

INPUT-TO-STATE STABLE SAMPLED-DATA SYNCHRONISATION OF MARKOVIAN JUMP LUR'E NETWORKS UNDER ACTUATOR SATURATION

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Submitted: 29th May 2025; Accepted: 11th August 2025

Abstract

This paper tackles the challenge of achieving Input-to-state stable (ISS) synchronization in actuator-saturated sampled-data control (SDC) networks for Markovian jump (MJ) Lur'e networks. We investigate the impacts of actuator saturation on system performance and stability, proposing a control strategy that ensures synchronization in the presence of external disturbances. Our analysis employs a Wirtinger-based integral inequality alongside a modified free matrix-based integral inequality (MFMBII), providing a framework for examining Lur'e networks. Initially, we create an MFMBII that combines the dynamics of MJ Lur'e networks and takes into consideration time-varying delays. Second, we formulate two sufficient conditions for the SDC design that ensure mean-square ISS error of specification for the hybrid closed-loop system. We do this by combining the MFMBII method with the Lyapunov-Krasovskii functional (LKF). Through a systematic methodology, we demonstrate that the proposed method maintains bounded state responses and converges to a common trajectory at an exponential rate. The results highlight the effectiveness of integrating ISS with SDC in managing complex dynamical networks. Finally, the proposed ISS method is validated through a numerical example, confirming its efficacy.

Keywords: input-to-state stability, Markovian jump, Lur'e networks, sampled-data control.

1 Introduction

In a MJ system, the dynamics can shift based on the probabilistic rules established by the Markov

process. This allows the system to transition between different sets of equations or parameters under specific conditions. MJ Lur'e systems are vital

across various fields due to their capacity to model intricate dynamic behaviors that encompass both continuous and discrete elements. Below are some key roles they fulfill. MJ Lur'e systems are essential for comprehending and managing complex systems that undergo both continuous evolution and discrete changes. Their capability to model uncertainties and switching behaviors renders them powerful tools in engineering, economics and beyond. The authors in [1] investigate the mean square exponential synchronization control of MJ chaotic Lur'e systems with multiple time delays under aperiodic sampling. In [2], they present event-triggered H_∞ control for Lur'e systems with switching exponential time-varying gains. The work presented in [3] focuses on the novel equivalent reciprocal convex combination technique to sampled-data master-slave synchronization for chaotic Lur'e systems with time-varying delays. In [4], the researcher tackles the issue of exponential stability analysis for switching time-delay Lur'e system with all modes unstable.

The stability and control of continuous-time systems via SDC where control inputs are applied at discrete intervals have been widely studied. This approach is particularly significant in digital control contexts involving discrete sensors and actuators. Classical work, such as that by Astrom has focused on linear systems with constant sampling intervals. Recently, attention has shifted toward aperiodic sampling, which presents notable challenges, especially for nonlinear systems. Exact discretization is often impractical and numerical approximations may fail to capture the true dynamics of the continuous-time system under sample-and-hold implementations. Stability analysis in such settings must consider the hybrid nature of the closed-loop system combining continuous state evolution between samples and discrete control updates. Several studies have extended SDC to complex system classes. For example, [5] examines Lur'e systems under SDC with sector and slope-restricted nonlinearities. In [6], sampled-data synchronization of chaotic Lur'e systems is explored under variable sampling rates. A SDC framework for exponential synchronization of such systems is proposed in [7], while [8] addresses master-slave synchronous criterion for uncertain chaotic Lur'e systems under time-delayed feedback control.

Actuator saturated exponential synchronization centers on the coordination of multiple dynamical systems while accounting for the impacts of actuator saturation and nonlinear feedback. Actuator saturation arises when a controller's output surpasses the physical limits of the actuator, potentially resulting in performance degradation and instability. Exponential synchronization is defined by the swift convergence of various system states to a common trajectory, a highly sought-after characteristic in control applications. Achieving actuator saturated exponential synchronization with Lur'e systems presents a formidable challenge in control theory. Nevertheless, through the meticulous design of controllers that effectively address both saturation and nonlinearities and efficient synchronization can be attained across dynamic systems. This field is continually evolving, adapting to the increasing complexity of real-world applications. The research publications mentioned as [9, 10, 11, 12] dive into the complexities of exponential synchronization and actuator saturation, providing information concerning its behavior and applications.

In practical implementations, external disturbances are often unavoidable and can lead to significant performance degradation. ISS introduced by Sontag for nonlinear control systems is particularly relevant in this context. It defines the property that the system state remains bounded when external disturbances are bounded. As a result, mean-square ISS for MJ systems has received a lot of interest, as it describes how limited external shocks affect state responses. The recent literature has seen substantial study on qualitative mean square ISS analysis and MJ systems stabilization design, showing its rising relevance in the control community. In their research published in [13], the authors explore the use of adaptive control to synchronize time delay Lur'e systems with ISS, while taking into account both outside interference and parameter uncertainty. In [14], they address use of SDC to synchronize delayed time delay Lur'e systems with ISS.

Time-varying delays are often encountered in real-world systems such as neural networks [15], control systems [16] and cyber-physical systems [17], where transmission lags are unpredictable and dependent on external factors. Understanding the impact of such delays on the chaotic behavior of Lur'e systems is crucial for designing

effective control strategies and ensuring robustness against disturbances. This study delves into the intricate effects of time-varying delays in chaotic Lur'e systems, addressing stability conditions, synchronization mechanisms and potential applications in advanced computational models. In [18],[19] researchers investigated the problem of synchronization for Lur'e systems in the presence of time delay.

The current state-of-the art in this field has seen the development of sampled-data control strategies for Markovian jump Lur'e systems, with a focus on ensuring synchronization in the presence of mode transitions and time-varying delays. These studies have leveraged Lyapunov-based techniques and linear matrix inequality (LMI) approaches to derive sufficient conditions for synchronization and stability. Furthermore, the concept of ISS has been employed to analyze the robustness of Markovian jump Lur'e systems to external disturbance and uncertainties. However, the existing literature has primarily focused on continuous-time or discrete-time formulations, without explicitly addressing the practical challenges posed by actuator saturation. The ability to handle actuator saturation is crucial in real-world applications, as it can significantly impact the system performance and stability. The proposed research aims to address this gap by developing a novel framework for the design and analysis of sampled-data control strategies for Markovian jump Lur'e system subject to actuator saturation, with the goal of achieving synchronization. By adopting an ISS-based approach, the study aims to provide rigorous guarantees on the stability and performance of the closed-loop system, offering a comprehensive and practical solution for the control of Markovian jump Lur'e systems in the presence of sampled-data, actuator saturation and external disturbances. This work represents the significance advancement in the field, as it tackles the more challenging and practical case of sampled-data control, while also considering the impact of actuator saturation. The proposed approach promises to enhance the reliability, stability and efficient operation of Markovian-jump Lur'e systems across various domains, paving the way for the broader adoption and implementation of these powerful systems in real-world applications.

