

Some new inequalities for generalized convex functions pertaining generalized fractional integral operators and their applications

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Abstract

In this paper, authors establish a new identity for a differentiable function using generic integral operators. By applying it, some new integral inequalities of trapezium, Ostrowski and Simpson type are obtained. Moreover, several special cases have been studied in detail. Finally, many useful applications have been found.

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1. INTRODUCTION AND PRELIMINARIES

DEFINITION 1.1. [49] $\Lambda : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called convex function on I , if

$$\Lambda((1-\zeta)b_1 + \zeta b_2) \leq (1-\zeta)\Lambda(b_1) + \zeta\Lambda(b_2), \quad \forall b_1, b_2 \in I, \zeta \in [0, 1].$$

THEOREM 1.2. (Trapezium inequality) Suppose that $\Lambda : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function, $b_1, b_2 \in I$ with $b_1 < b_2$, then

$$\Lambda\left(\frac{b_1 + b_2}{2}\right) \leq \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \Lambda(\ell) d\ell \leq \frac{\Lambda(b_1) + \Lambda(b_2)}{2}. \quad (1)$$

Interested readers are referred to [4]-[6],[15; 19; 20; 22; 25; 26; 28],[33]-[38],[45; 47; 52; 53; 55; 56].

THEOREM 1.3. (Ostrowski inequality) Assume that $\Lambda : I \rightarrow \mathbb{R}$ be a differentiable function on I° , $b_1, b_2 \in I^\circ$ with $b_1 < b_2$. If $|\Lambda'(\ell)| \leq \mathcal{K}$, then

$$\left| \Lambda(\ell) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \Lambda(\xi) d\xi \right| \leq \mathcal{K}(b_2 - b_1) \left[\frac{1}{4} + \frac{\left(\ell - \frac{b_1 + b_2}{2}\right)^2}{(b_2 - b_1)^2} \right], \quad \forall \ell \in [b_1, b_2]. \quad (2)$$

For other recent published papers about Ostrowski type inequalities, see [1]-[3],[7]-[14],[17],[29]-[32],[39]-[41],[43; 44; 50; 54; 57].

THEOREM 1.4. (Simpson inequality) Let $\Lambda : [b_1, b_2] \rightarrow \mathbb{R}$ be four times differentiable on (b_1, b_2) and suppose that

$$\|\Lambda^{(4)}\|_\infty := \sup_{\ell \in (b_1, b_2)} |\Lambda^{(4)}| < +\infty.$$

Then

$$\left| \int_{b_1}^{b_2} \Lambda(\ell) d\ell - \frac{b_2 - b_1}{3} \left[\frac{\Lambda(b_1) + \Lambda(b_2)}{2} + 2\Lambda\left(\frac{b_1 + b_2}{2}\right) \right] \right| \leq \frac{(b_2 - b_1)^5}{2880} \|\Lambda^{(4)}\|_\infty. \quad (3)$$

About Simpson type inequalities, see [27; 42; 51; 57].

In our paper we will establish some new trapezium, Ostrowski and Simpson type inequalities pertaining generalized convex functions with respect to another function. Moreover, many useful applications will be given. Hence, it is important to recall some essential facts relevant to fractional integrals.

DEFINITION 1.5. [34] Assume that $\Lambda \in \mathcal{L}[b_1, b_2]$, then κ -fractional integrals, $\eta, \kappa > 0$ with $b_1 \geq 0$ are

$$I_{b_1^+}^{\eta, \kappa} \Lambda(\xi_1) = \frac{1}{\kappa \Gamma_\kappa(\eta)} \int_{b_1}^{\xi_1} (\xi_1 - \xi)^{\frac{\eta}{\kappa} - 1} \Lambda(\xi) d\xi, \quad b_1 < \xi_1,$$

and

$$I_{b_2^-}^{\eta, \kappa} \Lambda(\xi_1) = \frac{1}{\kappa \Gamma_\kappa(\eta)} \int_{\xi_1}^{b_2} (\xi - \xi_1)^{\frac{\eta}{\kappa} - 1} \Lambda(\xi) d\xi, \quad b_2 > \xi_1, \quad (4)$$

respectively.

DEFINITION 1.6. [35; 36] S is called ϖ -convex set, if

$$\varpi(j)b_2 + (1 - \varpi(j))b_1 \in S, \quad \forall b_1, b_2 \in S, j \in [0, 1].$$

DEFINITION 1.7. $\Lambda : S \rightarrow \mathbb{R}$ on ϖ -convex set S is called ϖ -convex function, if

$$\Lambda(b_1 + je^{i\varpi}(b_2 - b_1)) \leq (1 - j)\Lambda(b_1) + j\Lambda(b_2), \quad \forall b_1, b_2 \in S, j \in [0, 1].$$

Raina, in [48], defined for $\rho, \delta > 0$ and $|z| < R$, the following class of functions:

$$\mathcal{F}_{\rho, \delta}^\sigma(z) = \mathcal{F}_{\rho, \delta}^{\sigma(0), \sigma(1), \dots}(z) = \sum_{k=0}^{+\infty} \frac{\sigma(k)}{\Gamma(\rho k + \delta)} z^k. \quad (5)$$

Choosing $|z| \leq 1$, we take hypergeometric function. For $\sigma = (1, 1, \dots)$ with $\rho = \eta$, $(\Re(\eta) > 0)$, $\delta = 1$ and $z \in \mathbb{C}$ in (5), we obtain Mittag-Leffler function

$$\mathcal{E}_\eta(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(1 + \eta k)}.$$

DEFINITION 1.8. $S \neq \emptyset$ is called generalized convex set, if

$$mb_1 + \zeta \mathcal{F}_{\rho, \delta}^\sigma(b_2 - mb_1) \in S, \quad \forall b_1, b_2 \in S, \quad m \in (0, 1], \quad \zeta \in [0, 1]. \quad (6)$$

DEFINITION 1.9. Λ is called generalized convex function, if

$$\Lambda(mb_1 + \zeta \mathcal{F}_{\rho, \delta}^\sigma(b_2 - mb_1)) \leq (1 - \zeta)\Lambda(mb_1) + \zeta\Lambda(b_2), \quad \forall b_1, b_2 \in S, \quad m \in (0, 1], \quad \zeta \in [0, 1]. \quad (7)$$

REMARK 1.10. Taking $m = 1$ and $\mathcal{F}_{\rho, \delta}^\sigma(b_2 - b_1) = b_2 - b_1 > 0$, we get definition 1.1. For suitable choice of $\mathcal{F}_{\rho, \delta}^\sigma(\cdot)$, we obtain definition 1.7. This describes the reasons and motivations of newly defined notions and the relation with these known definitions.

DEFINITION 1.11. [23; 24] Assume that $\Lambda_2 : [b_1, b_2] \rightarrow \mathbb{R}$ is an increasing and positive monotone function on $[b_1, b_2]$, with continuous derivative on (b_1, b_2) . The left-right- fractional integrals of Λ_1 with respect to Λ_2 on $[b_1, b_2]$, $\eta > 0$ are

$$I_{b_1+}^{\eta, \Lambda_2} \Lambda_1(\xi_1) = \frac{1}{\Gamma(\eta)} \int_{b_1}^{\xi_1} \frac{\Lambda_2'(\xi) \Lambda_1(\xi)}{[\Lambda_2(\xi_1) - \Lambda_2(\xi)]^{1-\eta}} d\xi, \quad b_1 < \xi_1, \quad (8)$$

and

$$I_{b_2-}^{\eta, \Lambda_2} \Lambda_1(\xi_1) = \frac{1}{\Gamma(\eta)} \int_{\xi_1}^{b_2} \frac{\Lambda_2'(\xi) \Lambda_1(\xi)}{[\Lambda_2(\xi) - \Lambda_2(\xi_1)]^{1-\eta}} d\xi, \quad b_2 > \xi_1, \quad (9)$$

respectively.

Function $\varpi : [0, +\infty) \rightarrow [0, +\infty)$ constructed from Sarikaya et al. in [45; 46], has the following properties:

$$\int_0^1 \frac{\varpi(\xi)}{\xi} d\xi < +\infty, \quad (10)$$

$$\frac{1}{A_1} \leq \frac{\varpi(\varepsilon)}{\varpi(\xi_1)} \leq A_1 \text{ for } \frac{1}{2} \leq \frac{\varepsilon}{\xi_1} \leq 2, \quad (11)$$

$$\frac{\varpi(\xi_1)}{\xi_1^2} \leq A_2 \frac{\varpi(\varepsilon)}{\varepsilon^2} \text{ for } \varepsilon \leq \xi_1, \quad (12)$$

$$\left| \frac{\varpi(\xi_1)}{\xi_1^2} - \frac{\varpi(\varepsilon)}{\varepsilon^2} \right| \leq A_3 |\xi_1 - \varepsilon| \frac{\varpi(\xi_1)}{\xi_1^2} \text{ for } \frac{1}{2} \leq \frac{\varepsilon}{\xi_1} \leq 2, \quad (13)$$

where $A_1, A_2, A_3 > 0$ are independent of $\varepsilon, \xi_1 > 0$. Moreover, Sarikaya et al. defined the following useful operators:

$${}_{b_1^+} I_{\varpi} \Lambda(\xi_1) = \int_{b_1}^{\xi_1} \frac{\varpi(\xi_1 - \xi)}{\xi_1 - \xi} \Lambda(\xi) d\xi, \quad b_1 < \xi_1, \quad (14)$$

$${}_{b_2^-} I_{\varpi} \Lambda(\xi_1) = \int_{\xi_1}^{b_2} \frac{\varpi(\xi - \xi_1)}{\xi - \xi_1} \Lambda(\xi) d\xi, \quad b_2 > \xi_1. \quad (15)$$

About their efficiency, see [18; 21; 45]. Finally, Farid in [16], defined the following generic operators:

$$G_{b_1^+}^{\varpi, \Lambda_2} \Lambda_1(\xi_1) = \int_{b_1}^{\xi_1} \frac{\varpi(\Lambda_2(\xi_1) - \Lambda_2(\xi))}{\Lambda_2(\xi_1) - \Lambda_2(\xi)} \Lambda_2'(\xi) \Lambda_1(\xi) d\xi, \quad b_1 < \xi_1, \quad (16)$$

and

$$G_{b_2^-}^{\varpi, \Lambda_2} \Lambda_1(\xi_1) = \int_{\xi_1}^{b_2} \frac{\varpi(\Lambda_2(\xi) - \Lambda_2(\xi_1))}{\Lambda_2(\xi) - \Lambda_2(\xi_1)} \Lambda_2'(\xi) \Lambda_1(\xi) d\xi, \quad b_2 > \xi_1, \quad (17)$$

respectively.

The paper is constructed in this way: In Section 2, we will find an interesting identity with parameter λ and using generic integral operators form auxiliary equality, some new integral inequalities of trapezium, Ostrowski and Simpson type will be obtain. Section 3 is devoted to useful applications.

