\right) + f(x_{i+1})\right) - \tfrac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} f(t)\,dt\right| \le \tfrac{x_{i+1}-x_i}{4(s+1)(s+2)}\left(\left(\tfrac{s-2}{4} + 2\left(\tfrac{3}{4}\right)^{s+2}\right)|f'(x_i)| + \left(\left(\tfrac{1}{4}\right)^{s+1} + \tfrac{3s+2}{2}\right)\left|f'\left(\tfrac{x_i+x_{i+1}}{2}\right)\right| + \left(\tfrac{s-2}{4} + 2\left(\tfrac{3}{4}\right)^{s+2}\right)|f'(x_{i+1})|\right). \quad (15)$$

Multiplying both sides of (15) by $(x_{i+1}-x_i)$, and then summing the obtained inequalities for all $i = 0,1,...,n-1$ and using the triangular inequality, we get the desired result.

Application to special means

For arbitrary real numbers a,b we have:

The Arithmetic mean: $A(a,b) = \frac{a+b}{2}$.

The Geometric mean: $G(a,b) = \sqrt{ab}$, $a,b > 0$.

The p-Logarithmic mean: $L_p(a,b) = \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}$, $a,b > 0, a \ne b$ and $p \in \mathbb{R}\setminus\{-1,0\}$.

PROPOSITION 4.2. Let $a,b \in \mathbb{R}$ with $0 < a < b$, then we have

$$\left|A\left(a^2,b^2\right) + 3A^2(a,b) - 4L_2^2(a,b)\right| \leq \tfrac{5(b-a)^2}{8}.$$

PROOF. The assertion follows from Theorem 3.1, applied to the function $f(x) = x^2$.

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