Generalized Lindley-Quasi Xgamma distribution

S. A. WANI AND S. SHAFI

Abstract

We obtained a new generalization of Lindley-Quasi Xgamma distribution by adding weight parameter to it through weighting technique and have shown the flexibility of proposed model. Expression for reliability measures, order statistics, Bonferroni curves & indices, Renyi entropy along with some other important properties are derived. Maximum likelihood estimation method is put to use for estimation of unknown parameters of proposed model. Simulation study for checking the performance of maximum likelihood estimates and for model comparison is carried out. Proposed model and its related models are fitted to real life data sets and goodness of fit measure Kolmogorov statistic & p-value, loss of information criteria's AIC, BIC, AICC & HQIC are computed through R software to check the applicability of proposed model in real life. The significance of weight parameter is also tested by using likelihood ratio test for both randomly generated data as well as real life data.

Mathematics Subject Classification 2010: 60E05, 62H10
Keywords: Weighting technique, Structural properties, Simulation study, Weight parameter, Bonferroni curves & indices, Renyi entropy, Maximum likelihood estimation, Application.

1. INTRODUCTION

Probability models are and have been generalized for providing more flexibility in terms of hazard rate, reliability, prediction and moments. Because of presence of extra parameter generalized probability models find greater applicability in real life. Probability models can be generalized by using various methods. One of the method employed for generalizing probability models by adding weight parameter is weighting technique. Weighted models are mostly used in those situations where observations are recorded with unequal probabilities which generally happens in encountered sampling. Weighted models find applications in many fields of real life like forestry, medical sciences, socioeconomic surveys etc. Warren [1] applied the size biased distributions in connection with sampling wood cells. Hassan, Dar and Para [2] introduced a new generalization of Ishita distribution and obtained vital mathematical properties of the distribution along with applications of the proposed model. Hassan, Wani and Para [3] formulated three parameter Quasi Lindley

10.2478/jamsi-2021-0001
© Sameer Ahmad Wani, Shaista Shafi
This is an open access article licensed under the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0).

distribution by using weighting technique and obtained various properties of that model. Das and Roy [4] studied the length biased weighted generalized Rayleigh distribution with properties and applications. Patil and Rao [5] introduced weighted distributions and size biased sampling with applications to wild life populations and human families and obtained its properties. Hassan, Wani and Shafi [6] introduced Poisson Pranav distribution and obtained its various mathematical properties along with obtaining applications of the proposed model. Rezaeia, Nadarajah and Tahghighnia [7] worked on a new three parameter life time distribution and studied its properties & applications. Hassan, Wani, Bilal and Akhtar [8] introduced weighted Quasi Xgamma distribution and studied its properties and applications.

Hassan, Wani, Shafi and Sheikh [9] introduced Lindley-Quasi Xgamma Distribution (LQXD). With p.d.f, c.d.f and c^{th} moment about origin $E(x^c)$ given below in (1.1), (1.2) and (1.3) respectively

$$f(x,\alpha,\theta)=\frac{\theta e^{-\theta x}}{(\alpha+\theta)^2}\left\{(\alpha+\theta)(\alpha+\frac{x^2\theta^2}{2})+\theta(\theta-1)(1+\alpha x)\right\} \qquad x>0,\theta>0,\alpha\geq 0 \tag{1.1}$$

$$F(X)=\frac{1}{2(\theta+\alpha)^2}\left[\begin{array}{l}(\alpha+\theta)\left\{2\alpha+2-\left(2\alpha+x^2\theta^2+2\theta x+2\right)e^{-\theta x}\right\}\\+2(\theta-1)\left\{\theta+\alpha-\left(\theta+\alpha\theta x+\alpha\right)e^{-\theta x}\right\}\end{array}\right] \qquad x>0,\theta>0,\alpha\geq 0 \tag{1.2}$$

$$E(x^c)=\frac{c!\left\{(\theta+\alpha)\left(\alpha+\frac{(c+1)(c+2)}{2}\right)+(\theta-1)(\theta+\alpha(c+1))\right\}}{\theta^c(\theta+\alpha)^2} \tag{1.3}$$

The important statistical properties along with application in real life were studied for the model given in (1.1).

In this paper we have obtained weighted version of Lindley-Quasi Xgamma (LQXD) distribution with p.d.f given in (1.1).

2. WEIGHTED LINDLEY-QUASI XGAMMA DISTRIBUTION

Consider X to be a non-negative random variable following Lindley Quasi Xgamma distribution with p.d.f $f(x)$. Suppose $W(x)$ is the non-negative weight function, then the probability density function of the weighted Lindley Quasi Xgamma distribution (WLQXD) is

$$f_w(x)=\frac{W(x)f(x)}{E(w(x))}, \quad x>0$$

Where $w(x)=x^c$ is a non-negative weight function and $f(x), E(x^c)$ are given in (1.1), (1.3) respectively.

$$f_w(x)=\frac{x^c f(x)}{E\left[x^c\right]} \tag{2.1}$$

$$f_w(x)=\frac{\theta^{c+1}x^c\left((\alpha+\theta)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)} \qquad x>0,\theta>0,\alpha\geq 0,c\geq 0$$

(2.2)

Proof of (2.2) being a probability density function is given below

PROOF: $\int_0^\infty f_w(x)=\int_0^\infty \frac{\theta^{c+1}x^c\left((\alpha+\theta)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}$

$$=\frac{\theta^{c+1}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\int_0^\infty x^c\left((\alpha+\theta)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}$$

$$=\frac{\theta^{c+1}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\times\frac{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}{\theta^{c+1}}$$

$$=1$$

$$\int_0^\infty f_w(x)=1$$

So (2.2) is a probability density function

where $\int_0^\infty x^{\beta-1}e^{-\alpha x}dx = \frac{\Gamma\beta}{\alpha^\beta}$ is a gamma function

The plots of p.d.f and c.d.f for different values of parameters are given in graphs 1 & 2 below. Graph 1 indicating that proposed model is positively skewed & platykurtic as well as leptokurtic. Graph 2 Indicating curve starts with zero and ends at one and showing c.d.f is non-decreasing function.