2. MAIN RESULTS

Let $\mathcal{P} = [mb_1, b_2]$, where $b_1 < b_2$ for some fixed $m \in (0, 1]$ with $\zeta \in [0, 1]$.

$$\Pi_m^{\varpi, \Upsilon}(\ell, \zeta) := \int_0^\zeta \frac{\varpi\left(\Upsilon\left(mb_1 + \xi \mathcal{F}_{\rho, \delta}^\sigma(\ell - mb_1)\right) - \Upsilon(mb_1)\right)}{\Upsilon\left(mb_1 + \xi \mathcal{F}_{\rho, \delta}^\sigma(\ell - mb_1)\right) - \Upsilon(mb_1)} \quad (18)$$

$$\times \Upsilon'\left(mb_1 + \xi \mathcal{F}_{\rho, \delta}^\sigma(\ell - mb_1)\right) d\xi < +\infty,$$

and

$$\begin{aligned} \Xi_m^{\sigma, \Upsilon}(\ell, \zeta) &:= \int_{\zeta}^1 \frac{\overline{\omega} \left(\Upsilon \left(m\ell + \mathcal{F}_{\rho, \delta}^{\sigma}(b_2 - m\ell) \right) - \Upsilon \left(m\ell + \xi \mathcal{F}_{\rho, \delta}^{\sigma}(b_2 - m\ell) \right) \right)}{\Upsilon \left(m\ell + \mathcal{F}_{\rho, \delta}^{\sigma}(b_2 - m\ell) \right) - \Upsilon \left(m\ell + \xi \mathcal{F}_{\rho, \delta}^{\sigma}(b_2 - m\ell) \right)} \quad (19) \\ &\quad \times \Upsilon' \left(m\ell + \xi \mathcal{F}_{\rho, \delta}^{\sigma}(b_2 - m\ell) \right) d\xi < +\infty. \end{aligned}$$

The following lemma will help us to find new results.

LEMMA 2.1. Let $\Lambda : \mathcal{P} \rightarrow \mathbb{R}$ be a differentiable function on $\mathcal{P}^\circ$ and $\lambda \in \mathbb{R}$. Assume that $\Lambda' \in \mathcal{L}(\mathcal{P})$ and $\mathcal{F}_{\rho, \lambda}^{\sigma}(b_2 - mb_1) > 0$, then

$$\begin{aligned} &\frac{\mathcal{F}_{\rho, \delta}^{\sigma}(\ell - mb_1)\Lambda \left(mb_1 + \zeta \mathcal{F}_{\rho, \delta}^{\sigma}(\ell - mb_1) \right) + \mathcal{F}_{\rho, \delta}^{\sigma}(b_2 - m\ell)\Lambda(m\ell)}{\mathcal{F}_{\rho, \delta}^{\sigma}(b_2 - mb_1)} \\ &- \frac{\lambda}{\mathcal{F}_{\rho, \delta}^{\sigma}(b_2 - mb_1)} \times \left[\frac{\mathcal{F}_{\rho, \delta}^{\sigma}(\ell - mb_1)\Lambda \left(mb_1 + \zeta \mathcal{F}_{\rho, \delta}^{\sigma}(\ell - mb_1) \right)}{\Pi_m^{\sigma, \Upsilon}(\ell, 1)} + \frac{\mathcal{F}_{\rho, \delta}^{\sigma}(b_2 - m\ell)\Lambda(m\ell)}{\Xi_m^{\sigma, \Upsilon}(\ell, 0)} \right] \\ &+ \frac{\lambda}{\mathcal{F}_{\rho, \delta}^{\sigma}(b_2 - mb_1)} \times \left[\frac{\mathcal{F}_{\rho, \delta}^{\sigma}(\ell - mb_1)\Lambda(mb_1)}{\Pi_m^{\sigma, \Upsilon}(\ell, 1)} + \frac{\mathcal{F}_{\rho, \delta}^{\sigma}(b_2 - m\ell)\Lambda(m\ell + \mathcal{F}_{\rho, \delta}^{\sigma}(b_2 - m\ell))}{\Xi_m^{\sigma, \Upsilon}(\ell, 0)} \right] \\ &- \frac{1}{\mathcal{F}_{\rho, \delta}^{\sigma}(b_2 - mb_1)} \times \left[\frac{G_{(mb_1 + \mathcal{F}_{\rho, \delta}^{\sigma}(\ell - mb_1))}^{\sigma, \Upsilon} - \Lambda(mb_1)}{\Pi_m^{\sigma, \Upsilon}(\ell, 1)} + \frac{G_{(m\ell) + \Lambda(m\ell + \mathcal{F}_{\rho, \delta}^{\sigma}(b_2 - m\ell))}^{\sigma, \Upsilon}}{\Xi_m^{\sigma, \Upsilon}(\ell, 0)} \right] \\ &= \frac{[\mathcal{F}_{\rho, \delta}^{\sigma}(\ell - mb_1)]^2}{\Pi_m^{\sigma, \Upsilon}(\ell, 1)\mathcal{F}_{\rho, \delta}^{\sigma}(b_2 - mb_1)} \times \int_0^1 \left[\Pi_m^{\sigma, \Upsilon}(\ell, \zeta) - \lambda \right] \Lambda' \left(mb_1 + \zeta \mathcal{F}_{\rho, \delta}^{\sigma}(\ell - mb_1) \right) d\zeta \quad (20) \\ &\quad - \frac{[\mathcal{F}_{\rho, \delta}^{\sigma}(b_2 - m\ell)]^2}{\Xi_m^{\sigma, \Upsilon}(\ell, 0)\mathcal{F}_{\rho, \delta}^{\sigma}(b_2 - mb_1)} \times \int_0^1 \left[\Xi_m^{\sigma, \Upsilon}(\ell, \zeta) - \lambda \right] \Lambda' \left(m\ell + \zeta \mathcal{F}_{\rho, \delta}^{\sigma}(b_2 - m\ell) \right) d\zeta. \end{aligned}$$

We denote

$$\begin{aligned} T_{\Lambda, \Pi_m^{\sigma, \Upsilon}, \Xi_m^{\sigma, \Upsilon}}(\lambda; \ell, b_1, b_2) &:= \frac{[\mathcal{F}_{\rho, \delta}^{\sigma}(\ell - mb_1)]^2}{\Pi_m^{\sigma, \Upsilon}(\ell, 1)\mathcal{F}_{\rho, \delta}^{\sigma}(b_2 - mb_1)} \quad (21) \\ &\quad \times \int_0^1 \left[\Pi_m^{\sigma, \Upsilon}(\ell, \zeta) - \lambda \right] \Lambda' \left(mb_1 + \zeta \mathcal{F}_{\rho, \delta}^{\sigma}(\ell - mb_1) \right) d\zeta \end{aligned}$$

$$-\frac{[\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)]^2}{\Xi_m^{\sigma,\Upsilon}(\ell, 0)\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \times \int_0^1 [\Xi_m^{\sigma,\Upsilon}(\ell, \zeta) - \lambda] \Lambda'(m\ell + \zeta\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)) d\zeta.$$

PROOF. By integrating by parts (21), we derive

$$\begin{aligned} T_{\Lambda, \Pi_m^{\sigma,\Upsilon}, \Xi_m^{\sigma,\Upsilon}}(\lambda; \ell, b_1, b_2) &= \frac{[\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)]^2}{\Pi_m^{\sigma,\Upsilon}(\ell, 1)\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \\ &\times \left\{ \int_0^1 \Pi_m^{\sigma,\Upsilon}(\ell, \zeta) \Lambda'(mb_1 + \zeta\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)) d\zeta - \lambda \int_0^1 \Lambda'(mb_1 + \zeta\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)) d\zeta \right\} \\ &\quad - \frac{[\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)]^2}{\Xi_m^{\sigma,\Upsilon}(\ell, 0)\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \\ &\times \left\{ \int_0^1 \Xi_m^{\sigma,\Upsilon}(\ell, \zeta) \Lambda'(m\ell + \zeta\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)) d\zeta - \lambda \int_0^1 \Lambda'(m\ell + \zeta\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)) d\zeta \right\} \\ &= \frac{[\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)]^2}{\Pi_m^{\sigma,\Upsilon}(\ell, 1)\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \times \left\{ \frac{\Pi_m^{\sigma,\Upsilon}(\ell, \zeta) \Lambda(mb_1 + \zeta\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1))}{\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)} \Big|_0^1 - \frac{1}{\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)} \right. \\ &\quad \times \int_0^1 \frac{\varpi\left(\Upsilon\left(mb_1 + \zeta\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)\right) - \Upsilon(mb_1)\right)}{\Upsilon\left(mb_1 + \zeta\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)\right) - \Upsilon(mb_1)} \\ &\quad \times \Upsilon'(mb_1 + \zeta\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)) \Lambda(mb_1 + \zeta\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)) d\zeta \\ &\quad \left. - \frac{\lambda}{\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)} \Lambda(mb_1 + \zeta\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)) \Big|_0^1 \right\} - \frac{[\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)]^2}{\Xi_m^{\sigma,\Upsilon}(\ell, 0)\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \\ &\quad \times \left\{ \frac{\Xi_m^{\sigma,\Upsilon}(\ell, \zeta) \Lambda(m\ell + \zeta\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell))}{\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)} \Big|_0^1 \right. \\ &\quad \left. - \frac{1}{\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)} \times \int_0^1 \frac{\varpi\left(\Upsilon\left(m\ell + \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)\right) - \Upsilon\left(m\ell + \zeta\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)\right)\right)}{\Upsilon\left(m\ell + \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)\right) - \Upsilon\left(m\ell + \zeta\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)\right)} \right. \\ &\quad \left. \times \Upsilon'(m\ell + \zeta\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)) \Lambda(m\ell + \zeta\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)) d\zeta \right\} \end{aligned}$$

$$\begin{aligned}
& - \frac{\lambda}{\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)} \Lambda(m\ell + \zeta \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)) \Big|_0^1 \Big\} \\
& = \frac{\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) \Lambda(mb_1 + \zeta \mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)) + \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) \Lambda(m\ell)}{\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \\
& - \frac{\lambda}{\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \times \left[\frac{\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) \Lambda(mb_1 + \zeta \mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1))}{\Pi_m^{\varpi,\Upsilon}(\ell, 1)} + \frac{\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) \Lambda(m\ell)}{\Xi_m^{\varpi,\Upsilon}(\ell, 0)} \right] \\
& + \frac{\lambda}{\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \times \left[\frac{\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) \Lambda(mb_1)}{\Pi_m^{\varpi,\Upsilon}(\ell, 1)} + \frac{\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) \Lambda(m\ell + \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell))}{\Xi_m^{\varpi,\Upsilon}(\ell, 0)} \right] \\
& - \frac{1}{\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \times \left[\frac{G_{(mb_1 + \mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1))}^{\varpi,\Upsilon} \Lambda(mb_1)}{\Pi_m^{\varpi,\Upsilon}(\ell, 1)} + \frac{G_{(m\ell + \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell))}^{\varpi,\Upsilon} \Lambda(m\ell + \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell))}{\Xi_m^{\varpi,\Upsilon}(\ell, 0)} \right].
\end{aligned}$$