The study of Markovian jump Lur'e systems has garnered significant attention due to their ability

to capture the complex dynamics and interactions inherent in a wide range of applications, including power systems, smart grids, biological neural networks, aerospace and flight control, industrial process control and medical systems. However, the practical implementation of these Lur'e systems often involves challenges such as actuator saturation, sampled-data control and the need for ISS to ensure coordinated behavior, consensus and resilience in the face of changing environments or disturbances. This work presents a novel framework for the design and analysis of sampled-data control strategies for Markovian jump Lur'e systems subject to actuator saturation, with the goal of achieving synchronization. The proposed approach leverages the powerful concept of ISS to provide rigorous guarantees on the stability and performance of the closed-loop system. Unlike previous studies that have primarily focused on continuous-time or discrete-time formulations, this research tackles the more challenging and practical case of sampled-data control, where the control updates are performed at discrete time instants. The key contributions are as follows:

1. A LKF is built on the notion that each sample has a unique activation function that is specially tailored to leverage the information contained within each sample state and mode during the entire sampling period. The LKF is designed to utilize the complete set of data available, ensuring that no aspect of the samples unique characteristics is overlooked. By tailoring the activation function specifically to each unique sample, the LKF can extract the maximum amount of knowledge from the data, leading to more robust and accurate models and predictions.
2. To overcome the limitations of Wirtinger-based integral inequalities, MFMBIs are incorporated into the synchronization framework. MFMBIs offer enhanced accuracy and computational efficiency, enabling the design of decentralized controllers suitable for large-scale synchronization systems.
3. Markovian jump Lur'e networks are important in real-world applications but face challenges like actuator saturation and discrete control updates. These issues make stability and synchronization difficult to achieve. This study devel-

ops an ISS-based approach to ensure synchronization under sampled-data control.

4. A sufficient condition for the synchronization error system in the context of ISS has been established using the LKF approach. This theoretical condition is more conservative than the criterion outlined in [13], particularly in scenarios without SDC in the Lur'e system.
5. Despite the vast research on mean-square ISS analysis and synthesis of MJ systems (see [13], [14]), no discoveries have been published that address how to identify the upper bound of the state based on constrained inputs. In contrast to some existing studies, this paper eases certain constraints related to external disturbances, making the proposed strategy more applicable in real-world scenarios. The role of MJ systems is particularly significant in the numerical analysis presented in this section, as illustrated in Example 1.

Notations: $\mathbb{R}^n$ and $\mathbb{R}^{m \times n}$ denote the n -dimensional Euclidean space and the set of all $m \times n$ real matrices. Symmetrical matrix elements are denoted by the symbol $\star$. $\lambda_{\max}(P)$ represents the greatest eigenvalue of a real symmetric matrix P . $\text{diag}\{\dots\}$ represents the block diagonal matrix.

2 Problem formulation and preliminaries

Let $\{r(\rho), \rho \geq 0\}$ be a right continuous Markovian chain on space $(\Omega, \mathcal{F}, \mathcal{P})$ with a natural filtration $\mathcal{F}$ taking values in a finite space $\mathcal{S} = \{1, 2, \dots, \mathcal{N}\}$ and governed by the following form:

$$\Pr\{r(\rho+h) = \bar{q} | r(\rho) = \bar{p}\} = \begin{cases} \phi_{\bar{p}\bar{q}}(l)h + o(h), & \bar{p} \neq \bar{q}, \\ 1 + \phi_{\bar{p}\bar{p}}(l)h + o(h), & \bar{p} = \bar{q}, \end{cases}$$

where $h > 0$ and $o(h)$ satisfies $\lim_{h \rightarrow 0} (o(h)/h) = 0$ and $\phi_{\bar{p}\bar{q}}(l) \geq 0$ for $\bar{p} \neq \bar{q}$ means the transition rate from mode $\bar{p}$ at time ρ to mode $\bar{q}$ at time $\rho + h$ and

$$\phi_{\bar{p}\bar{p}}(l) = - \sum_{\bar{q} \in \mathcal{S}, \bar{q} \neq \bar{p}} \phi_{\bar{p}\bar{q}}(l).$$

Consider master-slave type chaotic Lur'e networks, which are described by

$$\begin{aligned} \dot{x}(\vartheta) = & A(r(\vartheta))x(\vartheta) + B(r(\vartheta))x(\vartheta - \beta(\vartheta)) \\ & + W(r(\vartheta))\sigma(D(r(\vartheta))x(\vartheta)) \end{aligned} \quad (1)$$

$$q(\vartheta) = C(r(\vartheta))x(\vartheta)$$

$$\dot{y}(\vartheta) = A(r(\vartheta))y(\vartheta) + B(r(\vartheta))y(\vartheta - \beta(\vartheta)) \quad (2)$$

$$+ W(r(\vartheta))\sigma(D(r(\vartheta))y(\vartheta)) + \check{\kappa}(u(\vartheta)) + G(r(\vartheta))w(\vartheta) \quad (3)$$

$$\check{q}(\vartheta) = C(r(\vartheta))y(\vartheta)$$

$$u(\vartheta) = \mathcal{K}(q(\vartheta) - \check{q}(\vartheta)) \quad (4)$$

with $x(s) = \hat{p}(s)$ and $y(s) = \check{p}(s)$, $s \in [-\beta_2, 0]$, where $\hat{p}(s)$ and $\check{p}(s)$ are a continuously vector-valued function, $x(\vartheta) = [x_{11}(\vartheta), \dots, x_{1n}(\vartheta)]^T \in \mathbb{R}^n$ and $y(\vartheta) = [y_{21}(\vartheta), \dots, y_{2n}(\vartheta)] \in \mathbb{R}^n$ denotes state vectors, $u(\vartheta) \in \mathbb{R}^n$ denotes the control input, $q(\vartheta)$ and $\check{q}(\vartheta) \in \mathbb{R}^l$ represent the corresponding outputs, $w(\vartheta) \in \mathbb{R}^{n_1}$ denotes the external interference which is affiliated to $\mathcal{L}_2[0, \infty)$, $A(r(\vartheta))$, $B(r(\vartheta))$, $W(r(\vartheta))$, $D(r(\vartheta))$, $C(r(\vartheta))$ and $G(r(\vartheta))$ are real constant matrices with appropriate dimensions and $\beta(\vartheta)$ denotes the time varying delay satisfying $\beta_1 \leq \beta(\vartheta) \leq \beta_2$ and $\dot{\beta}(\vartheta) \leq \mu$, where β_1 , β_2 and μ are known constants. Assume that the non-linearity $\sigma(\cdot) = [\sigma_1(\cdot), \dots, \sigma_{n_h}(\cdot)]^T : \mathbb{R}^{n_h} \rightarrow \mathbb{R}^{n_h}$ satisfies incrementally sector-bounded within sector bounds $[0, k]$ as [14]; i.e.,

$$[\sigma_i(x_1) - \sigma_i(x_2)]^T [\sigma_i(x_1) - \sigma_i(x_2) - k(x_1 - x_2)] \leq 0, \quad (5)$$

$$\forall x_1, x_2.$$

$\check{\kappa}(u(\vartheta)) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the definition of the saturation function:

$$\check{\kappa}(u(\vartheta)) = [\check{\kappa}_1(u_1(\vartheta)), \check{\kappa}_2(u_2(\vartheta)), \dots, \check{\kappa}_n(u_n(\vartheta))]^T \quad (6)$$

where $\check{\kappa}_i(u_i(\vartheta)) = \text{sign}(u_i(\vartheta)) \min\{u_{0i}, |u_i(\vartheta)|\}$, where u_{0i} is the i th element of vector u_0 and u_0 is the known saturation level. Define $z(\cdot) = [z_1(\cdot), \dots, z_n(\cdot)]^T = x(\cdot) - y(\cdot)$. Then we can write the error system as

$$\begin{aligned} \dot{z}(\vartheta) = & A(r(\vartheta))z(\vartheta) + B(r(\vartheta))z(\vartheta - \beta(\vartheta)) \\ & + W(r(\vartheta))g(D(r(\vartheta))z(\vartheta), y(\vartheta)) \\ & - \check{\kappa}(\mathcal{K}z(\vartheta_k)) - G(r(\vartheta))w(\vartheta) \end{aligned} \quad (7)$$

with $g(D(r(\vartheta))z(\vartheta), y(\vartheta)) = \sigma(D(r(\vartheta))z(\vartheta) + D(r(\vartheta))y(\vartheta)) - \sigma(D(r(\vartheta))y(\vartheta))$. Let $D(r(\vartheta)) = [d_{1\bar{p}}, d_{2\bar{p}}, \dots, d_{n_h\bar{p}}]^T$ with $d_{i\bar{p}} \in \mathbb{R}^n$, $i = 1, 2, \dots, n_h$.