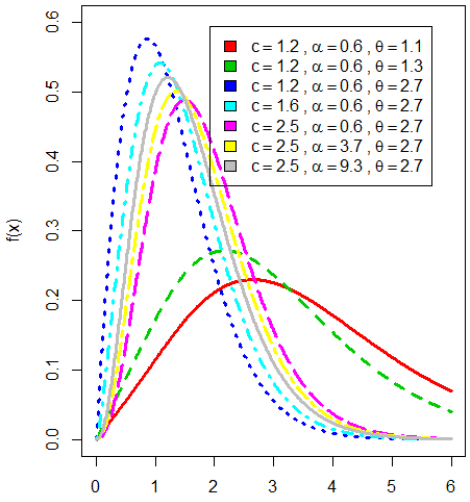


Figure 1 Graph of density function

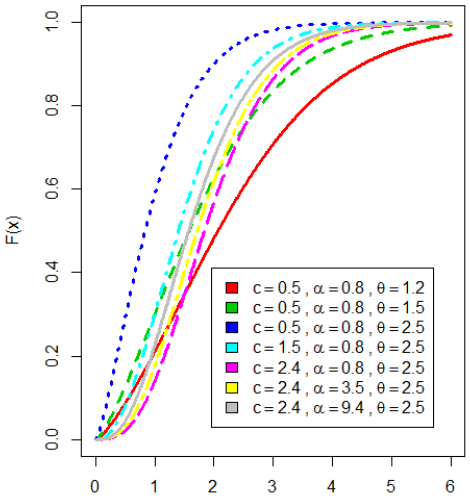


Figure 2 Graph of distribution function

The corresponding c.d.f of Weighted Lindley-Quasi Xgamma distribution is given in (2.3) & obtained as

$$F_w(x) = \int_0^x \frac{\theta^{c+1}x^c\left((\alpha+\theta)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}dx \qquad (2.3)$$

Put $\theta x = t$ in (2.3)

$\theta\, dx = dt$

$as\, x \to 0, t \to 0 \;\; and \;\; x \to x, t \to \theta x$

$$F_w(x)=\left[\frac{(\alpha+\theta)\left(2\alpha\gamma(c+1,\theta x)+\gamma(c+3,\theta x)\right)+2(\theta-1)\left(\theta\gamma(c+1,\theta x)+\alpha\gamma(c+2,\theta x)\right)}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right] \tag{2.4}$$

where $\gamma(s,x)=\int_0^x t^{s-1}e^{-t}dt$ is a lower incomplete gamma function

3. NEED OF PROPOSED MODEL AND MOTIVATION FOR DEVELOPING PROPOSED MODEL

There are many situations in real life where models with less parameters don't perform better. Here in this paper we added an extra parameter known as weight parameter to two parameter Lindley-Quasi Xgamma distribution to obtain a generalized model which will find greater applicability in dealing with complex data. It can be observed from Table 1 that proposed model can be used for over dispersed as well as under dispersed data. Addition of weight parameter also brings the flexibility in terms of moments as can be seen from graphs of probability density function and hazard rate. Complex data applicability and increased flexibility were two important points which motivated us to work on this model.

4. RELIABILITY MEASURES

This division of paper presents survival function, hazard rate, reverse hazard rate of the proposed Weighted Lindley-Quasi Xgamma distribution for random variable X, where X represents the lifetime of a system.

4.1. Reliability Function R(x)

The reliability function or survival function $R(x)$ gives the numerical value of odds of surviving of a system beyond a specified time (t).

Mathematically

$$R_w(x)=P(X>t)=1-F_w(x)$$

The reliability function or the survival function of Weighted Lindley-Quasi Xgamma distribution is obtained as:

$$R_w(x)=1-\left[\frac{(\alpha+\theta)\left(2\alpha\gamma(c+1,\theta x)+\gamma(c+3,\theta x)\right)+2(\theta-1)\left(\theta\gamma(c+1,\theta x)+\alpha\gamma(c+2,\theta x)\right)}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right]$$

4.2. Hazard Rate

The hazard rate which is defined as chance that a system which is surviving up to time *"t"* will fail in the small time interval after *"t"* is obtained for WLQXD as:

$$H.R = h_w(x) = \frac{f_w(x)}{R_w(x)}$$

$$= \left(\frac{\theta^{c+1} x^c \left((\alpha+\theta)\left(2\alpha + x^2\theta^2\right) + 2\theta(\theta-1)(1+\alpha x)\right) e^{-\theta x}}{\left(\begin{array}{l} c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2)) + 2(\theta-1)(\theta+\alpha(c+1))\right) - \\ (\alpha+\theta)\left(2\alpha\gamma(c+1,\theta x) + \gamma(c+3,\theta x)\right) + 2(\theta-1)\left(\theta\gamma(c+1,\theta x) + \alpha\gamma(c+2,\theta x)\right) \end{array} \right)} \right)$$

The graphs of reliability function and hazard rate of WLQXD for different values of parameter are given below

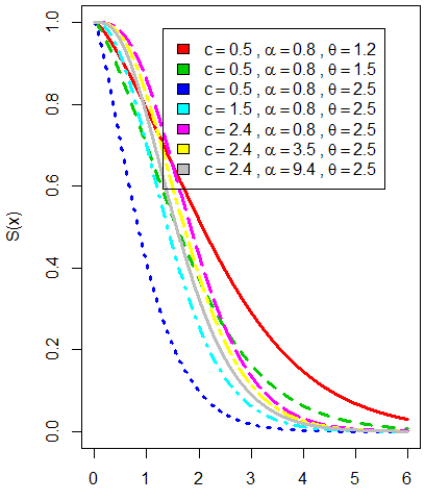


Figure 3 Graph of Reliability function

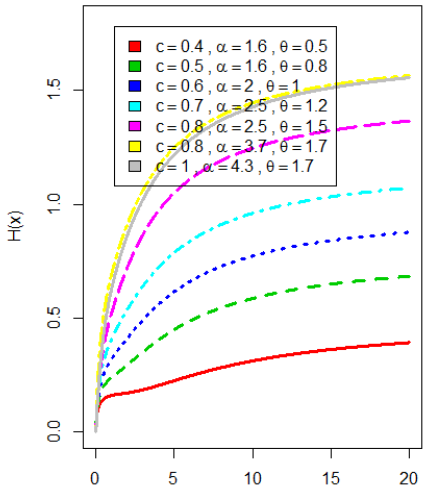


Figure 4 Graph of Hazard rate

From the above graph of hazard rate it is revealed that our proposed model possesses non-decreasing hazard rate and it can be also seen that hazard rate becomes constant as value of x increases. There are many situations in real life where hazard rate is non-decreasing, like lifetime of human beings, animals etc.

4.3. Reverse Hazard Rate

The reverse hazard rate of the Weighted Lindley-Quasi Xgamma distribution is given as:

$$R.H.R = h_{r\,w}(x) = \frac{f_w(x)}{F_w(x)}$$

$$= \frac{\theta^{c+1}x^c\left((\alpha+\theta)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}}{(\alpha+\theta)\left(2\alpha\gamma(c+1,\theta x)+\gamma(c+3,\theta x)\right)+2(\theta-1)\left(\theta\gamma(c+1,\theta x)+\alpha\gamma(c+2,\theta x)\right)}$$

5. STATISTICAL PROPERTIES

Moments, characteristic function, mean deviation, harmonic mean, coefficient of variation characterize probability models. Here we have obtained these statistical properties for proposed Weighted Lindley-Quasi Xgamma distribution.