REMARK 2.2. *a.* Taking $m = 1$, $\lambda = 0$, $\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) = \ell - mb_1$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) = b_2 - m\ell$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1) = b_2 - mb_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Lemma 2.1, then

$$T_\Lambda(\ell, b_1, b_2) := \Lambda(\ell) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \Lambda(\zeta) d\zeta.$$

b. Choosing $m = 1$, $\lambda = 1$, $\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) = \ell - mb_1$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) = b_2 - m\ell$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1) = b_2 - mb_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Lemma 2.1, then

$$\overline{T}_\Lambda(\ell, b_1, b_2) := \frac{(\ell - b_1)\Lambda(b_1) + (b_2 - \ell)\Lambda(b_2)}{b_2 - b_1} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \Lambda(\zeta) d\zeta.$$

c. Taking $m = 1$, $\ell = \frac{b_1 + b_2}{2}$, $\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) = \ell - mb_1$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) = b_2 - m\ell$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1) = b_2 - mb_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Lemma 2.1, then

$$T_\Lambda(\lambda; b_1, b_2) := \lambda \left[\frac{\Lambda(b_1) + \Lambda(b_2)}{2} \right] + (1 - \lambda) \Lambda\left(\frac{b_1 + b_2}{2}\right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \Lambda(\zeta) d\zeta.$$

THEOREM 2.3. Let $\Lambda: \mathcal{P} \rightarrow \mathbb{R}$ be a differentiable function on $\mathcal{P}^\circ$ and $\lambda \in [0, 1]$. If $|\Lambda'|^q$ is generalized convex function on $\mathcal{P}$ and $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1) > 0$, then for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$|T_{\Lambda, \Pi_m^{\varpi,\Upsilon}, \Xi_m^{\varpi,\Upsilon}}(\lambda; \ell, b_1, b_2)|$$

$$\begin{aligned}
&\leq \frac{[\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)]^2}{\sqrt[q]{2\Pi_m^{\sigma,\Upsilon}(\ell, 1)\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)}} \sqrt[p]{B_{\Pi_m}^{\sigma,\Upsilon}(\ell; \lambda, p)} \times \sqrt[q]{|\Lambda'(mb_1)|^q + |\Lambda'(\ell)|^q} \quad (22) \\
&+ \frac{[\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)]^2}{\sqrt[q]{2\Pi_m^{\sigma,\Upsilon}(\ell, 0)\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)}} \sqrt[p]{B_{\Xi_m}^{\sigma,\Upsilon}(\ell; \lambda, p)} \times \sqrt[q]{|\Lambda'(m\ell)|^q + |\Lambda'(b_2)|^q},
\end{aligned}$$

where

$$B_{\Pi_m}^{\sigma,\Upsilon}(\ell; \lambda, p) := \int_0^1 \left| \Pi_m^{\sigma,\Upsilon}(\ell, \zeta) - \lambda \right|^p d\zeta, \quad B_{\Xi_m}^{\sigma,\Upsilon}(\ell; \lambda, p) := \int_0^1 \left| \Xi_m^{\sigma,\Upsilon}(\ell, \zeta) - \lambda \right|^p d\zeta. \quad (23)$$

PROOF. Applying Lemma 2.1, generalized convexity of $|\Lambda'|^q$, Hölder's inequality, we get

$$\begin{aligned}
&\left| T_{\Lambda, \Pi_m^{\sigma,\Upsilon}, \Xi_m^{\sigma,\Upsilon}}(\lambda; \ell, b_1, b_2) \right| \\
&\leq \frac{[\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)]^2}{\Pi_m^{\sigma,\Upsilon}(\ell, 1)\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \times \int_0^1 \left| \Pi_m^{\sigma,\Upsilon}(\ell, \zeta) - \lambda \right| \left| \Lambda'(mb_1 + \zeta\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)) \right| d\zeta \\
&+ \frac{[\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)]^2}{\Xi_m^{\sigma,\Upsilon}(\ell, 0)\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \times \int_0^1 \left| \Xi_m^{\sigma,\Upsilon}(\ell, \zeta) - \lambda \right| \left| \Lambda'(m\ell + \zeta\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)) \right| d\zeta \\
&\leq \frac{[\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)]^2}{\Pi_m^{\sigma,\Upsilon}(\ell, 1)\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \\
&\times \left(\int_0^1 \left| \Pi_m^{\sigma,\Upsilon}(\ell, \zeta) - \lambda \right|^p d\zeta \right)^{\frac{1}{p}} \left(\int_0^1 \left| \Lambda'(mb_1 + \zeta\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)) \right|^q d\zeta \right)^{\frac{1}{q}} \\
&+ \frac{[\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)]^2}{\Xi_m^{\sigma,\Upsilon}(\ell, 0)\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \\
&\times \left(\int_0^1 \left| \Xi_m^{\sigma,\Upsilon}(\ell, \zeta) - \lambda \right|^p d\zeta \right)^{\frac{1}{p}} \left(\int_0^1 \left| \Lambda'(m\ell + \zeta\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)) \right|^q d\zeta \right)^{\frac{1}{q}} \\
&\leq \frac{[\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)]^2}{\Pi_m^{\sigma,\Upsilon}(\ell, 1)\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \sqrt[p]{B_{\Pi_m}^{\sigma,\Upsilon}(\ell; \lambda, p)} \left(\int_0^1 \left[(1 - \zeta)|\Lambda'(mb_1)|^q + \zeta|\Lambda'(\ell)|^q \right] d\zeta \right)^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
& + \frac{[\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)]^2}{\Xi_m^{\sigma,\Upsilon}(\ell, 0) \mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \sqrt[p]{B_{\Xi_m}^{\sigma,\Upsilon}(\ell; \lambda, p)} \left(\int_0^1 [(1 - \zeta)|\Lambda'(m\ell)|^q + \zeta|\Lambda'(b_2)|^q] d\zeta \right)^{\frac{1}{q}} \\
& = \frac{[\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)]^2}{\sqrt[q]{2\Pi_m^{\sigma,\Upsilon}(\ell, 1) \mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)}} \sqrt[p]{B_{\Pi_m}^{\sigma,\Upsilon}(\ell; \lambda, p)} \times \sqrt[q]{|\Lambda'(mb_1)|^q + |\Lambda'(\ell)|^q} \\
& + \frac{[\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)]^2}{\sqrt[q]{2\Pi_m^{\sigma,\Upsilon}(\ell, 0) \mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)}} \sqrt[p]{B_{\Xi_m}^{\sigma,\Upsilon}(\ell; \lambda, p)} \times \sqrt[q]{|\Lambda'(m\ell)|^q + |\Lambda'(b_2)|^q}.
\end{aligned}$$

COROLLARY 2.4. Taking $p = 2 = q$ in Theorem 2.3, we have

$$\begin{aligned}
& |T_{\Lambda, \Pi_m^{\sigma,\Upsilon}, \Xi_m^{\sigma,\Upsilon}}(\lambda; \ell, b_1, b_2)| \\
& \leq \frac{[\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)]^2}{\sqrt{2\Pi_m^{\sigma,\Upsilon}(\ell, 1) \mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)}} \sqrt{B_{\Pi_m}^{\sigma,\Upsilon}(\ell; \lambda, 2)} \times \sqrt{|\Lambda'(mb_1)|^2 + |\Lambda'(\ell)|^2} \quad (24) \\
& + \frac{[\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)]^2}{\sqrt{2\Pi_m^{\sigma,\Upsilon}(\ell, 0) \mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)}} \sqrt{B_{\Xi_m}^{\sigma,\Upsilon}(\ell; \lambda, 2)} \times \sqrt{|\Lambda'(m\ell)|^2 + |\Lambda'(b_2)|^2}.
\end{aligned}$$

COROLLARY 2.5. Choosing $|\Lambda'| \leq \mathcal{K}$ in Theorem 2.3, we get

$$\begin{aligned}
& |T_{\Lambda, \Pi_m^{\sigma,\Upsilon}, \Xi_m^{\sigma,\Upsilon}}(\lambda; \ell, b_1, b_2)| \leq \frac{\mathcal{K}}{\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \quad (25) \\
& \times \left[\frac{[\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)]^2}{\Pi_m^{\sigma,\Upsilon}(\ell, 1)} \sqrt[p]{B_{\Pi_m}^{\sigma,\Upsilon}(\ell; \lambda, p)} + \frac{[\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)]^2}{\Xi_m^{\sigma,\Upsilon}(\ell, 0)} \sqrt[p]{B_{\Xi_m}^{\sigma,\Upsilon}(\ell; \lambda, p)} \right].
\end{aligned}$$

COROLLARY 2.6. Taking $m = 1$, $\lambda = 0$, $\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) = \ell - mb_1$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) = b_2 - m\ell$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1) = b_2 - mb_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Theorem 2.3, we obtain

$$\begin{aligned}
& |T_\Lambda(\ell, b_1, b_2)| \leq \frac{1}{\sqrt[q]{2} \sqrt[q]{p+1} (b_2 - b_1)} \quad (26) \\
& \times \left\{ (\ell - b_1)^2 \sqrt[q]{|\Lambda'(b_1)|^q + |\Lambda'(\ell)|^q} + (b_2 - \ell)^2 \sqrt[q]{|\Lambda'(\ell)|^q + |\Lambda'(b_2)|^q} \right\}.
\end{aligned}$$

COROLLARY 2.7. Choosing $\ell = \frac{b_1 + b_2}{2}$ in Corollary 2.6, we have

$$|T_\Lambda(b_1, b_2)| \leq \frac{(b_2 - b_1)}{4 \sqrt[q]{2} \sqrt[q]{p+1}} \quad (27)$$

$$\times \left\{ \sqrt[q]{|\Lambda'(b_1)|^q + \left| \Lambda' \left(\frac{b_1 + b_2}{2} \right) \right|^q} + \sqrt[q]{\left| \Lambda' \left(\frac{b_1 + b_2}{2} \right) \right|^q + |\Lambda'(b_2)|^q} \right\}.$$

COROLLARY 2.8. Taking $m = 1$, $\lambda = 1$, $\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) = \ell - mb_1$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) = b_2 - m\ell$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1) = b_2 - mb_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Theorem 2.3, we get

$$|\overline{T}_\Lambda(\ell, b_1, b_2)| \leq \frac{1}{\sqrt[q]{2} \sqrt[p]{p+1} (b_2 - b_1)} \quad (28)$$

$$\times \left\{ (\ell - b_1)^2 \sqrt[q]{|\Lambda'(b_1)|^q + |\Lambda'(\ell)|^q} + (b_2 - \ell)^2 \sqrt[q]{|\Lambda'(\ell)|^q + |\Lambda'(b_2)|^q} \right\}.$$