It is naturally that $g(D(r(\vartheta))z(\vartheta), y(\vartheta))$ meets the following condition by (5):

$$g_i(d_{i\bar{p}}^T z(\vartheta), y(\vartheta))(g_i(d_{i\bar{p}}^T z(\vartheta), y(\vartheta) - k(d_{i\bar{p}}^T z(\vartheta))) \leq 0, \quad (8)$$

$$i = 1, 2, \dots, n_h.$$

The SDC can be given by

$$u(\vartheta) = \mathcal{K}z(\vartheta_k), \quad \vartheta_k \leq \vartheta \leq \vartheta_{k+1} \quad (9)$$

in which $\mathcal{K}$ denotes the gain matrix, ϑ_k ($k = 0, 1, 2, \dots$) means the sample instants satisfying

$$0 = \vartheta_0 < \vartheta_1 < \vartheta_2 < \dots < \vartheta_k < \dots < \lim_{k \rightarrow +\infty} \vartheta_k = +\infty.$$

The time gap between successive sample instants is indicated as

$$\vartheta_{k+1} - \vartheta_k = d_k \in (0, d].$$

It is worth mentioning that the maximum interval that can be sampled over an interval of $k \geq 0$ is known as d_k , while the upper limit of sampling intervals is called d .

The saturation nonlinearity is represented by a generalized sector bounded condition [20], it may be expressed as follows $\wp(\mathcal{K}z(\vartheta_k)) = \mathcal{K}z(\vartheta_k) - \check{\mathcal{K}}(\mathcal{K}z(\vartheta_k))$, we can express the error system (7) may be stated as follows:

Under controller (9), error system (7) can be expressed as

$$\begin{aligned} \dot{z}(\vartheta) = & A(r(\vartheta))z(\vartheta) + B(r(\vartheta))z(\vartheta - \beta(\vartheta)) \\ & + W(r(\vartheta))g(D(r(\vartheta))z(\vartheta), y(\vartheta)) - \mathcal{K}z(\vartheta_k) \\ & - G(r(\vartheta))w(\vartheta) + \wp(\mathcal{K}z(\vartheta_k)). \end{aligned} \quad (10)$$

For further analysis, the definitions of ISS behaviour and ISS synchronization will be given: Let $\tilde{G} \in \mathbb{R}^{n \times n}$ be a matrix. Define the following polyhedral sets:

$$\check{E}_\Omega = \{z \in \mathbb{R}^n : |(\mathcal{K}_i - \tilde{G}_i)z| \leq u_{0i}, i = 1, 2, \dots, n\}.$$

The terms $\tilde{G}_i$ and $\mathcal{K}_i$ represent the i th row of $\tilde{G}$ and $\mathcal{K}$, respectively. Regarding the non linearity $\wp(\mathcal{K}z(\vartheta_k))$, the following inequality holds if $z(\vartheta_k) \in \check{E}_\Omega$.

It is possible to consider any diagonal matrix $0 < N \in \mathbb{R}^{n \times n}$:

$$\wp^T(\mathcal{K}z(\vartheta_k))N(\wp(\mathcal{K}z(\vartheta_k)) - \tilde{G}z(\vartheta_k)) \leq 0.$$

Remark 1: Actuator saturation presents a significant challenge in control systems, particularly in scenarios employing SDC. This approach involves managing continuous-time systems through discrete measurements and control actions. When actuators reach their physical limits, they can no longer respond linearly to control inputs, resulting in performance degradation and potential instability. Integrating synchronization techniques can effectively mitigate these effects, ensuring that multiple interconnected systems maintain coordinated behavior despite saturation. It was decided that we would use a sigmoidal saturation function in our SDC design in order to address the special requirements of our scenario. Further, the interaction between actuator saturation and time delay adds an additional element of complexity to the control of the system. The saturation of the actuator limits the control input that can be applied, while the time delay affects the response time of the system. This combination may result in decreased performance, instability and control design difficulties. As far as the authors are aware, previous research has examined the relationship between actuator saturation and SDC in the context of synchronization issues. Consequently, our study represents a significant contribution to this field.

Definition 1: [14] If there exist a K -class function $\check{\alpha}(\cdot)$ and a KL -class function $\phi(\cdot, \cdot)$ such that, then system (10) is said to attain the mean square ISS

$$\begin{aligned} \mathbb{E}\{\|z(\vartheta)\|\} \leq & \mathbb{E}\left\{\phi\left(\sup_{-\beta_2 < s < 0} \|\theta(s)\|_{\beta}, \vartheta\right)\right\} \\ & + \check{\alpha}_1\left(\sup_{s \in [0, \vartheta]} \|w(s)\|\right) \end{aligned} \quad (11)$$

for each $\vartheta \geq 0$, where

$$\|\theta(s)\|_{\beta} = \sqrt{z^T(0)z(0) + \int_{-\beta_2}^0 z^T(s)z(s)ds + \int_{-\beta_2}^0 \dot{z}^T(s)\dot{z}(s)ds}.$$

Definition 2: [14] If the error system (10) of systems (1) and (3) is ISS, then those systems are considered ISS synchronized.

Remark 2: In physical systems, external disturbances are often inevitable, resulting in oscillations,

instability, or challenges in achieving synchronization. Typically, this interference is not predetermined but is generally bounded. According to Definition 1, the ISS property ensures that the system not only maintains a bounded state even when subjected to bounded disturbances but also exhibits Lyapunov asymptotic stability when no such interference is present. This establishes a crucial connection between input-output stability and Lyapunov stability. If $\theta(\vartheta)$ is chosen as $\theta e^{-\varepsilon\vartheta}$, where θ and ε are positive constants, then ISS implies exponential stability in the absence of disturbances. Therefore, investigating the synchronization problem of Lur'e systems affected by external interference is of considerable significance.