5.1. Moments and Related Measures

Assuming X to be a random variable having Weighted Lindley-Quasi Xgamma distribution with parameters θ, c and α .Then the r^{th} moment about origin for a given probability distribution is given by

$$\mu_r' = E\left(X^r\right) = \int_0^\infty x^r f_w(x)dx \qquad r=1, 2, 3\ldots$$

$$= \int_0^\infty x^r \frac{\theta^{c+1}x^c\left((\alpha+\theta)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\,dx$$

$$\mu_r' = \left[\frac{(r+c)!\left((\alpha+\theta)(2\alpha+(r+c+1)(r+c+2))+2(\theta-1)(\theta+\alpha(r+c+1))\right)}{\theta^r\left(c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)\right)}\right] \tag{5.1.1}$$

Put r=1 in equation (5.1.1) we get

$$\mu_1' = \left[\frac{(c+1)\left((\alpha+\theta)(2\alpha+(c+2)(c+3))+2(\theta-1)(\theta+\alpha(c+2))\right)}{\theta\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right]$$

Which is mean of the WLQXD.

Put r=2 in equation (5.1.1) we get

$$\mu_2{}' = \left[\frac{(c+1)(c+2)\left((\alpha+\theta)(2\alpha+(c+4)(c+3))+2(\theta-1)(\theta+\alpha(c+3))\right)}{\theta^2\left(((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right]$$

The variance μ_2 of WLQXD is

$$\mu_2 = \left(\left[\frac{(c+1)(c+2)\left((\alpha+\theta)(2\alpha+(c+4)(c+3))+2(\theta-1)(\theta+\alpha(c+3))\right)}{\theta^2\left(((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right] - \left[\frac{(c+1)\left((\alpha+\theta)(2\alpha+(c+2)(c+3))+2(\theta-1)(\theta+\alpha(c+2))\right)}{\theta\left(((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right]^2\right)$$

The coefficient of variation (C.V) of WLQXD is

$$C.V = \frac{(\mu_2)^{\frac{1}{2}}}{\mu_1{}'}$$

$$= \frac{\left(\left[\frac{(c+1)(c+2)\left((\alpha+\theta)(2\alpha+(c+4)(c+3))+2(\theta-1)(\theta+\alpha(c+3))\right)}{\theta^2\left(((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right] - \left[\frac{(c+1)\left((\alpha+\theta)(2\alpha+(c+2)(c+3))+2(\theta-1)(\theta+\alpha(c+2))\right)}{\theta\left(((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right]^2\right)^{\frac{1}{2}}}{\left[\frac{(c+1)\left((\alpha+\theta)(2\alpha+(c+2)(c+3))+2(\theta-1)(\theta+\alpha(c+2))\right)}{\theta\left(((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right]}$$

The index of dispersion (γ) of WLQXD is

$$\gamma = \frac{(\mu_2)}{\mu_1{}'}$$

$$= \frac{\left(\left[\frac{(c+1)(c+2)\left((\alpha+\theta)(2\alpha+(c+4)(c+3))+2(\theta-1)(\theta+\alpha(c+3))\right)}{\theta^2\left(((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right] - \left[\frac{(c+1)\left((\alpha+\theta)(2\alpha+(c+2)(c+3))+2(\theta-1)(\theta+\alpha(c+2))\right)}{\theta\left(((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right]^2\right)}{\left[\frac{(c+1)\left((\alpha+\theta)(2\alpha+(c+2)(c+3))+2(\theta-1)(\theta+\alpha(c+2))\right)}{\theta\left(((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right]}$$

Table 1: Description of WLQXD for different parameter values.

α	θ	c	Mean	Variance	Index of dispersion	Coefficient of variation
0.5	0.5	0.3	6.353521	14.5477	2.289707	0.6003198
2.5	0.5	0.5	4.704545	14.00362	2.976614	0.7954308
2.5	5.5	1.5	0.6199305	0.1368836	0.2208048	0.5968055
0.5	5.5	2.0	0.7485493	0.1655399	0.2211476	0.5435393

It can be observed from Table 1 that WLQXD is over dispersed as well as under dispersed for different parameter values.

5.2. Harmonic Mean of Weighted Lindley-Quasi Xgamma Distribution

The harmonic mean H of the WLQXD is determined as

$$\frac{1}{H} = E\left[\frac{1}{X}\right] = \int_0^\infty \frac{1}{x} f_w(x)dx$$

$$= \int_0^\infty \frac{1}{x} \frac{\theta^{c+1}x^c\left((\alpha+\theta)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}dx$$

$$H = \frac{c\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}{\theta\left((\alpha+\theta)(2\alpha+c(c+1))+2(\theta-1)(\theta+\alpha c)\right)}$$

5.3. Mean Deviation about Mean and Median of WLQXD

We have derived the expressions for mean deviation about mean and median of WLQXD in this section.

THEOREM 1. If X has the WLQXD (θ,α,c), then the mean deviation about mean $(\delta_1(X))$ and mean deviation about median $(\delta_2(X))$ are given as:

$$\delta_1(X) = \left[\begin{array}{l} 2\mu\left\{\frac{(\alpha+\theta)\left(2\alpha\gamma(c+1,\theta\mu)+\gamma(c+3,\theta\mu)\right)+2(\theta-1)\left(\theta\gamma(c+1,\theta\mu)+\alpha\gamma(c+2,\theta\mu)\right)}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right\} \\ -2\left\{\frac{(\alpha+\theta)\left(2\alpha\gamma(c+2,\theta\mu)+\gamma(c+4,\theta\mu)\right)+2(\theta-1)\left(\theta\gamma(c+2,\theta\mu)+\alpha\gamma(c+3,\theta\mu)\right)}{\theta c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right\} \end{array}\right]$$

And

$$\delta_2(X)=\left[\mu-2\left\{\frac{(\alpha+\theta)\left(2\alpha\gamma(c+2,\theta M)+\gamma(c+4,\theta M)\right)+2(\theta-1)\left(\theta\gamma(c+2,\theta M)+\alpha\gamma(c+3,\theta M)\right)}{\theta c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right\}\right]$$

respectively.

PROOF. Mean deviation about mean and mean deviation about median are defined as

$$\delta_1(X)=\int_0^{\infty}|x-\mu|f_w(x)dx$$

And $\delta_2(X)=\int_0^{\infty}|x-M|f_w(x)dx$

respectively.

Where μ and *M* are mean and median respectively of random variable *X~WLQXD.*

The measures $\delta_1(X)$ and $\delta_2(X)$ can be obtained by using the simplified relationships.

$$\delta_1(X)=\int_0^{\mu}(\mu-x)f_w(x)dx+\int_{\mu}^{\infty}(x-\mu)f_w(x)dx$$

$$\delta_1(X)=2\mu F_w(\mu)-2\int_0^{\mu}xf_w(x)dx \tag{5.3.1}$$

And

$$\delta_2(X)=\int_0^{M}(M-x)f_w(x)dx+\int_{M}^{\infty}(x-M)f_w(x)dx$$

$$\delta_2(X)=\mu-2\int_0^{M}xf_w(x)dx \tag{5.3.2}$$

Where $f_w(x)=\dfrac{\theta^{c+1}x^c\left((\alpha+\theta)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}$

Now

$$\int_0^{\mu} x f_w(x)dx = \left[\frac{(\alpha+\theta)\left(2\alpha\gamma(c+2,\theta\mu)+\gamma(c+4,\theta\mu)\right)+2(\theta-1)\left(\theta\gamma(c+2,\theta\mu)+\alpha\gamma(c+3,\theta\mu)\right)}{\theta c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)} \right] \quad (5.3.3)$$