COROLLARY 2.9. Choosing $m = 1$, $\lambda = \frac{1}{3}$, $\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) = \ell - mb_1$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) = b_2 - m\ell$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1) = b_2 - mb_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Theorem 2.3, we obtain

$$\left| T_\Lambda \left(\frac{1}{3}; b_1, b_2 \right) \right| \leq \frac{1}{\sqrt[q]{2} (b_2 - b_1)} \sqrt[p]{\frac{2^{p+1} + 1}{3^{p+1} (p+1)}} \quad (29)$$

$$\times \left\{ (\ell - b_1)^2 \sqrt[q]{|\Lambda'(b_1)|^q + |\Lambda'(\ell)|^q} + (b_2 - \ell)^2 \sqrt[q]{|\Lambda'(\ell)|^q + |\Lambda'(b_2)|^q} \right\}.$$

COROLLARY 2.10. Substituting $\lambda = 0$ and $\varpi(\zeta) = \zeta$ in Theorem 2.3, we have

$$|T_{\Lambda, \Pi_m^r, \Xi_m^r}(0; \ell, b_1, b_2)| \leq \frac{1}{\sqrt[q]{2} \mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \quad (30)$$

$$\begin{aligned} & \times \left\{ \sqrt[q]{\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)} \sqrt[p]{B_1^r(\ell; p)} \times \sqrt[q]{|\Lambda'(mb_1)|^q + |\Lambda'(\ell)|^q} \right. \\ & \left. + \sqrt[q]{\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)} \sqrt[p]{B_2^r(\ell; p)} \times \sqrt[q]{|\Lambda'(m\ell)|^q + |\Lambda'(b_2)|^q} \right\}, \end{aligned}$$

where

$$B_1^r(\ell; p) := \int_{mb_1}^{mb_1 + \mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)} [\Upsilon(\zeta) - \Upsilon(mb_1)]^p d\zeta \quad (31)$$

and

$$B_2^r(\ell; p) := \int_{m\ell}^{m\ell + \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)} [\Upsilon(m\ell + \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)) - \Upsilon(\zeta)]^p d\zeta. \quad (32)$$

COROLLARY 2.11. For $\lambda = 0$ and $\varpi(\zeta) = \frac{\zeta^\alpha}{\Gamma(\alpha)}$ in Theorem 2.3, we get

$$\begin{aligned} |T_{\Lambda, \Pi_m^{\mathbf{r}}, \Xi_m^{\mathbf{r}}}(0; \ell, b_1, b_2)| &\leq \frac{1}{\sqrt[q]{2} \mathcal{F}_{\rho, \delta}^\sigma(b_2 - mb_1)} \\ &\times \left\{ \sqrt[q]{\mathcal{F}_{\rho, \delta}^\sigma(\ell - mb_1)} \sqrt[p]{B_3^{\mathbf{r}}(\ell; p, \alpha)} \times \sqrt[q]{|\Lambda'(mb_1)|^q + |\Lambda'(\ell)|^q} \right. \\ &\left. + \sqrt[q]{\mathcal{F}_{\rho, \delta}^\sigma(b_2 - m\ell)} \sqrt[p]{B_4^{\mathbf{r}}(\ell; p, \alpha)} \times \sqrt[q]{|\Lambda'(m\ell)|^q + |\Lambda'(b_2)|^q} \right\}, \end{aligned} \quad (33)$$

where

$$B_3^{\mathbf{r}}(\ell; p, \alpha) := \int_{mb_1}^{mb_1 + \mathcal{F}_{\rho, \delta}^\sigma(\ell - mb_1)} [\Upsilon(\zeta) - \Upsilon(mb_1)]^{p\alpha} d\zeta \quad (34)$$

and

$$B_4^{\mathbf{r}}(\ell; p, \alpha) := \int_{m\ell}^{m\ell + \mathcal{F}_{\rho, \delta}^\sigma(b_2 - m\ell)} [\Upsilon(m\ell + \mathcal{F}_{\rho, \delta}^\sigma(b_2 - m\ell)) - \Upsilon(\zeta)]^{p\alpha} d\zeta. \quad (35)$$

COROLLARY 2.12. Substituting $\lambda = 0$ and $\varpi(\zeta) = \frac{\zeta^{\frac{\alpha}{\kappa}}}{\kappa \Gamma_\kappa(\alpha)}$ in Theorem 2.3, we obtain

$$\begin{aligned} |T_{\Lambda, \Pi_m^{\mathbf{r}}, \Xi_m^{\mathbf{r}}}(0; \ell, b_1, b_2)| &\leq \frac{1}{\sqrt[q]{2} \mathcal{F}_{\rho, \delta}^\sigma(b_2 - mb_1)} \\ &\times \left\{ \sqrt[q]{\mathcal{F}_{\rho, \delta}^\sigma(\ell - mb_1)} \sqrt[p]{B_5^{\mathbf{r}}(\ell; p, \alpha, \kappa)} \times \sqrt[q]{|\Lambda'(mb_1)|^q + |\Lambda'(\ell)|^q} \right. \\ &\left. + \sqrt[q]{\mathcal{F}_{\rho, \delta}^\sigma(b_2 - m\ell)} \sqrt[p]{B_6^{\mathbf{r}}(\ell; p, \alpha, \kappa)} \times \sqrt[q]{|\Lambda'(m\ell)|^q + |\Lambda'(b_2)|^q} \right\}, \end{aligned} \quad (36)$$

where

$$B_5^{\mathbf{r}}(\ell; p, \alpha, \kappa) := \int_{mb_1}^{mb_1 + \mathcal{F}_{\rho, \delta}^\sigma(\ell - mb_1)} [\Upsilon(\zeta) - \Upsilon(mb_1)]^{\frac{p\alpha}{\kappa}} d\zeta \quad (37)$$

and

$$B_6^{\mathbf{r}}(\ell; p, \alpha, \kappa) := \int_{m\ell}^{m\ell + \mathcal{F}_{\rho, \delta}^\sigma(b_2 - m\ell)} [\Upsilon(m\ell + \mathcal{F}_{\rho, \delta}^\sigma(b_2 - m\ell)) - \Upsilon(\zeta)]^{\frac{p\alpha}{\kappa}} d\zeta. \quad (38)$$

COROLLARY 2.13. For $\lambda = 0$, $\forall \xi \in [0, \zeta]$, $\varpi_{\mathbf{r}}(\ell, \zeta) = \zeta(\Upsilon(mb_1 + \mathcal{F}_{\rho, \delta}^\sigma(\ell - mb_1)) - \zeta)^{\alpha-1}$ and $\forall \xi \in [\zeta, 1]$, $\varpi_{\mathbf{r}}(\ell, \zeta) = \zeta(\Upsilon(m\ell + \mathcal{F}_{\rho, \delta}^\sigma(b_2 - m\ell)) - \zeta)^{\alpha-1}$ in Theorem 2.3, we

have

$$\begin{aligned}
 |T_{\Lambda, \Pi_m^Y, \Xi_m^Y}(0; \ell, b_1, b_2)| &\leq \frac{[\mathcal{F}_{\rho, \delta}^\sigma(\ell - mb_1)]^{\frac{q+1}{q}}}{\sqrt[q]{2} [\Upsilon(mb_1 + \mathcal{F}_{\rho, \delta}^\sigma(\ell - mb_1)) - \Upsilon(mb_1)] \mathcal{F}_{\rho, \delta}^\sigma(b_2 - mb_1)} \quad (39) \\
 &\times \sqrt[p]{B_7^Y(\ell; p)} \times \sqrt[q]{|\Lambda'(mb_1)|^q + |\Lambda'(\ell)|^q} \\
 &+ \frac{[\mathcal{F}_{\rho, \delta}^\sigma(b_2 - m\ell)]^{\frac{q+1}{q}}}{\sqrt[q]{2} [\Upsilon^\alpha(m\ell + \mathcal{F}_{\rho, \delta}^\sigma(b_2 - m\ell)) - \Upsilon^\alpha(m\ell)] \mathcal{F}_{\rho, \delta}^\sigma(b_2 - mb_1)} \\
 &\times \sqrt[p]{B_8^Y(\ell; p, \alpha)} \times \sqrt[q]{|\Lambda'(m\ell)|^q + |\Lambda'(b_2)|^q},
 \end{aligned}$$

where

$$B_7^Y(\ell; p) := \int_{mb_1}^{mb_1 + \mathcal{F}_{\rho, \delta}^\sigma(\ell - mb_1)} [\Upsilon(\zeta) - \Upsilon(mb_1)]^p d\zeta \quad (40)$$

and

$$B_8^Y(\ell; p, \alpha) := \int_{m\ell}^{m\ell + \mathcal{F}_{\rho, \delta}^\sigma(b_2 - m\ell)} [\Upsilon^\alpha(m\ell + \mathcal{F}_{\rho, \delta}^\sigma(b_2 - m\ell)) - \Upsilon^\alpha(\zeta)]^p d\zeta. \quad (41)$$

COROLLARY 2.14. Substituting $\lambda = 0$ and $\varpi(\zeta) = \frac{\zeta}{\alpha} \exp(-A\zeta)$, where $A = \frac{1-\alpha}{\alpha}$, in Theorem 2.3, we get

$$\begin{aligned}
 |T_{\Lambda, \Pi_m^Y, \Xi_m^Y}(0; \ell, b_1, b_2)| &\leq \frac{[\mathcal{F}_{\rho, \delta}^\sigma(\ell - mb_1)]^{\frac{q+1}{q}}}{\sqrt[q]{2} \mathcal{F}_{\rho, \delta}^\sigma(b_2 - mb_1)} \quad (42) \\
 &\times \sqrt[p]{B_9^Y(\ell; p, A)} \times \sqrt[q]{|\Lambda'(mb_1)|^q + |\Lambda'(\ell)|^q} \\
 &+ \frac{[\mathcal{F}_{\rho, \delta}^\sigma(b_2 - m\ell)]^{\frac{q+1}{q}}}{\sqrt[q]{2} \mathcal{F}_{\rho, \delta}^\sigma(b_2 - mb_1)} \sqrt[p]{B_{10}^Y(\ell; p, A)} \times \sqrt[q]{|\Lambda'(m\ell)|^q + |\Lambda'(b_2)|^q},
 \end{aligned}$$

where

$$B_9^Y(\ell; p, A) := \int_{mb_1}^{mb_1 + \mathcal{F}_{\rho, \delta}^\sigma(\ell - mb_1)} \left\{ 1 - \exp[A(\Upsilon(mb_1) - \Upsilon(\zeta))] \right\}^p d\zeta \quad (43)$$

and

$$B_{10}^Y(\ell; p, A) := \int_{m\ell}^{m\ell + \mathcal{F}_{\rho, \delta}^\sigma(b_2 - m\ell)} \left\{ 1 - \exp[A(\Upsilon(\zeta) - \Upsilon(m\ell + \mathcal{F}_{\rho, \delta}^\sigma(b_2 - m\ell)))] \right\}^p d\zeta. \quad (44)$$