Lemma 1: [20] If two scalars $\beta_2 \geq \beta_1 > 0$ and a constant matrix $\mathcal{U} \in \mathbb{R}^{n \times n}$, $\mathcal{U} = \mathcal{U}^T > 0$, such that the integrations involved are well-defined, then

$$(1) - \frac{\beta_2^2 - \beta_1^2}{2} \int_{\vartheta - \beta_2}^{\vartheta - \beta_1} \int_{\vartheta + \check{\gamma}}^{\vartheta} x^T(v) \mathcal{U} x(v) dv d\check{\gamma} \leq$$

$$- \left(\int_{\vartheta - \beta_2}^{\vartheta - \beta_1} \int_{\vartheta + \check{\gamma}}^{\vartheta} x(v) dv d\check{\gamma} \right)^T \mathcal{U} \left(\int_{\vartheta - \beta_2}^{\vartheta - \beta_1} \int_{\vartheta + \check{\gamma}}^{\vartheta} x(v) dv d\check{\gamma} \right)$$

$$(2) - (\beta_2 - \beta_1) \int_{\vartheta - \beta_2}^{\vartheta - \beta_1} x^T(v) \mathcal{U} x(v) dv \leq$$

$$- \left(\int_{\vartheta - \beta_2}^{\vartheta - \beta_1} x(v) dv \right)^T \mathcal{U} \left(\int_{\vartheta - \beta_2}^{\vartheta - \beta_1} x(v) dv \right).$$

Lemma 2: [20] In $[\psi_1, \psi_2] \rightarrow \mathbb{R}^n$, for every given constant matrix $\check{G} < 0$ and continuously differential function τ . Then the inequality shown below is true:

$$- \int_{\psi_1}^{\psi_2} \dot{\tau}^T(s) \check{G} \dot{\tau}(s) ds$$

$$\leq - \frac{1}{\psi_2 - \psi_1} [\tau(\psi_2) - \tau(\psi_1)]^T \check{G} [\tau(\psi_2) - \tau(\psi_1)]$$

$$- \frac{3}{\psi_2 - \psi_1} \hat{\omega}^T \check{G} \hat{\omega}$$

where $\hat{\omega} = \tau(\psi_2) + \tau(\psi_1) - \frac{2}{\psi_2 - \psi_1} \int_{\psi_1}^{\psi_2} \tau(s) ds$.

Lemma 3: [21] Consider a function $[\kappa, \underline{\vee}] \rightarrow \mathbb{R}^n$ is differentiable. The inequality holds for all matrices $M_1, M_2 \in \mathbb{R}^{3n \times n}$ given a symmetric matrix

$$R \in \mathbb{R}^{n \times n} > 0,$$

$$- \int_{\kappa}^{\underline{\vee}} \dot{\tau}^T(s) R \dot{\tau}(s) ds \leq \mathfrak{W}^T(\kappa, \underline{\vee}) \Psi(\kappa, \underline{\vee}) \mathfrak{W}(\kappa, \underline{\vee})$$

where

$$\mathfrak{W}(\kappa, \underline{\vee}) = \begin{bmatrix} \tau^T(\underline{\vee}) & \tau^T(\kappa) & \int_{\kappa}^{\underline{\vee}} \tau^T(s) ds \end{bmatrix}^T,$$

$$b_1 = [\hat{I} \quad 0 \quad 0],$$

$$b_2 = [0 \quad \hat{I} \quad 0],$$

$$b_3 = [0 \quad 0 \quad \hat{I}],$$

$$\Psi(\kappa, \underline{\vee}) = (\underline{\vee} - \kappa) \left(P_1 + \frac{1}{3} P_2 \right) + \text{sym}\{M_1(b_1 - b_2) + M_2(2b_3 - b_1 - b_2)\},$$

$$P_1 = M_1 R^{-1} M_1^T,$$

$$P_2 = M_2 R^{-1} M_2^T.$$

Lemma 4: [21] Let $U \in \mathbb{R}^{2n \times 2n}$ be a symmetric matrix such that

$$U = \begin{bmatrix} U_{11} & U_{12} \\ \star & U_{22} \end{bmatrix} \geq 0, U_{pq} \in \mathbb{R}^{n \times n}, p, q = 1, 2,$$

It is trustworthy to use the integral inequality shown below:

$$-(\vartheta - \vartheta_k) \int_{\vartheta_k}^{\vartheta} \dot{\tau}^T(s) U_{11} \dot{\tau}(s) ds \leq$$

$$\begin{bmatrix} \tau(\vartheta) \\ \tau(\vartheta_k) \end{bmatrix}^T \begin{bmatrix} S_{11} & S_{12} \\ \star & S_{22} \end{bmatrix} \begin{bmatrix} \tau(\vartheta) \\ \tau(\vartheta_k) \end{bmatrix}$$

where

$$S_{11} = U_{12} U_{22}^{-1} U_{12}^T - U_{11} + (\vartheta - \vartheta_k)^2 U_{22} + 2(\vartheta - \vartheta_k) U_{12},$$

$$S_{22} = U_{12} U_{22}^{-1} U_{12}^T - U_{11}$$

$$S_{12} = -U_{12} U_{22}^{-1} U_{12}^T + U_{11} - (\vartheta - \vartheta_k) U_{12}^T.$$

3 Main results

In this section, we present the proof and performance analysis of ISS for the system (10). The analysis is initially conducted under the SDC-based control gains structured as in (9) by constructing a LKF that explicitly incorporates the sampling instants.

Theorem 3.1: Given scalars α, d, μ satisfy $\alpha > 0$ and $d > 0$, matrices $E, F, P_{\bar{p}}, \bar{U}_1, N, L_1, \begin{bmatrix} L_1 & L_2 \\ \star & L_3 \end{bmatrix}$ and any matrices $X_s (s = 1, \dots, 5), Q_s (s = 3, \dots, 10), M_1, M_2, \tilde{\mathcal{K}}$ with appropriate dimensions $Y, Y_1, R_0 \in \mathbb{R}^{n \times n}$ satisfy a set of LMIs for any $d_k \in (0, d]$:

$$\begin{aligned} \Delta_1(d_k) &= \\ &= \begin{bmatrix} \Xi_1 + d_k \Xi_2 & \sqrt{2}e^{-\alpha \beta_2}(e_1^T L_2 - e_8^T L_2) \\ \star & -L_3 \end{bmatrix} < 0 \\ \Delta_2(d_k) &= \\ &= \begin{bmatrix} \Xi_1 + d_k \Xi_3 & \sqrt{d_k}e^{-\alpha d} \Pi_2^T M_1 & \sqrt{d_k}e^{-\alpha d} \Pi_2^T M_2 & \sqrt{2}e^{-\alpha \beta_2}(e_1^T L_2 - e_8^T L_2) \\ \star & -\bar{U}_1 & 0 & 0 \\ \star & \star & -3\bar{U}_1 & 0 \\ \star & \star & \star & -L_3 \end{bmatrix} < 0 \end{aligned} \quad (12)$$

$$\Xi_4 = \bar{P} + d_k W - \hat{I} > 0,$$

$$\bar{P} = \begin{bmatrix} P_{\bar{p}} & \check{O}_{n \times n} \\ \star & \check{O}_{n \times n} \end{bmatrix},$$

$$\hat{I} = \begin{bmatrix} \hat{I}_n & \check{O}_{n \times n} \\ \star & \check{O}_{n \times n} \end{bmatrix},$$

$$\begin{bmatrix} P_{\bar{p}} & (\mathcal{K} - \tilde{G}_1)^T \\ \star & u_{01}^2 \end{bmatrix} \geq 0$$