And

$$\int_0^{M} x f_w(x)dx = \left[\frac{(\alpha+\theta)\left(2\alpha\gamma(c+2,\theta M)+\gamma(c+4,\theta M)\right)+2(\theta-1)\left(\theta\gamma(c+2,\theta M)+\alpha\gamma(c+3,\theta M)\right)}{\theta c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)} \right] \quad (5.3.4)$$

Using expressions (5.3.1), (5.3.2), (5.3.3) and (5.3.4) and expression for c.d.f (2.4) we obtain mean deviation about mean $(\delta_1(X))$ and mean deviation about median

$$(\delta_2(X)) \quad \delta_1(X) = \left[\begin{array}{l} 2\mu\left\{ \frac{(\alpha+\theta)\left(2\alpha\gamma(c+1,\theta\mu)+\gamma(c+3,\theta\mu)\right)+2(\theta-1)\left(\theta\gamma(c+1,\theta\mu)+\alpha\gamma(c+2,\theta\mu)\right)}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)} \right\} \\ -2\left\{ \frac{(\alpha+\theta)\left(2\alpha\gamma(c+2,\theta\mu)+\gamma(c+4,\theta\mu)\right)+2(\theta-1)\left(\theta\gamma(c+2,\theta\mu)+\alpha\gamma(c+3,\theta\mu)\right)}{\theta c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)} \right\} \end{array} \right]$$

&

$$\delta_2(X) = \left[\mu - 2\left\{ \frac{(\alpha+\theta)\left(2\alpha\gamma(c+2,\theta M)+\gamma(c+4,\theta M)\right)+2(\theta-1)\left(\theta\gamma(c+2,\theta M)+\alpha\gamma(c+3,\theta M)\right)}{\theta c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)} \right\} \right]$$

5.4. Generating functions of Weighted Lindley-Quasi Xgamma Distribution

We will derive moment generating function and characteristic function of WLQXD in this segment of paper.

THEOREM 2. If $X \sim WLQXD(\theta,\alpha,c)$ then the moment generating function $M_X(t)$ and characteristic generating function $\phi_X(t)$ are

$$M_X(t) = \left(\begin{array}{l} \frac{\theta^{c+1}}{\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)} \\ \left((\alpha+\theta)\left(\frac{2\alpha}{(\theta-t)^{c+1}} + \frac{\theta^2(c+1)(c+2)}{(\theta-t)^{c+3}} \right) + 2\theta(\theta-1)\left(\frac{1}{(\theta-t)^{c+1}} + \frac{(c+1)\alpha}{(\theta-t)^{c+2}} \right) \right) \end{array} \right)$$

And

$$\phi_X(t)=\left(\frac{\theta^{c+1}}{((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1)))}\left((\alpha+\theta)\left(\frac{2\alpha}{(\theta-it)^{c+1}}+\frac{\theta^2(c+1)(c+2)}{(\theta-it)^{c+3}}\right)+2\theta(\theta-1)\left(\frac{1}{(\theta-it)^{c+1}}+\frac{(c+1)\alpha}{(\theta-it)^{c+2}}\right)\right)\right)$$

respectively.

PROOF. We begin with the well-known definition of the moment generating function given by

$$M_X(t)=E\left(e^{tX}\right)=\int_0^\infty e^{tx}\,f_w(x)\,dx$$

$$=\int_0^\infty e^{tx}\,\frac{\theta^{c+1}x^c\left((\alpha+\theta)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}}{c!((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1)))}\,dx$$

$$M_X(t)=\left(\frac{\theta^{c+1}}{((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1)))}\left((\alpha+\theta)\left(\frac{2\alpha}{(\theta-t)^{c+1}}+\frac{\theta^2(c+1)(c+2)}{(\theta-t)^{c+3}}\right)+2\theta(\theta-1)\left(\frac{1}{(\theta-t)^{c+1}}+\frac{(c+1)\alpha}{(\theta-t)^{c+2}}\right)\right)\right) \tag{5.4.1}$$

Which is the m.g.f of Weighted Lindley-Quasi Xgamma distribution.

Also we know that $\phi_X(t)=M_X(it)$

Therefore,

$$\phi_X(t)=\left(\frac{\theta^{c+1}}{((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1)))}\left((\alpha+\theta)\left(\frac{2\alpha}{(\theta-it)^{c+1}}+\frac{\theta^2(c+1)(c+2)}{(\theta-it)^{c+3}}\right)+2\theta(\theta-1)\left(\frac{1}{(\theta-it)^{c+1}}+\frac{(c+1)\alpha}{(\theta-it)^{c+2}}\right)\right)\right)$$

(5.4.2)

Which is the characteristic function of WLQXD distribution.

6. ORDER STATISTICS OF WEIGHTED LINDLEY-QUASI XGAMMA DISTRIBUTION

Consider $X_{(1)}, X_{(2)}, X_{(3)}...., X_{(n)}$ to be the ordered statistics of the random sample $x_1, x_2, x_3, ...x_n$ obtained from the Weighted Lindley-Quasi Xgamma distribution with cumulative distribution function $F_w(x)$ and probability density function $f_w(x)$, then the probability density function of v^{th} order statistics $X_{(v)}$ is given by:

$$f_{vw}(x) = \frac{n!}{(v-1)!(n-v)!} f_w(x)[F_w(x)]^{v-1}[1-F_w(x)]^{n-v} \qquad v=1, 2, 3\ldots n$$

Using the equations (2.2) and (2.3), the probability density function of v^{th} order statistics of Weighted Lindley-Quasi Xgamma distribution is given by:

$$f_{(v)w}(x) = \left[\begin{array}{l} \dfrac{n!}{(v-1)!(n-v)!} \left\{ \dfrac{\theta^{c+1}x^c\left((\alpha+\theta)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)} \right\} \\ \left[\dfrac{(\alpha+\theta)\left(2\alpha\gamma(c+1,\theta x)+\gamma(c+3,\theta x)\right)+2(\theta-1)\left(\theta\gamma(c+1,\theta x)+\alpha\gamma(c+2,\theta x)\right)}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)} \right]^{v-1} \\ \left[1-\left(\dfrac{(\alpha+\theta)\left(2\alpha\gamma(c+1,\theta x)+\gamma(c+3,\theta x)\right)+2(\theta-1)\left(\theta\gamma(c+1,\theta x)+\alpha\gamma(c+2,\theta x)\right)}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)} \right) \right]^{n-v}. \end{array} \right]$$