THEOREM 2.15. Let $\Lambda: \mathcal{P} \rightarrow \mathbb{R}$ be a differentiable function on $\mathcal{P}^\circ$ and $\lambda \in [0, 1]$. If $|\Lambda'|^q$ is generalized convex on $\mathcal{P}$ and $\mathcal{F}_{\rho, \delta}^\sigma(b_2 - mb_1) > 0$, then for $q \geq 1$, we have

$$\begin{aligned} |T_{\Lambda, \Pi_m^{\sigma, \Upsilon}, \Xi_m^{\sigma, \Upsilon}}(\lambda; \ell, b_1, b_2)| &\leq \frac{[\mathcal{F}_{\rho, \delta}^\sigma(\ell - mb_1)]^2}{\Pi_m^{\sigma, \Upsilon}(\ell, 1)\mathcal{F}_{\rho, \delta}^\sigma(b_2 - mb_1)} \left[B_{\Pi_m}^{\sigma, \Upsilon}(\ell; \lambda, 1)\right]^{1-\frac{1}{q}} \\ &\times \sqrt[q]{\left[B_{\Pi_m}^{\sigma, \Upsilon}(\ell; \lambda, 1) - E_{\Pi_m}^{\sigma, \Upsilon}(\ell; \lambda)\right] |\Lambda'(mb_1)|^q + E_{\Pi_m}^{\sigma, \Upsilon}(\ell; \lambda) |\Lambda'(\ell)|^q} \\ &+ \frac{[\mathcal{F}_{\rho, \delta}^\sigma(b_2 - m\ell)]^2}{\Xi_m^{\sigma, \Upsilon}(\ell, 0)\mathcal{F}_{\rho, \delta}^\sigma(b_2 - mb_1)} \left[B_{\Xi_m}^{\sigma, \Upsilon}(\ell; \lambda, 1)\right]^{1-\frac{1}{q}} \\ &\times \sqrt[q]{\left[B_{\Xi_m}^{\sigma, \Upsilon}(\ell; \lambda, 1) - G_{\Xi_m}^{\sigma, \Upsilon}(\ell; \lambda)\right] |\Lambda'(m\ell)|^q + G_{\Xi_m}^{\sigma, \Upsilon}(\ell; \lambda) |\Lambda'(b_2)|^q}, \end{aligned} \quad (45)$$

where

$$E_{\Pi_m}^{\sigma, \Upsilon}(\ell; \lambda) := \int_0^1 \zeta \left| \Pi_m^{\sigma, \Upsilon}(\ell, \zeta) - \lambda \right| d\zeta, \quad G_{\Xi_m}^{\sigma, \Upsilon}(\ell; \lambda) := \int_0^1 \zeta \left| \Xi_m^{\sigma, \Upsilon}(\ell, \zeta) - \lambda \right| d\zeta. \quad (46)$$

PROOF. By Lemma 2.1, generalized convexity of $|\Lambda'|^q$ and power mean inequality, we get

$$\begin{aligned} &|T_{\Lambda, \Pi_m^{\sigma, \Upsilon}, \Xi_m^{\sigma, \Upsilon}}(\lambda; \ell, b_1, b_2)| \\ &\leq \frac{[\mathcal{F}_{\rho, \delta}^\sigma(\ell - mb_1)]^2}{\Pi_m^{\sigma, \Upsilon}(\ell, 1)\mathcal{F}_{\rho, \delta}^\sigma(b_2 - mb_1)} \times \int_0^1 \left| \Pi_m^{\sigma, \Upsilon}(\ell, \zeta) - \lambda \right| \left| \Lambda'(mb_1 + \zeta\mathcal{F}_{\rho, \delta}^\sigma(\ell - mb_1)) \right| d\zeta \\ &+ \frac{[\mathcal{F}_{\rho, \delta}^\sigma(b_2 - m\ell)]^2}{\Xi_m^{\sigma, \Upsilon}(\ell, 0)\mathcal{F}_{\rho, \delta}^\sigma(b_2 - mb_1)} \times \int_0^1 \left| \Xi_m^{\sigma, \Upsilon}(\ell, \zeta) - \lambda \right| \left| \Lambda'(m\ell + \zeta\mathcal{F}_{\rho, \delta}^\sigma(b_2 - m\ell)) \right| d\zeta \\ &\leq \frac{[\mathcal{F}_{\rho, \delta}^\sigma(\ell - mb_1)]^2}{\Pi_m^{\sigma, \Upsilon}(\ell, 1)\mathcal{F}_{\rho, \delta}^\sigma(b_2 - mb_1)} \\ &\times \left(\int_0^1 \left| \Pi_m^{\sigma, \Upsilon}(\ell, \zeta) - \lambda \right| d\zeta \right)^{1-\frac{1}{q}} \left(\int_0^1 \left| \Pi_m^{\sigma, \Upsilon}(\ell, \zeta) - \lambda \right| \left| \Lambda'(mb_1 + \zeta\mathcal{F}_{\rho, \delta}^\sigma(\ell - mb_1)) \right|^q d\zeta \right)^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
& + \frac{[\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)]^2}{\Xi_m^{\sigma,\Upsilon}(\ell, 0) \mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \\
& \times \left(\int_0^1 |\Xi_m^{\sigma,\Upsilon}(\ell, \zeta) - \lambda| d\zeta \right)^{1-\frac{1}{q}} \left(\int_0^1 |\Xi_m^{\sigma,\Upsilon}(\ell, \zeta) - \lambda| |\Lambda'(m\ell + \zeta \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell))|^q d\zeta \right)^{\frac{1}{q}} \\
& \leq \frac{[\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)]^2}{\Pi_m^{\sigma,\Upsilon}(\ell, 1) \mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \sqrt[p]{B_{\Pi_m}^{\sigma,\Upsilon}(\ell; \lambda, p)} \\
& \times \left(\int_0^1 |\Pi_m^{\sigma,\Upsilon}(\ell, \zeta) - \lambda| \left[(1 - \zeta) |\Lambda'(mb_1)|^q + \zeta |\Lambda'(\ell)|^q \right] d\zeta \right)^{\frac{1}{q}} \\
& + \frac{[\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)]^2}{\Xi_m^{\sigma,\Upsilon}(\ell, 0) \mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \sqrt[p]{B_{\Xi_m}^{\sigma,\Upsilon}(\ell; \lambda, p)} \\
& \times \left(\int_0^1 |\Xi_m^{\sigma,\Upsilon}(\ell, \zeta) - \lambda| \left[(1 - \zeta) |\Lambda'(m\ell)|^q + \zeta |\Lambda'(b_2)|^q \right] d\zeta \right)^{\frac{1}{q}} \\
& = \frac{[\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)]^2}{\Pi_m^{\sigma,\Upsilon}(\ell, 1) \mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \left[B_{\Pi_m}^{\sigma,\Upsilon}(\ell; \lambda, 1) \right]^{1-\frac{1}{q}} \\
& \times \sqrt[q]{\left[B_{\Pi_m}^{\sigma,\Upsilon}(\ell; \lambda, 1) - E_{\Pi_m}^{\sigma,\Upsilon}(\ell; \lambda) \right] |\Lambda'(mb_1)|^q + E_{\Pi_m}^{\sigma,\Upsilon}(\ell; \lambda) |\Lambda'(\ell)|^q} \\
& + \frac{[\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)]^2}{\Xi_m^{\sigma,\Upsilon}(\ell, 0) \mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \left[B_{\Xi_m}^{\sigma,\Upsilon}(\ell; \lambda, 1) \right]^{1-\frac{1}{q}} \\
& \times \sqrt[q]{\left[B_{\Xi_m}^{\sigma,\Upsilon}(\ell; \lambda, 1) - G_{\Xi_m}^{\sigma,\Upsilon}(\ell; \lambda) \right] |\Lambda'(m\ell)|^q + G_{\Xi_m}^{\sigma,\Upsilon}(\ell; \lambda) |\Lambda'(b_2)|^q}.
\end{aligned}$$

COROLLARY 2.16. For $q = 1$ in Theorem 2.15, we get

$$\begin{aligned}
& |T_{\Lambda, \Pi_m^{\sigma,\Upsilon}, \Xi_m^{\sigma,\Upsilon}}(\lambda; \ell, b_1, b_2)| \leq \frac{[\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)]^2}{\Pi_m^{\sigma,\Upsilon}(\ell, 1) \mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \\
& \times \left[\left(B_{\Pi_m}^{\sigma,\Upsilon}(\ell; \lambda, 1) - E_{\Pi_m}^{\sigma,\Upsilon}(\ell; \lambda) \right) |\Lambda'(mb_1)| + E_{\Pi_m}^{\sigma,\Upsilon}(\ell; \lambda) |\Lambda'(\ell)| \right]
\end{aligned} \tag{47}$$

$$\begin{aligned}
& + \frac{[\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)]^2}{\Xi_m^{\varpi,\Upsilon}(\ell, 0) \mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \\
& \times \left[(B_{\Xi_m}^{\varpi,\Upsilon}(\ell; \lambda, 1) - G_{\Xi_m}^{\varpi,\Upsilon}(\ell; \lambda)) |\Lambda'(m\ell)| + G_{\Xi_m}^{\varpi,\Upsilon}(\ell; \lambda) |\Lambda'(b_2)| \right].
\end{aligned}$$

COROLLARY 2.17. Taking $|\Lambda'| \leq \mathcal{H}$ in Theorem 2.15, we have

$$|T_{\Lambda, \Pi_m^{\varpi,\Upsilon}, \Xi_m^{\varpi,\Upsilon}}(\lambda; \ell, b_1, b_2)| \leq \frac{\mathcal{H}}{\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \quad (48)$$

$$\times \left[\frac{[\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)]^2}{\Pi_m^{\varpi,\Upsilon}(\ell, 1)} B_{\Pi_m}^{\varpi,\Upsilon}(\ell; \lambda, 1) + \frac{[\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)]^2}{\Xi_m^{\varpi,\Upsilon}(\ell, 0)} B_{\Xi_m}^{\varpi,\Upsilon}(\ell; \lambda, 1) \right].$$

COROLLARY 2.18. Choosing $m = 1$, $\lambda = 0$, $\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) = \ell - mb_1$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) = b_2 - m\ell$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1) = b_2 - mb_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Theorem 2.15, we get

$$|T_\Lambda(\ell, b_1, b_2)| \leq \frac{1}{2\sqrt[q]{3}(b_2 - b_1)} \quad (49)$$

$$\times \left\{ (\ell - b_1)^2 \sqrt[q]{|\Lambda'(b_1)|^q + 2|\Lambda'(\ell)|^q} + (b_2 - \ell)^2 \sqrt[q]{2|\Lambda'(\ell)|^q + |\Lambda'(b_2)|^q} \right\}.$$

COROLLARY 2.19. Taking $\ell = \frac{b_1 + b_2}{2}$ in Corollary 2.18, we obtain

$$|T_\Lambda(b_1, b_2)| \leq \frac{(b_2 - b_1)}{8\sqrt[q]{3}} \quad (50)$$

$$\times \left\{ \sqrt[q]{|\Lambda'(b_1)|^q + 2\left|\Lambda'\left(\frac{b_1 + b_2}{2}\right)\right|^q} + \sqrt[q]{2\left|\Lambda'\left(\frac{b_1 + b_2}{2}\right)\right|^q + |\Lambda'(b_2)|^q} \right\}.$$