where

$$\begin{aligned} \Xi_1 &= \sum_{\bar{q}=1}^m \bar{\theta}_{\bar{p}\bar{q}}(l) z_1^T P_{\bar{q}} z_1 + z_1^T \alpha P_{\bar{p}} z_1 + z_1^T P_{\bar{p}} z_2 \\ &\quad - z_1^T \text{sym}\left(\frac{X_1}{2}\right) z_1 - z_1^T (X_2 - X_1) z_3 \\ &\quad - z_1^T X_3 z_4 - z_3^T \text{sym}\left(\frac{X_1}{2} - X_2\right) z_3 \\ &\quad - z_3^T X_4 z_4 - z_4^T \text{sym}\left(\frac{X_5}{2}\right) z_4 - 2z_5^T N z_5 \\ &\quad + z_3^T \tilde{G}^T N z_5 + 2\beta_2^2 z_2^T L_1 z_2 \\ &\quad + e^{-\alpha d} (\Pi_2^T M_1 \Pi_3 + \Pi_2^T M_2 \Pi_4) + z_1^T Q_3 z_1 \\ &\quad - e^{-\alpha \beta_1} z_7^T Q_3 z_7 + z_1^T Q_4 z_1 \\ &\quad - (1 - \mu) e^{-\alpha \beta_2} z_9^T Q_4 z_9 + e^{-\alpha \beta_1} z_7^T Q_5 z_7 \\ &\quad - e^{-\alpha \beta_2} z_8^T Q_5 z_8 + z_2^T Q_6 z_2 \\ &\quad - e^{-\alpha \beta_1} z_6^T Q_6 z_6 + \wp_2 z_1^T Q_7 z_1 \\ &\quad - z_{11}^T Q_7 z_{11} + \wp_1 z_1^T Q_8 z_1 - z_{10}^T Q_8 z_{10} \\ &\quad + \wp_4 z_2^T Q_{10} z_2 - \beta_1^2 z_1^T Q_{10} z_1 + \beta_1 z_1^T Q_{10} z_{10} \\ &\quad - z_{10}^T Q_{10} z_{10} + \wp_3 z_6^T Q_9 z_6 - 4z_7^T Q_9 z_7 - 2z_7^T Q_9 z_8, \end{aligned}$$

$$\begin{aligned} &+ \frac{6}{\beta_{21}} z_7^T Q_9 z_{11} - 4z_8^T Q_9 z_8 \\ &+ \frac{6}{\beta_{21}} z_8^T Q_9 z_{11} - \frac{12}{\beta_{21}^2} z_{11}^T Q_9 z_{11} \\ &- 2e^{-\alpha \beta_2} z_1^T L_1 z_1 + 2\beta_2^2 e^{-\alpha \beta_2} z_1^T L_3 z_1 \\ &+ 4\beta_2 e^{-\alpha \beta_2} z_1^T L_2 z_1 + 2e^{-\alpha \beta_2} z_1^T L_1 z_8 \\ &- 2\beta_2 e^{-\alpha \beta_2} z_1^T L_2 z_8 - 2e^{-\alpha \beta_2} z_8^T L_1 z_8 \\ &- z_1^T \text{sym}\left(\frac{Y}{2}\right) z_1 - z_1^T (-Y + Y_1) z_3 \\ &- z_3^T \text{sym}\left(\frac{Y}{2} - Y_1\right) z_3 - z_1^T R_0^T z_2 \\ &+ 2z_1^T R_0^T A_{\bar{p}} z_1 + z_1^T R_0^T B_{\bar{p}} z_9 + z_1^T R_0^T W_{\bar{p}} z_{12} \\ &- z_1^T \mathcal{K} z_3 - z_1^T R_0^T G_{\bar{p}} z_{13} + z_1^T R_0^T z_5 \\ &- 2z_2^T R_0^T z_2 + z_2^T R_0^T A_{\bar{p}} z_1 + z_2^T R_0^T B_{\bar{p}} z_9 \\ &+ z_2^T R_0^T W_{\bar{p}} z_{12} - z_2^T \mathcal{K} z_3 - z_2^T R_0^T G_{\bar{p}} z_{13} \\ &+ z_2^T R_0^T z_5 - 2z_{12}^T U z_{12} + k z_1^T D_{\bar{p}} U z_{12} \\ &+ z_1^T E z_1 - z_{13}^T F z_{13}. \end{aligned}$$

$$\begin{aligned} \Xi_2 &= z_1^T \text{sym}\left(\frac{X_1}{2}\right) z_2 + z_2^T (X_2 - X_1)^T z_3 \\ &\quad + z_2^T X_3 z_4 + 2z_1^T X_3^T z_1 + z_1^T X_4^T z_3 \\ &\quad + z_1^T \text{sym}\left(\frac{X_5}{2}\right) z_4 + \alpha z_1^T \text{sym}\left(\frac{X_1}{2}\right) z_1 \\ &\quad + \alpha z_1^T (X_2 - X_1) z_3 + \alpha z_1^T X_3 z_4 \\ &\quad + \alpha z_3^T \text{sym}\left(\frac{X_1}{2} - X_2\right) z_3 \\ &\quad + \alpha z_3^T X_4 z_4 + \alpha z_4^T \text{sym}\left(\frac{X_5}{2}\right) z_4 \\ &\quad + z_2^T \bar{U}_1 z_2 + \Phi_1 \\ &\quad + z_1^T \left(\frac{Y+Y^T}{2}\right) z_2 + z_2^T (-Y + Y_1) z_3, \end{aligned}$$

$$\Xi_3 = -2\alpha z_5^T N z_5 + \alpha z_3^T \tilde{G}^T N z_5,$$

$$\Pi_1 = [z_1^T \quad z_8^T]^T,$$

$$\Pi_2 = [z_1^T \quad z_3^T \quad z_4^T]^T,$$

$$\Pi_3 = (b_1 - b_2) \Pi_2,$$

$$\Pi_4 = (2b_3 - b_1 - b_2) \Pi_2,$$

$$\begin{aligned} \Phi_1 &= \alpha \left[z_1^T \text{sym}\left(\frac{Y}{2}\right) z_1 \right. \\ &\quad \left. + z_1^T (-Y + Y_1) z_3 \right. \\ &\quad \left. + z_3^T \text{sym}\left(\frac{Y}{2} - Y_1\right) z_3 \right]. \end{aligned}$$

$$\begin{aligned}
\wp_1 &= \frac{\beta_1 e^{\alpha\beta_1} - \beta_1}{\alpha}, \\
\wp_2 &= \beta_{21} \frac{e^{\alpha\beta_2} - e^{\alpha\beta_1}}{\alpha}, \\
\wp_3 &= \frac{\beta_{21} e^{\alpha\beta_{21}} - \beta_{21}}{\alpha}, \\
\wp_4 &= \frac{\beta_1^2 e^{\alpha\beta_1} - \alpha\beta_1^3 - \beta_1^2}{2\alpha^2}, \\
\eta_1(\vartheta) &= \begin{bmatrix} z(\vartheta) \\ z(\vartheta_k) \end{bmatrix}, \\
\xi(\vartheta) &= \text{col} \left\{ z(\vartheta), \dot{z}(\vartheta), z(\vartheta_k), \right. \\
&\quad \int_{\vartheta_k}^{\vartheta} z(s) ds, \wp(\mathcal{K}z(\vartheta_k)), \\
&\quad \dot{z}(\vartheta - \beta_1), z(\vartheta - \beta_1), \\
&\quad z(\vartheta - \beta_2), z(\vartheta - \beta(\vartheta)), \\
&\quad \int_{\vartheta - \beta_1}^{\vartheta} z(s) ds, \int_{\vartheta - \beta_2}^{\vartheta - \beta_1} z(s) ds, \\
&\quad \left. g(D(r(\vartheta))z(\vartheta), y(\vartheta)), w(\vartheta) \right\}, \\
z_{\tilde{s}} &= [0_{n \times (\tilde{s}-1)n} \quad \tilde{I}_n \quad 0_{n \times (13-\tilde{s})n}], \\
\tilde{s} &= 1, 2, \dots, 13.
\end{aligned}$$

Then for the initial conditions $z(\vartheta_0)$ belonging to the ellipsoidal set $\check{E}_{P_{\tilde{p}}} = \{z \in \mathbb{R}^n : z^T P_{\tilde{p}} z \leq 1\}$, the ISS synchronization of chaotic Lur'e networks (1) and (2) can be achieved under the controller (8) with gain $\mathcal{K} = R_0^{-1} \tilde{\mathcal{K}}$.