Then, the p.d.f of first order statistic $X_{(1)}$ of Weighted Lindley-Quasi Xgamma distribution is given by:

$$f_{(1)w}(x) = \left[\begin{array}{l} n\left\{ \dfrac{\theta^{c+1}x^c\left((\alpha+\theta)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)} \right\} \\ \left[1-\left(\dfrac{(\alpha+\theta)\left(2\alpha\gamma(c+1,\theta x)+\gamma(c+3,\theta x)\right)+2(\theta-1)\left(\theta\gamma(c+1,\theta x)+\alpha\gamma(c+2,\theta x)\right)}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)} \right) \right]^{n-1}. \end{array} \right]$$

and the pdf of n^{th} order statistic $X_{(n)}$ of Weighted Lindley-Quasi Xgamma distribution is given as:

$$f_{(n)w}(x)=\left[n\left\{\frac{\theta^{c+1}x^{c}\left((\alpha+\theta)\left(2\alpha+x^{2}\theta^{2}\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right\}\right.$$
$$\left.\left[\frac{(\alpha+\theta)\left(2\alpha\gamma(c+1,\theta x)+\gamma(c+3,\theta x)\right)+2(\theta-1)\left(\theta\gamma(c+1,\theta x)+\alpha\gamma(c+2,\theta x)\right)}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right]^{n-1}\right].$$

7. BONFERRONI AND LORENZ CURVES AND INDICES OF WLQXD

The Bonferroni curve $(B(p))$, Lorenz curve $(L(p))$, Bonferroni index (B) and Gini index (G) have find applicability in fields of economics, demography, reliability, life testing and medical sciences. The Bonferroni and Lorenz curves are defined as

$$B(p)=\frac{1}{p\mu}\int_{0}^{q}xf_{w}(x)dx \tag{7.1}$$

$$L(p)=\frac{1}{\mu}\int_{0}^{q}xf_{w}(x)dx \tag{7.2}$$

Where $\mu=E(x)$ is the mean of WLQXD and $q=F^{-1}(p)$.

The Bonferroni and Gini indices are defined as

$$B=1-\int_{0}^{1}B(p)dp \tag{7.3}$$

$$G=1-2\int_{0}^{1}L(p)dp \tag{7.4}$$

Using the p.d.f (2.2) of WLQXD we get

$$\int_{0}^{q}xf_{w}(x)dx=\left[\frac{(\alpha+\theta)\left(2\alpha\gamma(c+2,\theta q)+\gamma(c+4,\theta q)\right)+2(\theta-1)\left(\theta\gamma(c+2,\theta q)+\alpha\gamma(c+3,\theta q)\right)}{\theta c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right] \tag{7.5}$$

Using equation (7.5) in (7.1) & (7.2) we get

$$B(p)=\frac{1}{p\mu}\left[\frac{(\alpha+\theta)\left(2\alpha\gamma(c+2,\theta q)+\gamma(c+4,\theta q)\right)+2(\theta-1)\left(\theta\gamma(c+2,\theta q)+\alpha\gamma(c+3,\theta q)\right)}{\theta c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right] \tag{7.6}$$

And

$$L(p)=\frac{1}{\mu}\left[\frac{(\alpha+\theta)\left(2\alpha\gamma(c+2,\theta q)+\gamma(c+4,\theta q)\right)+2(\theta-1)\left(\theta\gamma(c+2,\theta q)+\alpha\gamma(c+3,\theta q)\right)}{\theta c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right] \tag{7.7}$$

Using (7.6) & (7.7) in (7.3) & (7.4) we get

$$B=1-\frac{1}{\mu}\left[\frac{(\alpha+\theta)\left(2\alpha\gamma(c+2,\theta q)+\gamma(c+4,\theta q)\right)+2(\theta-1)\left(\theta\gamma(c+2,\theta q)+\alpha\gamma(c+3,\theta q)\right)}{\theta c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right] \tag{7.8}$$

$$L=1-\frac{2}{\mu}\left[\frac{(\alpha+\theta)\left(2\alpha\gamma(c+2,\theta q)+\gamma(c+4,\theta q)\right)+2(\theta-1)\left(\theta\gamma(c+2,\theta q)+\alpha\gamma(c+3,\theta q)\right)}{\theta c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right] \tag{7.9}$$

8. RENYI ENTROPY

The information associated with various values of random variable following a particular distribution is called entropy. Renyi entropy $T_R(\delta)$ of random variable X following Weighted Lindley-Quasi Xgamma distribution is obtained as

$$T_R(\gamma)=\frac{1}{1-\delta}\log\left(\int_0^\infty {f_w}^{\delta}(x)dx\right)$$

Where $\delta>0$ and $\delta\neq 1$

$$T_R(\delta)=\frac{1}{1-\delta}\log\left(\int_0^\infty\left(\frac{\theta^{c+1}x^c\left((\alpha+\theta)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right)^{\delta}dx\right)$$

$$T_R(\delta)=\frac{1}{1-\delta}\log\left(\int_0^\infty\frac{\theta^{\delta(c+1)}x^{\delta c}\left((\alpha+\theta)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)^{\delta}e^{-\delta\theta x}}{\left(c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)\right)^{\delta}}dx\right) \tag{8.1}$$

Using the binomial expansion

$$\left((\alpha+\theta)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)^{\delta}$$

$$=\sum_{k=0}^{\delta}\binom{\delta}{k}2^k\theta^k(\theta-1)^k(1+\alpha x)^k(\alpha+\theta)^{\delta-k}\left(2\alpha+x^2\theta^2\right)^{\delta-k}$$

$$=\sum_{k=0}^{\delta}\binom{\delta}{k}2^k\theta^k(\theta-1)^k(\alpha+\theta)^{\delta-k}\sum_{p=0}^{k}\binom{k}{p}\alpha^p x^p\sum_{l=0}^{\delta-k}\binom{\delta-k}{l}x^{2l}\theta^{2l}2^{\delta-k-l}\alpha^{\delta-k-l} \tag{8.2}$$

Using (8.2) in (8.1) we get

$$T_R(\delta)=\frac{1}{1-\delta}\log\left\{\frac{\left(\sum_{k=0}^{\delta}\binom{\delta}{k}(\theta-1)^k(\alpha+\theta)^{\delta-k}\sum_{p=0}^{k}\binom{k}{p}\sum_{l=0}^{\delta-k}\binom{\delta-k}{l}\theta^{c\delta+\delta+k+2l}2^{\delta-l}\alpha^{p+\delta-k-l}\int_0^{\infty}x^{c\delta+p+2l}e^{-\delta\theta x}dx\right)}{\left(c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)\right)^{\delta}}\right\}$$

$$T_R(\delta)=\frac{1}{1-\delta}\log\left\{\frac{\left(\sum_{k=0}^{\delta}\binom{\delta}{k}(\theta-1)^k(\alpha+\theta)^{\delta-k}\sum_{p=0}^{k}\binom{k}{p}\sum_{l=0}^{\delta-k}\binom{\delta-k}{l}\theta^{\delta+k-p-1}2^{\delta-l}\alpha^{p+\delta-k-l}(c\delta+p+2l)!\right)}{\left(c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)\right)^{\delta}\delta^{c\delta+p+2l+1}}\right\}$$