COROLLARY 2.20. Choosing $m = 1$, $\lambda = 1$, $\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) = \ell - mb_1$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) = b_2 - m\ell$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1) = b_2 - mb_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Theorem 2.15, we have

$$|\overline{T}_\Lambda(\ell, b_1, b_2)| \leq \frac{1}{2\sqrt[q]{3}(b_2 - b_1)} \quad (51)$$

$$\times \left\{ (\ell - b_1)^2 \sqrt[q]{2|\Lambda'(b_1)|^q + |\Lambda'(\ell)|^q} + (b_2 - \ell)^2 \sqrt[q]{|\Lambda'(\ell)|^q + 2|\Lambda'(b_2)|^q} \right\}.$$

COROLLARY 2.21. Taking $m = 1, \lambda = \frac{1}{3}, \mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) = \ell - mb_1, \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) = b_2 - m\ell, \mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1) = b_2 - mb_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Theorem 2.15, we get

$$\left| T_\Lambda \left(\frac{1}{3}; b_1, b_2 \right) \right| \leq \frac{1}{2\sqrt[4]{243}(b_2 - b_1)} \quad (52)$$

$$\times \left\{ (\ell - b_1)^2 \sqrt[4]{185|\Lambda'(b_1)|^q + 58|\Lambda'(\ell)|^q} + (b_2 - \ell)^2 \sqrt[4]{195|\Lambda'(\ell)|^q + 48|\Lambda'(b_2)|^q} \right\}.$$

COROLLARY 2.22. Substituting $\lambda = 0$ and $\varpi(\zeta) = \zeta$ in Theorem 2.15, we obtain

$$\left| T_{\Lambda, \Pi_m^r, \Xi_m^r}(0; \ell, b_1, b_2) \right| \leq \frac{1}{[\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)]^{\frac{q+1}{q}} \mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \quad (53)$$

$$\begin{aligned} & \times \left[B_1^r(\ell; 1) \right]^{1-\frac{1}{q}} \sqrt[4]{ \left[B_1^r(\ell; 1) \mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) - C_1^r(\ell) \right] |\Lambda'(mb_1)|^q + C_1^r(\ell) |\Lambda'(\ell)|^q } \\ & + \frac{1}{[\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)]^{\frac{q+1}{q}} \mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \left[B_2^r(\ell; 1) \right]^{1-\frac{1}{q}} \\ & \times \sqrt[4]{ \left[B_2^r(\ell; 1) \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) - E_1^r(\ell) \right] |\Lambda'(m\ell)|^q + E_1^r(\ell) |\Lambda'(b_2)|^q }, \end{aligned}$$

where

$$C_1^r(\ell) := \int_{mb_1}^{mb_1 + \mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)} (\zeta - mb_1)(\Upsilon(\zeta) - \Upsilon(mb_1)) d\zeta, \quad (54)$$

$$E_1^r(\ell) := \int_{m\ell}^{m\ell + \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)} (\zeta - m\ell)(\Upsilon(m\ell + \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)) - \Upsilon(\zeta)) d\zeta. \quad (55)$$

COROLLARY 2.23. For $\lambda = 0$ and $\varpi(\zeta) = \frac{\zeta^\alpha}{\Gamma(\alpha)}$ in Theorem 2.15, we have

$$\begin{aligned} \left| T_{\Lambda, \Pi_m^r, \Xi_m^r}(0; \ell, b_1, b_2) \right| & \leq \frac{1}{[\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)]^{\frac{q+1}{q}} \mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \quad (56) \\ & \times \left[B_3^r(\ell; 1, \alpha) \right]^{1-\frac{1}{q}} \\ & \times \sqrt[4]{ \left[B_3^r(\ell; 1, \alpha) \mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) - C_1^r(\ell, \alpha) \right] |\Lambda'(mb_1)|^q + C_1^r(\ell, \alpha) |\Lambda'(\ell)|^q } \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{[\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)]^{\frac{q+1}{q}} \mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \left[B_4^\Upsilon(\ell; 1, \alpha) \right]^{1-\frac{1}{q}} \\
& \times \sqrt[q]{\left[B_4^\Upsilon(\ell; 1, \alpha) \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) - E_1^\Upsilon(\ell, \alpha) \right] |\Lambda'(m\ell)|^q + E_1^\Upsilon(\ell, \alpha) |\Lambda'(b_2)|^q},
\end{aligned}$$

where

$$C_1^\Upsilon(\ell, \alpha) := \int_{mb_1}^{mb_1 + \mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)} (\zeta - mb_1) [\Upsilon(\zeta) - \Upsilon(mb_1)]^\alpha d\zeta, \quad (57)$$

$$E_1^\Upsilon(\ell, \alpha) := \int_{m\ell}^{m\ell + \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)} (\zeta - m\ell) [\Upsilon(m\ell + \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)) - \Upsilon(\zeta)]^\alpha d\zeta. \quad (58)$$

COROLLARY 2.24. Substituting $\lambda = 0$ and $\varpi(\zeta) = \frac{\zeta^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.15, we get

$$|T_{\Lambda, \Pi_m^\Upsilon, \Xi_m^\Upsilon}(0; \ell, b_1, b_2)| \leq \frac{1}{[\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)]^{\frac{q+1}{q}} \mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \left[B_5^\Upsilon(\ell; 1, \alpha, \kappa) \right]^{1-\frac{1}{q}} \quad (59)$$

$$\times \sqrt[q]{\left[B_5^\Upsilon(\ell; 1, \alpha, \kappa) \mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) - C_1^\Upsilon(\ell, \alpha, \kappa) \right] |\Lambda'(mb_1)|^q + C_1^\Upsilon(\ell, \alpha, \kappa) |\Lambda'(\ell)|^q}$$

$$+ \frac{1}{[\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)]^{\frac{q+1}{q}} \mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \left[B_6^\Upsilon(\ell; 1, \alpha, \kappa) \right]^{1-\frac{1}{q}}$$

$$\times \sqrt[q]{\left[B_6^\Upsilon(\ell; 1, \alpha, \kappa) \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) - E_1^\Upsilon(\ell, \alpha, \kappa) \right] |\Lambda'(m\ell)|^q + E_1^\Upsilon(\ell, \alpha, \kappa) |\Lambda'(b_2)|^q},$$

where

$$C_1^\Upsilon(\ell, \alpha, \kappa) := \int_{mb_1}^{mb_1 + \mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)} (\zeta - mb_1) [\Upsilon(\zeta) - \Upsilon(mb_1)]^{\frac{\alpha}{k}} d\zeta, \quad (60)$$

$$E_1^\Upsilon(\ell, \alpha, \kappa) := \int_{m\ell}^{m\ell + \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)} (\zeta - m\ell) [\Upsilon(m\ell + \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)) - \Upsilon(\zeta)]^{\frac{\alpha}{k}} d\zeta. \quad (61)$$

COROLLARY 2.25. For $\lambda = 0$, $\forall \xi \in [0, \zeta]$, $\varpi_\Upsilon(\ell, \xi) = \zeta (\Upsilon(mb_1 + \mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)) - \xi)^{\alpha-1}$ and $\forall \xi \in [\zeta, 1]$, $\varpi_\Upsilon(\ell, \xi) = \zeta (\Upsilon(m\ell + \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)) - \xi)^{\alpha-1}$ in Theorem 2.15, we obtain

$$|T_{\Lambda, \Pi_m^\Upsilon, \Xi_m^\Upsilon}(0; \ell, b_1, b_2)| \leq \frac{1}{[\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1)]^{\frac{q+1}{q}} \mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1)} \quad (62)$$

$$\begin{aligned}
& \times \left[B_7^Y(\ell; 1, \alpha) \right]^{1-\frac{1}{q}} \sqrt[q]{\left[B_7^Y(\ell; 1, \alpha) \mathcal{F}_{\rho, \delta}^\sigma(\ell - mb_1) - C_1^Y(\ell) \right] |\Lambda'(mb_1)|^q + C_1^Y(\ell) |\Lambda'(\ell)|^q} \\
& + \frac{1}{\left[\mathcal{F}_{\rho, \delta}^\sigma(b_2 - m\ell) \right]^{\frac{q+1}{q}} \mathcal{F}_{\rho, \delta}^\sigma(b_2 - mb_1)} \left[B_8^Y(\ell; 1, \alpha) \right]^{1-\frac{1}{q}} \\
& \times \sqrt[q]{\left[B_8^Y(\ell; 1, \alpha) \mathcal{F}_{\rho, \delta}^\sigma(b_2 - m\ell) - L_2^Y(\ell, \alpha) \right] |\Lambda'(m\ell)|^q + L_2^Y(\ell, \alpha) |\Lambda'(b_2)|^q},
\end{aligned}$$

where

$$L_2^Y(\ell, \alpha) := \int_{m\ell}^{m\ell + \mathcal{F}_{\rho, \delta}^\sigma(b_2 - m\ell)} (\zeta - m\ell) \left[\Upsilon^\alpha(m\ell + \mathcal{F}_{\rho, \delta}^\sigma(b_2 - m\ell)) - \Upsilon^\alpha(\zeta) \right] d\zeta. \quad (63)$$

COROLLARY 2.26. Substituting $\lambda = 0$ and $\varpi(\zeta) = \frac{\zeta}{\alpha} \exp(-A\zeta)$, where $A = \frac{1-\alpha}{\alpha}$ in Theorem 2.15, we have

$$\begin{aligned}
& \left| T_{\Lambda, \Pi_m^Y, \Theta_m^Y}(0; \ell, b_1, b_2) \right| \leq \frac{1}{(1-\alpha) \left[\mathcal{F}_{\rho, \delta}^\sigma(\ell - mb_1) \right]^{\frac{q+1}{q}} \mathcal{F}_{\rho, \delta}^\sigma(b_2 - mb_1)} \quad (64) \\
& \times \left\{ \left[B_9^Y(\ell; 1, A) \right]^{1-\frac{1}{q}} \sqrt[q]{L_3^Y(\ell, A) |\Lambda'(mb_1)|^q + L_4^Y(\ell, A) |\Lambda'(\ell)|^q} \right. \\
& + \frac{1}{(1-\alpha) \left[\mathcal{F}_{\rho, \delta}^\sigma(b_2 - m\ell) \right]^{\frac{q+1}{q}} \mathcal{F}_{\rho, \delta}^\sigma(b_2 - mb_1)} \\
& \left. \times \left[B_{10}^Y(\ell; 1, A) \right]^{1-\frac{1}{q}} \sqrt[q]{L_5^Y(\ell, A) |\Lambda'(m\ell)|^q + L_6^Y(\ell, A) |\Lambda'(b_2)|^q} \right\},
\end{aligned}$$

where

$$L_3^Y(\ell, A) := \int_{mb_1}^{mb_1 + \mathcal{F}_{\rho, \delta}^\sigma(\ell - mb_1)} (mb_1 + \mathcal{F}_{\rho, \delta}^\sigma(\ell - mb_1) - \zeta) \quad (65)$$

$$\times \left\{ 1 - \exp \left[A (\Upsilon(mb_1) - \Upsilon(\zeta)) \right] \right\} d\zeta,$$

$$L_4^Y(\ell, A) := \int_{mb_1}^{mb_1 + \mathcal{F}_{\rho, \delta}^\sigma(\ell - mb_1)} (\zeta - mb_1) \left\{ 1 - \exp \left[A (\Upsilon(mb_1) - \Upsilon(\zeta)) \right] \right\} d\zeta, \quad (66)$$

$$L_5^Y(\ell, A) := \int_{m\ell}^{m\ell + \mathcal{F}_{\rho, \delta}^\sigma(b_2 - m\ell)} (m\ell + \mathcal{F}_{\rho, \delta}^\sigma(b_2 - m\ell) - \zeta) \quad (67)$$

$$\times \left\{ 1 - \exp \left[A \left(\Upsilon(\zeta) - \Upsilon(m\ell + \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)) \right) \right] \right\} d\zeta,$$

$$L_6^r(\ell, A) := \int_{m\ell}^{m\ell + \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)} (\zeta - m\ell) \left\{ 1 - \exp \left[A \left(\Upsilon(\zeta) - \Upsilon(m\ell + \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell)) \right) \right] \right\} d\zeta. \quad (68)$$