Proof: Building the error system's (10) time-dependent LKF in the manner described below:

$$\begin{aligned}
\mathcal{V}(z(\vartheta), \vartheta, r(\vartheta)) &= \\
&\mathcal{V}_1(z(\vartheta), \vartheta, r(\vartheta)) + \sum_{i=2}^9 \mathcal{V}_i(z(\vartheta), \vartheta) \quad (13)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{V}_1(z(\vartheta), \vartheta, \bar{p}) &= e^{\alpha\vartheta} z^T(\vartheta) P_{\bar{p}} z(\vartheta), \\
\mathcal{V}_2(z(\vartheta), \vartheta) &= (\vartheta_{k+1} - \vartheta) e^{\alpha\vartheta} \begin{bmatrix} z(\vartheta) \\ z(\vartheta_k) \\ \int_{\vartheta_k}^{\vartheta} z(s) ds \end{bmatrix}^T Q_2 \begin{bmatrix} z(\vartheta) \\ z(\vartheta_k) \\ \int_{\vartheta_k}^{\vartheta} z(s) ds \end{bmatrix}, \\
\mathcal{V}_3(z(\vartheta), \vartheta) &= (\vartheta_{k+1} - \vartheta) \int_{\vartheta_k}^{\vartheta} e^{\alpha s} \dot{z}^T(s) \tilde{U}_1 \dot{z}(s) ds, \\
\mathcal{V}_4(z(\vartheta), \vartheta) &= -2e^{\alpha\vartheta} (\vartheta - \vartheta_k) \wp^T(\mathcal{K}z(\vartheta_k)) N \\
&\quad \times [\wp(\mathcal{K}z(\vartheta_k)) - \tilde{G}z(\vartheta_k)], \\
\mathcal{V}_5(z(\vartheta), \vartheta) &= (\vartheta_{k+1} - \vartheta) e^{\alpha\vartheta} \eta_1^T(\vartheta) W \eta_1(\vartheta), \\
\mathcal{V}_6(z(\vartheta), \vartheta) &= \int_{\vartheta - \beta_1}^{\vartheta} e^{\alpha s} z^T(s) Q_3 z(s) ds \\
&\quad + \int_{\vartheta - \beta(\vartheta)}^{\vartheta} e^{\alpha s} z^T(s) Q_4 z(s) ds \\
&\quad + \int_{\vartheta - \beta_2}^{\vartheta - \beta_1} e^{\alpha s} z^T(s) Q_5 z(s) ds \\
&\quad + \int_{\vartheta - \beta_1}^{\vartheta} e^{\alpha s} \dot{z}^T(s) Q_6 \dot{z}(s) ds, \\
\mathcal{V}_7(z(\vartheta), \vartheta) &= 2\beta_2 \int_{-\beta_2}^0 \int_{\vartheta+\theta}^{\vartheta} e^{\alpha s} \dot{z}^T(s) L_1 \dot{z}(s) ds d\theta, \\
\mathcal{V}_8(z(\vartheta), \vartheta) &= \beta_{21} \int_{-\beta_2}^{-\beta_1} \int_{\vartheta+\theta}^{\vartheta} e^{\alpha(s-\theta)} z^T(s) Q_7 z(s) ds d\theta \\
&\quad + \beta_1 \int_{-\beta_1}^0 \int_{\vartheta+\theta}^{\vartheta} e^{\alpha(s-\theta)} z^T(s) Q_8 z(s) ds d\theta \\
&\quad + \beta_{21} \int_{-\beta_2}^{-\beta_1} \int_{\vartheta+\theta}^{\vartheta - \beta_1} e^{\alpha(s-\theta)} \dot{z}^T(s) Q_9 \dot{z}(s) ds d\theta, \\
\mathcal{V}_9(z(\vartheta), \vartheta) &= \frac{\beta_1^2}{2} \int_{-\beta_1}^0 \int_{\theta}^0 \int_{\vartheta+\lambda}^{\vartheta} e^{\alpha(s-\lambda)} \dot{z}^T(s) Q_{10} \dot{z}(s) ds d\lambda d\theta.
\end{aligned}$$

with

$$\begin{aligned}
W &= \begin{bmatrix} \frac{Y+Y^T}{2} & -Y + Y_1 \\ \star & -Y_1 - Y_1^T + \frac{Y+Y^T}{2} \end{bmatrix}, \\
Q_2 &= \begin{bmatrix} \frac{X_1+X_1^T}{2} & -X_1 + X_2 & X_3 \\ \star & \frac{X_1+X_1^T}{2} - X_2 - X_2^T & X_4 \\ \star & \star & \frac{X_5+X_5^T}{2} \end{bmatrix}.
\end{aligned}$$

The weak infinitesimal generator is indicated, as $\mathcal{L} \mathcal{V}(z(\vartheta), \vartheta, r(\vartheta)) =$

$$\begin{aligned}
&\lim_{\tilde{v} \rightarrow 0^+} \frac{1}{\tilde{v}} \left[\mathbb{E} \{ \mathcal{V}(z(\vartheta + \tilde{v}), \vartheta + \tilde{v}, r(\vartheta + \tilde{v})) | z(\vartheta), \vartheta, r(\vartheta) \} \right. \\
&\quad \left. - \mathcal{V}(z(\vartheta), \vartheta, r(\vartheta)) \right]. \quad (14)
\end{aligned}$$

It is derived by calculating the $\mathcal{L}$ along the trajectories of system (10) over the interval $\vartheta \in [\vartheta_k, \vartheta_{k+1})$

$$\begin{aligned} \mathcal{L}\mathcal{V}(z(\vartheta), \vartheta, r(\vartheta)) &= \\ &= \mathcal{L}\mathcal{V}_1(z(\vartheta), \vartheta, r(\vartheta)) + \sum_{i=2}^9 \mathcal{L}\mathcal{V}_i(z(\vartheta), \vartheta) \end{aligned} \quad (15)$$

$$\begin{aligned} \mathcal{L}\mathcal{V}_1(z(\vartheta), \vartheta, r(\vartheta)) &= \\ &= e^{\alpha\vartheta} z^T(\vartheta) \left(\sum_{\bar{q}=1}^m \bar{\vartheta}_{\bar{p}\bar{q}}(l) P_{\bar{q}} \right) z(\vartheta) \\ &\quad + \alpha e^{\alpha\vartheta} z^T(\vartheta) P_{\bar{p}} z(\vartheta) \\ &\quad + 2e^{\alpha\vartheta} z^T(\vartheta) P_{\bar{p}} \dot{z}(\vartheta) \\ &= \xi^T(\vartheta) \left\{ e^{\alpha\vartheta} z_1^T \left(\sum_{\bar{q}=1}^m \bar{\vartheta}_{\bar{p}\bar{q}}(l) P_{\bar{q}} \right) z_1 \right. \\ &\quad + \alpha e^{\alpha\vartheta} z_1^T P_{\bar{p}} z_1 \\ &\quad \left. + 2e^{\alpha\vartheta} z_1^T P_{\bar{p}} z_2 \right\} \xi(\vartheta) \end{aligned} \quad (16)$$