9. ESTIMATION OF PARAMETERS OF WEIGHTED LINDLEY-QUASI-XGAMMA DISTRIBUTION

Parameters are estimated by using method of maximum likelihood estimation. Considering $x_1, x_2, \ldots x_n$ to be the random sample of size n drawn from Weighted Lindley-Quasi Xgamma distribution having density function given by (2.2), then the likelihood function of WLQXD is given as:

$$L(x\mid\theta,\alpha,c)=\prod_{i=1}^{n}\left(\frac{\theta^{c+1}x_i^{\,c}\left((\alpha+\theta)\left(2\alpha+x_i^{\,2}\theta^2\right)+2\theta(\theta-1)(1+\alpha x_i)\right)e^{-\theta x_i}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right)$$

Taking log on both sides of likelihood function we get log likelihood function as:

$$\log L = \begin{pmatrix} n(c+1)\log\theta + c\sum_{i=1}^{n}\log x_i - \theta\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\log\left((\alpha+\theta)\left(2\alpha+{x_i}^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x_i)\right) \\ n\log c! - n\log\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right) \end{pmatrix} \tag{9.1}$$

Now partially differentiate (9.1) with respect to θ, α, c and equating the result to zero, we obtain the following normal equations,

$$\frac{\partial \log L}{\partial \theta} = \left[\begin{array}{l} \frac{(c+1)n}{\theta} - \frac{n(2\alpha+(c+1)(c+2)+4\theta+2\alpha(c+1)-2}{\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)} + \\ \sum_{i=1}^{n}\left(\frac{2\alpha\theta{x_i}^2+2\alpha+3{x_i}^2\theta^2+2(1+\alpha x_i)(2\theta-1)}{\left((\alpha+\theta)\left(2\alpha+{x_i}^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x_i)\right)}\right) - \sum_{i=1}^{n}x_i \end{array} \right] = 0 \tag{9.2}$$

$$\frac{\partial \log L}{\partial \alpha} = \left[\begin{array}{l} -\frac{n(2\theta+(c+1)(c+2)+4\alpha+2\theta(c+1)-2(c+1)}{\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)} + \\ \sum_{i=1}^{n}\left(\frac{4\alpha+{x_i}^2\theta^2+2\theta+2\theta(\theta-1)x_i}{\left((\alpha+\theta)\left(2\alpha+{x_i}^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x_i)\right)}\right) \end{array} \right] = 0 \tag{9.3}$$

$$\frac{\partial \log L}{\partial c} = \left[n\log\theta - \frac{n\left((\theta+\alpha)(2c+3)+(2\alpha\theta-2\alpha)\right)}{\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)} - n\left(\log(c+1)-\frac{1}{2(c+1)}\right)+\sum_{i=1}^{n}\log x_i \right] = 0 \tag{9.4}$$

MLEs of θ, α, c cannot be obtained by solving above complex equations as these equations are not in closed form. So we solve above equations by using iteration method through R software.

10. SIMULATION STUDY

In this part of paper we have carried out the simulation study for checking the performance of maximum likelihood (ML) estimates by taking different sample sizes (n=30, 50, 70, 110). We have used the inverse CDF technique for data simulation for WLQXD using R software. The process was repeated 1000 times for calculation of bias, variance & mean square error (MSE) as are given values in Table 2. For two parameter combinations of WLQXD, decreasing trend is being observed in average bias, variance and MSE as we increase the sample size. Hence, the performance of ML estimators is quite good and consistent in case of Weighted Lindley-Quasi Xgamma Distribution.

Table 2: Simulation Study of ML estimators for WLQXD.

Parameter	n (sample size)	$\alpha = 0.8, \theta = 0.7, c = 0.8$			$\alpha = 0.5, \theta = 1.7, c = 0.6$		
		Bias	Variance	MSE	Bias	Variance	MSE
α	30	0.4862598	0.4406534	0.677102	0.40865	0.540489	0.7074845
θ		1.625651	2.056811	4.699553	0.6685304	1.925161	2.372094
c		0.4383506	0.5523539	0.7445051	0.5508202	0.5480579	0.8514608
α	50	0.478908	0.438007	0.6673606	0.36540	0.454492	0.5880101
θ		1.434994	0.9519631	3.011171	0.4003302	1.001894	1.162159
c		0.3290241	0.2367339	0.3449908	0.4239412	0.2483344	0.4280606
α	70	0.467299	0.4364103	0.6547787	0.30907	0.447506	0.5430311
θ		1.261992	0.6351194	2.227743	0.338809	0.6839647	0.7987562
c		0.2534664	0.1573207	0.2215659	0.3918621	0.1738424	0.3273983
α	110	0.459871	0.4333168	0.6447982	0.26780	0.398899	0.4706166
θ		1.17978	0.3789148	1.770795	0.2331814	0.4179511	0.4723247
c		0.2156018	0.09346848	0.1399526	0.3397029	0.1026403	0.2180383

11. TESTING SIGNIFICANCE OF WEIGHT PARAMETER ON BASIS OF SIMULATED DATA FROM WLQXD

For comparing proposed model with base model and for testing the significance of weighted parameter we generated random samples of sizes (50, 100, 200, 500) from WLQXD using inverse CDF technique. It is evident from the Table 3 that weighted parameter plays a highly significant role for large samples. Even though in small samples, the AIC, AICC, BIC and Negative Log likelihood values are also minimum in case of Weighted model as compared to base model. LR statistic for testing H_0 versus H_1 is $\psi = 2(L(\hat{\Theta}) - L(\hat{\Theta}_0))$, where $\hat{\Theta}$ and $\hat{\Theta}_0$ are the MLEs under H_1 and H_0. The statistic ψ is asymptotically $(as\ n \to \infty)$ distributed as χ^2_k, with k degrees of freedom which is equal to the difference in dimensionality of $\hat{\Theta}$ and $\hat{\Theta}_0$. H_0 will be rejected if the LR-test p-value is <0.01 (or LR Statistic value >3.841) at 95% confidence level

Table 3: Model Comparison Based On Simulated Data from WLQXD.

$\hat{c} = 0.6, \hat{\alpha} = 0.9, \hat{\theta} = 1.5$				Parameter Estimates		Likelihood Ratio Statistic
Criterion	WLQXD	LQXD	Sample Size (n)	WLQXD	LQXD	
-logL	13.02580	17.33376	50	$\hat{c} = 0.7756921$ (0.3682644) $\hat{\alpha} = 0.7541315$ (3.0450987) $\hat{\theta} = 4.8614277$ (0.9328238)	$\hat{\alpha} = 1.317147$ (1.466524) $\hat{\theta} = 3.107664$ (0.352400)	8.615
AIC	32.05161	38.66752				
AICC	24.05161	32.66752				
BIC	37.78767	42.49157				
-logL	19.48443	31.00223	100	$\hat{c} = 0.9303426$ (0.2523067) $\hat{\alpha} = 1.9131216$ (3.0526314) $\hat{\theta} = 5.3480262$ (0.6873024)	$\hat{\alpha} = 1.6114539$ (1.1262244) $\hat{\theta} = 3.1965086$ (0.2635093)	23.035
AIC	44.96885	66.00447				
AICC	36.96885	60.00447				
BIC	52.78436	71.21481				

-logL	33.61176	61.80406	200	$\hat{c}=1.1748909$ (0.2734827). $\hat{\alpha}=0.4499443$ (1.9152323) $\hat{\theta}=5.9318359$ (0.6560791)	$\hat{\alpha}=1.5711807$ (0.7844924) $\hat{\theta}=3.2011676$ (0.1854941)	56.384
AIC	73.22353	127.60812				
AICC	65.22353	121.6081				
BIC	83.11848	134.2048				
-logL	106.5939	134.1149	500	$\hat{c}=0.54764349$ (0.08917304) $\hat{\alpha}=1.50125869$ (1.05332253) $\hat{\theta}=4.64187400$ (0.26435388)	$\hat{\alpha}=1.5740119$ (0.5568915) $\hat{\theta}=3.2970302$ (0.1221850)	55.042
AIC	219.1878	272.2297				
AICC	211.1878	266.2297				
BIC	231.8316	280.6589				

12. APPLICATIONS OF WEIGHTED LINDLEY-QUASI XGAMMA DISTRIBUTION

Proposed model, its base model and some other lifetime models are fitted to two real life data sets for testing how fine proposed model fits to real life data sets as compared to other mentioned models.