3. APPLICATIONS

For $b_1, b_2 \in \mathbb{R}$ and $0 < b_1 < b_2$ we recall:

(1) arithmetic mean:

$$A(b_1, b_2) = \frac{b_1 + b_2}{2};$$

(2) harmonic mean:

$$H(b_1, b_2) = \frac{2}{\frac{1}{b_1} + \frac{1}{b_2}};$$

(3) logarithmic mean:

$$L(b_1, b_2) = \frac{b_2 - b_1}{\ln|b_2| - \ln|b_1|};$$

(4) generalized log-mean:

$$L_r(b_1, b_2) = \left[\frac{b_2^{r+1} - b_1^{r+1}}{(r+1)(b_2 - b_1)} \right]^{\frac{1}{r}}; \quad r \in \mathbb{R} \setminus \{-1, 0\}.$$

From the main results in Section 2, we get

PROPOSITION 3.1. Let $r, b_1, b_2 \in \mathbb{R}$ with $0 < b_1 < b_2$, then for $r, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\left| A^r(b_1, b_2) - L_r^r(b_1, b_2) \right| \leq \frac{r(b_2 - b_1)}{4\sqrt[p]{p+1}} \quad (69)$$

$$\times \left\{ \sqrt[q]{A \left(b_1^{q(r-1)}, \left(\frac{b_1 + b_2}{2} \right)^{q(r-1)} \right)} + \sqrt[q]{A \left(\left(\frac{b_1 + b_2}{2} \right)^{q(r-1)}, b_2^{q(r-1)} \right)} \right\}.$$

PROOF. Taking $m = 1$, $\lambda = 0$, $\ell = \frac{b_1 + b_2}{2}$, $\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) = \ell - mb_1$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) = b_2 - m\ell$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1) = b_2 - mb_1$, $\Lambda(\zeta) = \zeta^r$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Theorem 2.3, we get (69).

PROPOSITION 3.2. Let $r, b_1, b_2 \in \mathbb{R}$ with $0 < b_1 < b_2$, then for $r, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we obtain

$$\left| A(b_1^r, b_2^r) - L_r^r(b_1, b_2) \right| \leq \frac{r(b_2 - b_1)}{4\sqrt[p]{p+1}} \quad (70)$$

$$\times \left\{ \sqrt[q]{A\left(b_1^{q(r-1)}, \left(\frac{b_1+b_2}{2}\right)^{q(r-1)}\right)} + \sqrt[q]{A\left(\left(\frac{b_1+b_2}{2}\right)^{q(r-1)}, b_2^{q(r-1)}\right)} \right\}.$$

PROOF. Choosing $m = 1, \lambda = 1, \ell = \frac{b_1+b_2}{2}, \mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) = \ell - mb_1, \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) = b_2 - m\ell, \mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1) = b_2 - mb_1, \Lambda(\zeta) = \zeta^r$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Theorem 2.3, we have (70).

PROPOSITION 3.3. Let $b_1, b_2 \in \mathbb{R}$ with $0 < b_1 < b_2$, then for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we get

$$\left| \frac{1}{A(b_1, b_2)} - \frac{1}{L(b_1, b_2)} \right| \leq \frac{(b_2 - b_1)}{4\sqrt[p]{p+1}} \quad (71)$$

$$\times \left\{ \frac{1}{\sqrt[q]{H\left(b_1^{2q}, \left(\frac{b_1+b_2}{2}\right)^{2q}\right)}} + \frac{1}{\sqrt[q]{H\left(\left(\frac{b_1+b_2}{2}\right)^{2q}, b_2^{2q}\right)}} \right\}.$$

PROOF. Taking $m = 1, \lambda = 0, \ell = \frac{b_1+b_2}{2}, \mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) = \ell - mb_1, \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) = b_2 - m\ell, \mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1) = b_2 - mb_1, \Lambda(\zeta) = \frac{1}{\zeta}$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Theorem 2.3, we obtain (71).

PROPOSITION 3.4. Let $b_1, b_2 \in \mathbb{R}$ with $0 < b_1 < b_2$, then for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\left| \frac{1}{H(b_1, b_2)} - \frac{1}{L(b_1, b_2)} \right| \leq \frac{(b_2 - b_1)}{4\sqrt[p]{p+1}} \quad (72)$$

$$\times \left\{ \frac{1}{\sqrt[q]{H\left(b_1^{2q}, \left(\frac{b_1+b_2}{2}\right)^{2q}\right)}} + \frac{1}{\sqrt[q]{H\left(\left(\frac{b_1+b_2}{2}\right)^{2q}, b_2^{2q}\right)}} \right\}.$$

PROOF. Choosing $m = 1, \lambda = 1, \ell = \frac{b_1+b_2}{2}, \mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) = \ell - mb_1, \mathcal{F}_{\rho,\delta}^\sigma(b_2 -$

$m\ell) = b_2 - m\ell$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1) = b_2 - mb_1$, $\Lambda(\zeta) = \frac{1}{\zeta}$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Theorem 2.3, we get (72).

PROPOSITION 3.5. Let $r, b_1, b_2 \in \mathbb{R}$ with $0 < b_1 < b_2$ and $r > 1$, then for $q \geq 1$, we obtain

$$\begin{aligned} & \left| A^r(b_1, b_2) - L_r^r(b_1, b_2) \right| \leq \sqrt[q]{\frac{2}{3}} \frac{r(b_2 - b_1)}{8} \\ & \times \left\{ \sqrt[q]{A\left(b_1^{q(r-1)}, 2\left(\frac{b_1+b_2}{2}\right)^{q(r-1)}\right)} + \sqrt[q]{A\left(2\left(\frac{b_1+b_2}{2}\right)^{q(r-1)}, b_2^{q(r-1)}\right)} \right\}. \end{aligned} \quad (73)$$

PROOF. Taking $m = 1$, $\lambda = 0$, $\ell = \frac{b_1+b_2}{2}$, $\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) = \ell - mb_1$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) = b_2 - m\ell$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1) = b_2 - mb_1$, $\Lambda(\zeta) = \zeta^r$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Theorem 2.15, we have (73).

PROPOSITION 3.6. Let $r, b_1, b_2 \in \mathbb{R}$ with $0 < b_1 < b_2$ and $r > 1$, then for $q \geq 1$, we get

$$\begin{aligned} & \left| A(b_1^r, b_2^r) - L_r^r(b_1, b_2) \right| \leq \sqrt[q]{\frac{2}{3}} \frac{r(b_2 - b_1)}{8} \\ & \times \left\{ \sqrt[q]{A\left(2b_1^{q(r-1)}, \left(\frac{b_1+b_2}{2}\right)^{q(r-1)}\right)} + \sqrt[q]{A\left(\left(\frac{b_1+b_2}{2}\right)^{q(r-1)}, 2b_2^{q(r-1)}\right)} \right\}. \end{aligned} \quad (74)$$

PROOF. Choosing $m = 1$, $\lambda = 1$, $\ell = \frac{b_1+b_2}{2}$, $\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) = \ell - mb_1$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) = b_2 - m\ell$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1) = b_2 - mb_1$, $\Lambda(\zeta) = \zeta^r$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Theorem 2.15, we obtain (74).

PROPOSITION 3.7. Let $b_1, b_2 \in \mathbb{R}$ with $0 < b_1 < b_2$, then for $q \geq 1$, we have

$$\begin{aligned} & \left| \frac{1}{A(b_1, b_2)} - \frac{1}{L(b_1, b_2)} \right| \leq \sqrt[q]{\frac{4}{3}} \frac{(b_2 - b_1)}{8} \\ & \times \left\{ \frac{1}{\sqrt[q]{H\left(2b_1^{2q}, \left(\frac{b_1+b_2}{2}\right)^{2q}\right)}} + \frac{1}{\sqrt[q]{H\left(\left(\frac{b_1+b_2}{2}\right)^{2q}, 2b_2^{2q}\right)}} \right\}. \end{aligned} \quad (75)$$

PROOF. Taking $m = 1$, $\lambda = 0$, $\ell = \frac{b_1+b_2}{2}$, $\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) = \ell - mb_1$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) =$

$b_2 - m\ell$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1) = b_2 - mb_1$, $\Lambda(\zeta) = \frac{1}{\zeta}$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Theorem 2.15, we get (75).

PROPOSITION 3.8. Let $b_1, b_2 \in \mathbb{R}$ with $0 < b_1 < b_2$, then for $q \geq 1$, we obtain

$$\left| \frac{1}{H(b_1, b_2)} - \frac{1}{L(b_1, b_2)} \right| \leq \sqrt[q]{\frac{4}{3}} \frac{(b_2 - b_1)}{8} \quad (76)$$

$$\times \left\{ \frac{1}{\sqrt[q]{H\left(b_1^{2q}, 2\left(\frac{b_1+b_2}{2}\right)^{2q}\right)}} + \frac{1}{\sqrt[q]{H\left(2\left(\frac{b_1+b_2}{2}\right)^{2q}, b_2^{2q}\right)}} \right\}.$$

PROOF. Choosing $m = 1$, $\lambda = 1$, $\ell = \frac{b_1+b_2}{2}$, $\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) = \ell - mb_1$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) = b_2 - m\ell$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1) = b_2 - mb_1$, $\Lambda(\zeta) = \frac{1}{\zeta}$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Theorem 2.15, we have (76).