$$\begin{aligned} \mathcal{L}\mathcal{V}_2(z(\vartheta), \vartheta) &= \\ &= \xi^T(\vartheta) \left\{ -e^{\alpha\vartheta} \left[z_1^T \operatorname{sym} \left\{ \frac{X_1}{2} \right\} z_1 \right. \right. \\ &\quad + z_1^T (X_2 - X_1) z_3 + z_1^T X_3 z_4 \\ &\quad + z_3^T \operatorname{sym} \left\{ \frac{X_1}{2} - X_2 \right\} z_3 \\ &\quad + z_3^T X_4 z_4 + z_4^T \operatorname{sym} \left\{ \frac{X_5}{2} \right\} z_4 \Big] \\ &\quad + 2(\vartheta_{k+1} - \vartheta) e^{\alpha\vartheta} z_1^T \operatorname{sym} \left\{ \frac{X_1}{2} \right\} z_2 \\ &\quad + 2(\vartheta_{k+1} - \vartheta) e^{\alpha\vartheta} z_3^T (X_2 - X_1) z_2 \\ &\quad + 2(\vartheta_{k+1} - \vartheta) e^{\alpha\vartheta} z_4^T X_3^T z_2 \\ &\quad + 2(\vartheta_{k+1} - \vartheta) e^{\alpha\vartheta} z_3^T X_4 z_1 \\ &\quad + 2(\vartheta_{k+1} - \vartheta) e^{\alpha\vartheta} z_4^T \operatorname{sym} \left\{ \frac{X_5}{2} \right\} z_1 \\ &\quad + 2(\vartheta_{k+1} - \vartheta) e^{\alpha\vartheta} z_1^T X_3^T z_1 \\ &\quad + \alpha(\vartheta_{k+1} - \vartheta) e^{\alpha\vartheta} z_1^T \operatorname{sym} \left\{ \frac{X_1}{2} \right\} z_1 \\ &\quad + \alpha(\vartheta_{k+1} - \vartheta) e^{\alpha\vartheta} z_1^T (X_2 - X_1) z_3 \\ &\quad + \alpha(\vartheta_{k+1} - \vartheta) e^{\alpha\vartheta} z_1^T X_3 z_4 \\ &\quad + \alpha(\vartheta_{k+1} - \vartheta) e^{\alpha\vartheta} z_3^T \operatorname{sym} \left\{ \frac{X_1}{2} - X_2 \right\} z_3 \\ &\quad + \alpha(\vartheta_{k+1} - \vartheta) e^{\alpha\vartheta} z_3^T X_4 z_4 \\ &\quad + \alpha(\vartheta_{k+1} - \vartheta) e^{\alpha\vartheta} z_4^T \operatorname{sym} \left\{ \frac{X_5}{2} \right\} z_4 \\ &\quad + \alpha(\vartheta_{k+1} - \vartheta) e^{\alpha\vartheta} z_3^T (X_2 - X_1)^T z_1 \\ &\quad + \alpha(\vartheta_{k+1} - \vartheta) e^{\alpha\vartheta} z_4^T X_3^T z_1 \\ &\quad \left. + \alpha(\vartheta_{k+1} - \vartheta) e^{\alpha\vartheta} z_4^T X_4^T z_3 \right\} \xi(\vartheta) \end{aligned} \quad (17)$$

$$\begin{aligned} \mathcal{L}\mathcal{V}_3(z(\vartheta), \vartheta) &= \\ &= (\vartheta_{k+1} - \vartheta) e^{\alpha\vartheta} \dot{z}^T(\vartheta) \bar{U}_1 \dot{z}(\vartheta) \\ &\quad - \int_{\vartheta_k}^{\vartheta} e^{\alpha s} \dot{z}^T(s) \bar{U}_1 \dot{z}(s) ds \\ &\leq (\vartheta_{k+1} - \vartheta) e^{\alpha\vartheta} \dot{z}^T(\vartheta) \bar{U}_1 \dot{z}(\vartheta) \\ &\quad - e^{\alpha\vartheta} e^{-\alpha d} \int_{\vartheta_k}^{\vartheta} \dot{z}^T(s) \bar{U}_1 \dot{z}(s) ds \end{aligned} \quad (18)$$

According to Lemma 3

$$\begin{aligned} &\leq \xi^T(\vartheta) \left\{ (\vartheta_{k+1} - \vartheta) e^{\alpha\vartheta} z_2^T \bar{U}_1 z_2 \right. \\ &\quad + \Pi_2^T e^{\alpha\vartheta} e^{-\alpha d} \left[(\vartheta - \vartheta_k) \left(M_1 \bar{U}_1^{-1} M_1^T + \frac{1}{3} M_2 \bar{U}_1^{-1} M_2^T \right) \right. \\ &\quad \left. \left. + \operatorname{sym} \{ M_1(b_1 - b_2) + M_2(2b_3 - b_1 - b_2) \} \right] \Pi_2 \right\} \xi(\vartheta) \end{aligned} \quad (19)$$

$$\begin{aligned} \mathcal{L}\mathcal{V}_4(z(\vartheta), \vartheta) &= \\ &= -2e^{\alpha\vartheta} \varphi^T(\mathcal{K}z(\vartheta_k)) N \left[\varphi(\mathcal{K}z(\vartheta_k)) - \tilde{G}z(\vartheta_k) \right] \\ &\quad - 2\alpha e^{\alpha\vartheta} (\vartheta - \vartheta_k) \varphi^T(\mathcal{K}z(\vartheta_k)) N \left[\varphi(\mathcal{K}z(\vartheta_k)) - \tilde{G}z(\vartheta_k) \right] \\ &= \xi^T(\vartheta) \left\{ -2e^{\alpha\vartheta} z_5^T N [z_5 - \tilde{G}z_3] \right. \\ &\quad \left. - 2\alpha e^{\alpha\vartheta} (\vartheta - \vartheta_k) z_5^T N [z_5 - \tilde{G}z_3] \right\} \xi(\vartheta) \end{aligned} \quad (20)$$

$$\begin{aligned} \mathcal{L}\mathcal{V}_5(z(\vartheta), \vartheta) &= \\ &= \xi^T(\vartheta) \left\{ \alpha(\vartheta_{k+1} - \vartheta) e^{\alpha\vartheta} \begin{bmatrix} z_1 \\ z_3 \end{bmatrix}^T W \begin{bmatrix} z_1 \\ z_3 \end{bmatrix} \right. \\ &\quad + e^{\alpha\vartheta} (\vartheta_{k+1} - \vartheta) 2z_2^T (-Y + Y_1) z_3 \\ &\quad + e^{\alpha\vartheta} (\vartheta_{k+1} - \vartheta) z_1^T (Y + Y^T) z_2 \\ &\quad \left. - e^{\alpha\vartheta} \begin{bmatrix} z_1 \\ z_3 \end{bmatrix}^T W \begin{bmatrix} z_1 \\ z_3 \end{bmatrix} \right\} \xi(\vartheta) \end{aligned} \quad (21)$$