DATA SET 1. The data set given in Table 4 represents the tensile strength measures in GPa of 69 carbon fibres tested under tension at gauge lengths of 20mm and has been taken from Bader and Priest [10].

Table 4: Tensile strength measures in GPa of 69 carbon fibres.

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865
1.944	1.958	1.966	1.997	2.006	2.021	2.027	2.055
2.063	2.098	2.140	2.179	2.224	2.240	2.253	2.270
2.272	2.274	2.301	2.301	2.359	2.382	2.382	2.426
2.434	2.435	2.478	2.490	2.511	2.514	2.535	2.554
2.566	2.570	2.586	2.629	2.633	2.642	2.648	2.684
2.697	2.726	2.770	2.773	2.800	2.809	2.818	2.821
2.848	2.880	2.954	3.012	3.067	3.084	3.090	3.096
3.128	3.233	3.433	3.585	3.585			

DATA SET 2. This data set given in Table 5 is due to Smith and Naylor [11] consists of 63 observations of the strengths of 1.5 cm glass fibres, originally obtained by workers at the UK National Physical Laboratory.

Table 5: The strengths of 1.5 cm glass fibres of 63 observations.

0.55	0.74	0.77	0.81	0.84	0.93	1.04	1.11
1.13	1.24	1.25	1.27	1.28	1.29	1.30	1.36
1.39	1.42	1.48	1.48	1.49	1.49	1.50	1.50
1.51	1.52	1.53	1.54	1.55	1.55	1.58	1.59
1.60	1.61	1.61	1.61	1.61	1.62	1.62	1.63
1.64	1.66	1.66	1.66	1.67	1.68	1.68	1.69
1.70	1.70	1.73	1.76	1.76	1.77	1.78	1.81
1.82	1.84	1.84	1.89	2.00	2.01	2.24	

For the analysis of both the data sets we used R Software version 3.5.3.

Table 6: Summary Statistic of data set 1 and 2.

Data set	Number of observations	Min.	Mean	First Quartile	Median	Third Quartile	Max.
1	69	1.312	2.451	2.098	2.478	2.773	3.585
2	63	0.550	1.507	1.375	1.590	1.685	2.240

Table 7: Model function of fitted models.

Distribution	Model function
Weighted Lindley Quasi Xgamma Distribution (WLQXD)	$\frac{\theta^{c+1}x^{c}\left((\alpha+\theta)\left(2\alpha+x^{2}\theta^{2}\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}$
Lindley Quasi Xgamma Distribution (LQXD)	$\frac{\theta e^{-\theta x}}{(\alpha+\theta)^{2}}\left\{(\alpha+\theta)(\alpha+\frac{x^{2}\theta^{2}}{2})+\theta(\theta-1)(1+\alpha x)\right\}$
Quasi Lindley Distribution (QLD)	$\frac{\theta(\alpha+\theta x)}{\alpha+1}e^{-\theta x}$
Exponential Distribution (ED)	$\frac{1}{\theta}e^{-\frac{x}{\theta}}$
Quasi Akash Distribution (QAD)	$\frac{\theta^{2}}{(\alpha\theta+2)}\left(\alpha+\theta x^{2}\right)e^{-\theta x}$

Table 8: ML estimates, -logL, AIC, AICC, BIC, HQIC, KS-distance, for fitted WLQXD and other mentioned models for data set 1.

Distribution	WLQXD	LQXD	QLD	ED	QAD
-logL	49.9985	96.20767	105.7322	130.8676	92.28544
AIC	105.997	196.4153	215.4644	263.7352	188.5709
AICC	106.3662	196.5972	215.6462	263.7949	188.7527
BIC	112.6993	200.8836	219.9326	265.9693	193.0391
HQIC	108.656	198.188	217.2371	264.6216	190.3436
KS-Distance (D)	0.057772	0.355	0.36154	0.44828	0.31119
Likelihood ratio statistic	92.418				
ML Estimates	$\hat{c} = 21.760292$ (5.169830) $\hat{\alpha} = 125.672241$ (958.215785) $\hat{\theta} = 9.793707$ (1.716879)	$\hat{\alpha} = 0.001000$ $\hat{\theta} = 0.98255426$ (0.05831841)	$\hat{\alpha} = 0.001000$ $\hat{\theta} = 0.8156695$ (0.06359975)	$\hat{\theta} = 2.4513330$ (0.2951056)	$\hat{\alpha} = 0.001000$ $\hat{\theta} = 1.22351031$ (0.06779732)

Table 9: ML estimates, -logL, AIC, AICC, BIC, HQIC, KS-distance, for fitted WLQXD and other mentioned models for data set 2.

Distribution	WLQXD	LQXD	QLD	ED	QAD
-logL	23.75059	73.63075	66.34654	88.83032	54.55285
AIC	53.50119	151.2615	136.6931	179.6606	113.1057
AICC	53.90797	151.4615	136.8931	179.7262	113.3057
BIC	59.93059	155.5478	140.9793	181.8038	117.392
HQIC	56.0299	152.9473	138.3789	180.5035	114.7915
KS-Distance (D)	0.2156	0.30991	0.34707	0.418	0.30549
Likelihood ratio statistic	99.760				
ML Estimates	$\hat{c} = 14.829858$ $\hat{\alpha} = 0.001000$ $\hat{\theta} = 11.732504$ (1.464477)	$\hat{\alpha} = 0.001000$ $\hat{\theta} = 1.4647349$ (0.1004913)	$\hat{\alpha} = 0.001000$ $\hat{\theta} = 1.3269391$ (0.1075059)	$\hat{\theta} = 1.506826$ (0.189842)	$\hat{\alpha} = 0.001000$ $\hat{\theta} = 1.9900949$ (0.1132172)

Different criteria's of goodness of fit like AIC, BIC, AICC, HQIC and K-S distance have been computed by using R software for both the data sets and it has been observed from Tables 8 and 9 that proposed model possesses lesser AIC, BIC, AICC, HQIC & K-S distance values as compared to Lindley Quasi Xgamma distribution, Quasi Lindley distribution, Exponential distribution and Quasi Akash distribution for both the data sets. Hence proposed model provides a good fit to both the mentioned data sets.