Finally, we will find some new error estimations pertaining quadrature formula. Let denote $\mathcal{Q} : b_1 = \varsigma_0 < \varsigma_1 < \dots < \varsigma_k = b_2$. The following quadrature formulas are very useful in the sequel.

$$\int_{b_1}^{b_2} \Lambda(\ell) d\ell = M(\Lambda, \mathcal{Q}) + E(\Lambda, \mathcal{Q}), \quad \int_{b_1}^{b_2} \Lambda(\ell) d\ell = T(\Lambda, \mathcal{Q}) + E^*(\Lambda, \mathcal{Q})$$

where

$$M(\Lambda, \mathcal{Q}) := \sum_{j=0}^{k-1} \Lambda\left(\frac{\varsigma_j + \varsigma_{j+1}}{2}\right) (\varsigma_{j+1} - \varsigma_j), \quad T(\Lambda, \mathcal{Q}) := \sum_{j=0}^{k-1} \frac{\Lambda(\varsigma_j) + \Lambda(\varsigma_{j+1})}{2} (\varsigma_{j+1} - \varsigma_j),$$

and $E(\Lambda, \mathcal{Q})$, $E^*(\Lambda, \mathcal{Q})$ are denoted their corresponding errors.

PROPOSITION 3.9. Let $\Lambda : [b_1, b_2] \rightarrow \mathbb{R}$ be a differentiable function on (b_1, b_2) , where $b_1 < b_2$. If $|\Lambda'|^q$ is convex on $[b_1, b_2]$, then for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$|E(\Lambda, \mathcal{Q})| \leq \frac{1}{4\sqrt[q]{2}\sqrt[p]{p+1}} \times \sum_{j=0}^{k-1} (\varsigma_{j+1} - \varsigma_j)^2 \quad (77)$$

$$\times \left\{ \sqrt[q]{|\Lambda'(\varsigma_j)|^q + \left|\Lambda'\left(\frac{\varsigma_j + \varsigma_{j+1}}{2}\right)\right|^q} + \sqrt[q]{\left|\Lambda'\left(\frac{\varsigma_j + \varsigma_{j+1}}{2}\right)\right|^q + |\Lambda'(\varsigma_{j+1})|^q} \right\}.$$

PROOF. From Theorem 2.3 for $m = 1$, $\lambda = 0$, $\ell = \frac{b_1+b_2}{2}$, $\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) = \ell - mb_1$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) = b_2 - m\ell$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1) = b_2 - mb_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ on

$[\varsigma_j, \varsigma_{j+1}]$ ($j = 0, \dots, k-1$) of $\mathcal{Q}$, we get

$$\left| \Lambda \left(\frac{\varsigma_j + \varsigma_{j+1}}{2} \right) - \frac{1}{\varsigma_{j+1} - \varsigma_j} \int_{\varsigma_j}^{\varsigma_{j+1}} \Lambda(\ell) d\ell \right| \leq \frac{(\varsigma_{j+1} - \varsigma_j)}{4\sqrt[q]{2}\sqrt[q]{p+1}} \quad (78)$$

$$\times \left\{ \sqrt[q]{|\Lambda'(\varsigma_j)|^q + \left| \Lambda' \left(\frac{\varsigma_j + \varsigma_{j+1}}{2} \right) \right|^q} + \sqrt[q]{\left| \Lambda' \left(\frac{\varsigma_j + \varsigma_{j+1}}{2} \right) \right|^q + |\Lambda'(\varsigma_{j+1})|^q} \right\}.$$

From (78), we have

$$\begin{aligned} |\mathbf{E}(\Lambda, \mathcal{Q})| &= \left| \int_{b_1}^{b_2} \Lambda(\ell) d\ell - \mathbf{M}(\Lambda, \mathcal{Q}) \right| \\ &\leq \left| \sum_{j=0}^{k-1} \left\{ \int_{\varsigma_j}^{\varsigma_{j+1}} \Lambda(\ell) d\ell - \Lambda \left(\frac{\varsigma_j + \varsigma_{j+1}}{2} \right) (\varsigma_{j+1} - \varsigma_j) \right\} \right| \\ &\leq \sum_{j=0}^{k-1} \left| \left\{ \int_{\varsigma_j}^{\varsigma_{j+1}} \Lambda(\ell) d\ell - \Lambda \left(\frac{\varsigma_j + \varsigma_{j+1}}{2} \right) (\varsigma_{j+1} - \varsigma_j) \right\} \right| \\ &\leq \frac{1}{4\sqrt[q]{2}\sqrt[q]{p+1}} \times \sum_{j=0}^{k-1} (\varsigma_{j+1} - \varsigma_j)^2 \\ &\times \left\{ \sqrt[q]{|\Lambda'(\varsigma_j)|^q + \left| \Lambda' \left(\frac{\varsigma_j + \varsigma_{j+1}}{2} \right) \right|^q} + \sqrt[q]{\left| \Lambda' \left(\frac{\varsigma_j + \varsigma_{j+1}}{2} \right) \right|^q + |\Lambda'(\varsigma_{j+1})|^q} \right\}. \end{aligned}$$

PROPOSITION 3.10. Let $\Lambda : [b_1, b_2] \rightarrow \mathbb{R}$ be a differentiable function on (b_1, b_2) , where $b_1 < b_2$. If $|\Lambda'|^q$ is convex on $[b_1, b_2]$, then for $q \geq 1$, we obtain

$$\begin{aligned} |\mathbf{E}(\Lambda, \mathcal{Q})| &\leq \frac{1}{8\sqrt[q]{3}} \times \sum_{j=0}^{k-1} (\varsigma_{j+1} - \varsigma_j)^2 \quad (79) \\ &\times \left\{ \sqrt[q]{|\Lambda'(\varsigma_j)|^q + 2 \left| \Lambda' \left(\frac{\varsigma_j + \varsigma_{j+1}}{2} \right) \right|^q} + \sqrt[q]{2 \left| \Lambda' \left(\frac{\varsigma_j + \varsigma_{j+1}}{2} \right) \right|^q + |\Lambda'(\varsigma_{j+1})|^q} \right\}. \end{aligned}$$

PROOF. The proof is analogous as to that of Proposition 3.9 taking $m = 1$, $\lambda = 0$, $\ell = \frac{b_1+b_2}{2}$, $\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) = \ell - mb_1$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) = b_2 - m\ell$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1) = b_2 - mb_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ using Theorem 2.15.

PROPOSITION 3.11. Let $\Lambda : [b_1, b_2] \rightarrow \mathbb{R}$ be a differentiable function on (b_1, b_2) ,

where $b_1 < b_2$. If $|\Lambda'|^q$ is convex on $[b_1, b_2]$, then for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$|\mathbf{E}^*(\Lambda, \mathcal{Q})| \leq \frac{1}{4\sqrt[q]{2}\sqrt[q]{p+1}} \times \sum_{j=0}^{k-1} (\varsigma_{j+1} - \varsigma_j)^2 \quad (80)$$

$$\times \left\{ \sqrt[q]{|\Lambda'(\varsigma_j)|^q + \left| \Lambda' \left(\frac{\varsigma_j + \varsigma_{j+1}}{2} \right) \right|^q} + \sqrt[q]{\left| \Lambda' \left(\frac{\varsigma_j + \varsigma_{j+1}}{2} \right) \right|^q + |\Lambda'(\varsigma_{j+1})|^q} \right\}.$$

PROOF. By Theorem 2.3 for $m = 1, \lambda = 1, \ell = \frac{b_1+b_2}{2}, \mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) = \ell - mb_1, \mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) = b_2 - m\ell, \mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1) = b_2 - mb_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ on $[\varsigma_j, \varsigma_{j+1}]$ ($j = 0, \dots, k-1$) of $\mathcal{Q}$, we get

$$\left| \frac{\Lambda(\varsigma_j) + \Lambda(\varsigma_{j+1})}{2} - \frac{1}{\varsigma_{j+1} - \varsigma_j} \int_{\varsigma_j}^{\varsigma_{j+1}} \Lambda(\ell) d\ell \right| \leq \frac{(\varsigma_{j+1} - \varsigma_j)}{4\sqrt[q]{2}\sqrt[q]{p+1}} \quad (81)$$

$$\times \left\{ \sqrt[q]{|\Lambda'(\varsigma_j)|^q + \left| \Lambda' \left(\frac{\varsigma_j + \varsigma_{j+1}}{2} \right) \right|^q} + \sqrt[q]{\left| \Lambda' \left(\frac{\varsigma_j + \varsigma_{j+1}}{2} \right) \right|^q + |\Lambda'(\varsigma_{j+1})|^q} \right\}.$$

From (81), we get

$$|\mathbf{E}^*(\Lambda, \mathcal{Q})| = \left| \int_{b_1}^{b_2} \Lambda(\ell) d\ell - \mathbf{T}(\Lambda, \mathcal{Q}) \right|$$

$$\leq \left| \sum_{j=0}^{k-1} \left\{ \int_{\varsigma_j}^{\varsigma_{j+1}} \Lambda(\ell) d\ell - \frac{\Lambda(\varsigma_j) + \Lambda(\varsigma_{j+1})}{2} (\varsigma_{j+1} - \varsigma_j) \right\} \right|$$

$$\leq \sum_{j=0}^{k-1} \left| \left\{ \int_{\varsigma_j}^{\varsigma_{j+1}} \Lambda(\ell) d\ell - \frac{\Lambda(\varsigma_j) + \Lambda(\varsigma_{j+1})}{2} (\varsigma_{j+1} - \varsigma_j) \right\} \right|$$

$$\leq \frac{1}{4\sqrt[q]{2}\sqrt[q]{p+1}} \times \sum_{j=0}^{k-1} (\varsigma_{j+1} - \varsigma_j)^2$$

$$\times \left\{ \sqrt[q]{|\Lambda'(\varsigma_j)|^q + \left| \Lambda' \left(\frac{\varsigma_j + \varsigma_{j+1}}{2} \right) \right|^q} + \sqrt[q]{\left| \Lambda' \left(\frac{\varsigma_j + \varsigma_{j+1}}{2} \right) \right|^q + |\Lambda'(\varsigma_{j+1})|^q} \right\}.$$

PROPOSITION 3.12. Let $\Lambda : [b_1, b_2] \rightarrow \mathbb{R}$ be a differentiable function on (b_1, b_2) ,

where $b_1 < b_2$. If $|\Lambda'|^q$ is convex on $[b_1, b_2]$, then for $q \geq 1$, we obtain

$$|E^*(\Lambda, Q)| \leq \frac{1}{8\sqrt[4]{3}} \times \sum_{j=0}^{k-1} (\varsigma_{j+1} - \varsigma_j)^2 \quad (82)$$

$$\times \left\{ \sqrt[q]{2|\Lambda'(\varsigma_j)|^q + \left| \Lambda' \left(\frac{\varsigma_j + \varsigma_{j+1}}{2} \right) \right|^q} + \sqrt[q]{\left| \Lambda' \left(\frac{\varsigma_j + \varsigma_{j+1}}{2} \right) \right|^q + 2|\Lambda'(\varsigma_{j+1})|^q} \right\}.$$

PROOF. The proof is analogous as to that of Proposition 3.11 taking $m = 1$, $\lambda = 1$, $\ell = \frac{b_1+b_2}{2}$, $\mathcal{F}_{\rho,\delta}^\sigma(\ell - mb_1) = \ell - mb_1$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - m\ell) = b_2 - m\ell$, $\mathcal{F}_{\rho,\delta}^\sigma(b_2 - mb_1) = b_2 - mb_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ using Theorem 2.15.

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