AIC = 2k-2log$L$ AICC = AIC+ $\frac{2k(k+1)}{n-k-1}$

BIC = k log$n$-2log$L$ HQIC=2k log(log(n))-2 log L

Where k= number of parameters in model

n= size of the sample (number of observations in data set)

logL= value of likelihood function of model

Likelihood ratio statistic is used for testing the significance of weight parameter in both the data sets. For testing $H_0 : c = 0$ versus $H_1 : c \neq 0$ the LR statistic for testing H_0 is $\omega_1 = 2\{L(\hat{\Theta}) - L(\hat{\Theta}_0)\} = 92.418$ for data set 1 as in Table 8, $\omega_2 = 2\{L(\hat{\Theta}) - L(\hat{\Theta}_0)\} = 99.760$ for data set 2 as in Table 9, where $\hat{\Theta}$ and $\hat{\Theta}_0$ are MLEs under H_1 and H_0 .LR statistic $\omega \sim (\chi_{(1)}{}^2(\alpha = 0.05) = 3.841)$ as $n \to \infty$, where 1= degrees of freedom is the difference in dimensionality. From Table 8 $\omega_1 = 92.418 > 3.841$ & from Table 9 $\omega_2 = 99.760 > 3.841$ at 5% level of significance for both the data sets, so we reject H_0 and conclude that weight parameter c plays statistically a significant role.

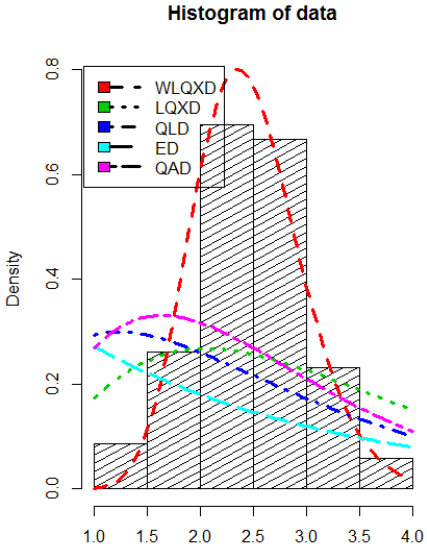


Figure 5 curve fitting of data set 1

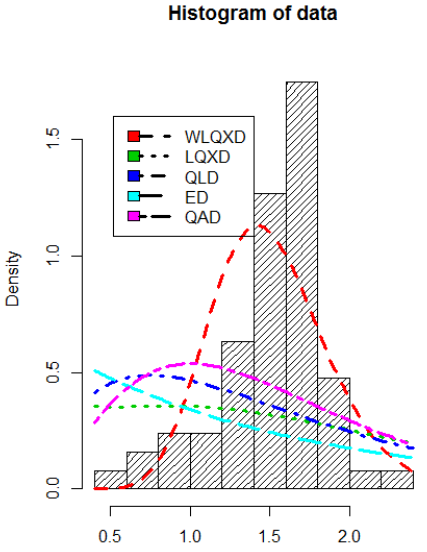


Figure 6 curve fitting of data set 2

13. CONCLUSION

Generalized version of Lindley-Quasi Xgamma distribution by using weighting technique has been proposed in this paper. We obtained the important statistical properties like moments, reliability, moment generating function, order statistics, Bonferroni and Gini indices of formulated model. Expression for Renyi entropy has been derived. For obtaining the estimates of unknown parameters maximum likelihood estimation method is used. For testing the suitability of ML estimates simulation study has been carried which showed that ML estimation method performs well for proposed model. For testing the goodness of fit of proposed model and for investigating the application of proposed model in real life we fitted our proposed model and its related models to two real life data sets and computed log-likelihood values, AIC, AICC, HQIC, BIC and Kolmogorov statistic (D). We observed that our model possesses lesser values of AIC, BIC, AICC, HQIC and D values. Hence our model finds greater applicability in modeling survival times. From generated data as well as from real life data significant role of weight parameter has been observed.

REFERENCES

[1] W. G. Warren. "Statistical Distributions in Forestry and Forest Products Research", *A Modern Course on Statistical Distributions in Scientific Work*, 369-384.

[2] A. Hassan, S. A. Dar and B. A. Para. "A New Generalization of Ishita Distribution: Properties and Applications", *Journal of Applied Probability and Statistics,* 13(1), 2019.

[3] A. Hassan, S. A. Wani and B. A. Para. "On three Parameter Weighted Quasi Lindley Distribution: Properties and Applications", *International Journal of Scientific Research in Mathematical and Statistical Sciences*, 5(5), 210-224, 2018.

[4] K. K. Das and T. B. Roy. "Applicability of Length Biased Weighted Generalized Rayleigh Distribution", *Advances in Applied Science Research,* 2(4), 320-327, 2011.

[5] G. P. Patil and C. R. Rao. "Weighted Distributions and Size Biased Sampling with Applications to Wild Life Populations and Human Families", *Biometric,* 34, 179-189, 1978.

[6] A. Hassan, S. A. Wani and S. Shafi. "Poisson Pranav Distribution and its Applications", *Pak. J. Statist*, 36(1), 57-72, 2020.

[7] S. Rezaeia, S. Nadarajah and N. Tahghighnia. "A New three Parameter Lifetime Distribution", *J Theor App Statist,* 47(4), 835-860, 2013.

[8] A. Hassan, S. A. Wani, B. A. Sheikh and N. Akhtar. "A New Generalized Quasi Xgamma Distribution Applicable to Survival Times*", Journal of Xi'an University of Architecture and Technology*, XII(IV), 3720-3736, 2020.

[9] A. Hassan, S. A. Wani, S. Shafi and B. A. Sheikh. "Lindley-Quasi Xgamma Distribution: Properties and Applications", *Pak. J. Statist*, 36(1), 73-89, 2020.

[10] M. G. Badar and A. M. Priest. "Statistical aspects of fiber and bundle strength in hybrid composites", *Progress in Science and Engineering Composites, Tokyo*, ICCM-IV, 1129-1136, 1982.

[11] R. L. Smith and J. C. Naylor. "A comparison of maximum likelihood and Bayesian estimators for three-parameter weibull distribution", *Applied Stat.,* 36, 358-369, 1987.

[12] S. Mudasir and S. P. Ahmad. "Structural Properties of Length Biased Nakagami Distribution", *International journal of Modern Mathematical Sciences*, 13(3), 217-227, 2015.

[13] R Core Team. R version 3.5.3, "A language and environment for statistical computing. R Foundation for Statistical Computing", *Vienna, Austria,* 2019, URL https://www.R-project.org/.

Sameer Ahmad Wani
Department of Statistics University of Kashmir,
Srinagar 190006, Jammu & Kashmir, India
email: wanisameer199@gmail.com (corresponding author)

Shaista Shafi
Department of Statistics University of Kashmir,
Srinagar 190006, Jammu & Kashmir, India
email: sannabhat02@gmail.com

corresponding author: Sameer Ahmad Wani