$$\mathcal{LV}_6(z(\vartheta), \vartheta) =$$

$$\begin{aligned} &= \leq \xi^T(\vartheta) \left\{ e^{\alpha\vartheta} z_1^T Q_3 z_1 - e^{\alpha(\vartheta-\beta_1)} z_7^T Q_3 z_7 \right. \\ &\quad + e^{\alpha\vartheta} z_1^T Q_4 z_1 - (1-\mu) e^{\alpha(\vartheta-\beta_2)} z_9^T Q_4 z_9 \\ &\quad + e^{\alpha(\vartheta-\beta_1)} z_7^T Q_5 z_7 - e^{\alpha(\vartheta-\beta_2)} z_8^T Q_5 z_8 \\ &\quad \left. + e^{\alpha\vartheta} z_2^T Q_6 z_2 - e^{\alpha(\vartheta-\beta_1)} z_6^T Q_6 z_6 \right\} \xi(\vartheta) \end{aligned}$$

$$\mathcal{LV}_7(z(\vartheta), \vartheta) =$$

$$\begin{aligned} &= 2\beta_2^2 e^{\alpha\vartheta} \dot{z}^T(\vartheta) L_1 \dot{z}(\vartheta) \\ &\quad - 2\beta_2 \int_{\vartheta-\beta_2}^{\vartheta} e^{\alpha s} \dot{z}^T(s) L_1 \dot{z}(s) ds \\ &\leq 2\beta_2^2 e^{\alpha\vartheta} \dot{z}^T(\vartheta) L_1 \dot{z}(\vartheta) \\ &\quad - 2\beta_2 e^{\alpha\vartheta} e^{-\alpha\beta_2} \int_{\vartheta-\beta_2}^{\vartheta} \dot{z}^T(s) L_1 \dot{z}(s) ds \quad (22) \end{aligned}$$

According to Lemma 4, we get

$$\leq \xi^T(\vartheta) \left\{ 2\beta_2^2 e^{\alpha\vartheta} z_2^T L_1 z_2 - 2e^{\alpha\vartheta} e^{-\alpha\beta_2} \Pi_1^T \Upsilon \Pi_1 \right\} \xi(\vartheta) \quad (23)$$

where

$$\Upsilon_{11} = L_2 L_3^{-1} L_2^T - L_1 + \beta_2^2 L_3 + 2\beta_2 L_2, \quad \Upsilon_{12} = -L_2 L_3^{-1} L_2^T + L_1 - \beta_2 L_2^T, \quad \Upsilon_{22} = L_2 L_3^{-1} L_2^T - L_1.$$

Using Lemma 1, we get

$$\mathcal{LV}_8(z(\vartheta), \vartheta) \leq$$

$$\begin{aligned} &\wp_2 e^{\alpha\vartheta} z^T(\vartheta) Q_7 z(\vartheta) \\ &\quad - e^{\alpha\vartheta} \left(\int_{\vartheta-\beta_2}^{\vartheta-\beta_1} z(s) ds \right)^T Q_7 \left(\int_{\vartheta-\beta_2}^{\vartheta-\beta_1} z(s) ds \right) \\ &\quad + \wp_1 e^{\alpha\vartheta} z^T(\vartheta) Q_8 z(\vartheta) \\ &\quad - e^{\alpha\vartheta} \left(\int_{\vartheta-\beta_1}^{\vartheta} z(s) ds \right)^T Q_8 \left(\int_{\vartheta-\beta_1}^{\vartheta} z(s) ds \right) \\ &\quad + e^{\alpha\vartheta} \wp_3 \dot{z}^T(\vartheta - \beta_1) Q_9 \dot{z}(\vartheta - \beta_1) \\ &\quad - e^{\alpha\vartheta} \beta_{21} \int_{\vartheta-\beta_2}^{\vartheta-\beta_1} \dot{z}^T(s) Q_9 \dot{z}(s) ds \end{aligned}$$

Given Lemma 2, it can be concluded that

$$\begin{aligned} &- \beta_{21} \int_{\vartheta-\beta_2}^{\vartheta-\beta_1} \dot{z}^T(s) Q_9 \dot{z}(s) ds \leq \\ &\quad - \left[\begin{array}{c} z(\vartheta - \beta_1) \\ z(\vartheta - \beta_2) \\ \frac{1}{\beta_{21}} \int_{\vartheta-\beta_2}^{\vartheta-\beta_1} z(s) ds \end{array} \right]^T \left[\begin{array}{ccc} 4Q_9 & 2Q_9 & -6Q_9 \\ * & 4Q_9 & -6Q_9 \\ * & * & 12Q_9 \end{array} \right] \left[\begin{array}{c} z(\vartheta - \beta_1) \\ z(\vartheta - \beta_2) \\ \frac{1}{\beta_{21}} \int_{\vartheta-\beta_2}^{\vartheta-\beta_1} z(s) ds \end{array} \right] \\ &\leq \xi^T(\vartheta) \left\{ \wp_2 e^{\alpha\vartheta} z_1^T Q_7 z_1 - e^{\alpha\vartheta} z_{11}^T Q_7 z_{11} \right. \\ &\quad + \wp_1 e^{\alpha\vartheta} z_1^T Q_8 z_1 - e^{\alpha\vartheta} z_{10}^T Q_8 z_{10} \\ &\quad + \wp_3 e^{\alpha\vartheta} z_6^T Q_9 z_6 \\ &\quad + e^{\alpha\vartheta} \left[-4z_7^T Q_9 z_7 - 2z_7^T Q_9 z_8 + \frac{6}{\beta_{21}} z_7^T Q_9 z_{11} \right. \\ &\quad \left. \left. - 4z_8^T Q_9 z_8 + \frac{6}{\beta_{21}} z_8^T Q_9 z_{11} - \frac{12}{\beta_{21}^2} z_{11}^T Q_9 z_{11} \right] \right\} \xi(\vartheta). \quad (24) \end{aligned}$$

According to Lemma 1, we have

$$\begin{aligned} \mathcal{LV}_9(e(\vartheta), \vartheta) &= \xi^T(\vartheta) \left\{ \wp_4 e^{\alpha\vartheta} z_2^T Q_{10} z_2 \right. \\ &\quad - \beta_1^2 e^{\alpha\vartheta} z_1^T Q_{10} z_1 + \beta_1 e^{\alpha\vartheta} z_1^T Q_{10} z_{10} \\ &\quad + \beta_1 e^{\alpha\vartheta} z_{10}^T Q_{10} z_1 \\ &\quad \left. - e^{\alpha\vartheta} z_{10}^T Q_{10} z_{10} \right\} \xi(\vartheta). \quad (25) \end{aligned}$$

The null assertion from (10) that follows always true for every suitably dimensional matrix R_0 :

$$\begin{aligned} &- \left[\begin{array}{c} z(\vartheta - \beta_1) \\ z(\vartheta - \beta_2) \\ \frac{1}{\beta_{21}} \int_{\vartheta-\beta_2}^{\vartheta-\beta_1} z(s) ds \end{array} \right]^T \left[\begin{array}{ccc} 4Q_9 & 2Q_9 & -6Q_9 \\ * & 4Q_9 & -6Q_9 \\ * & * & 12Q_9 \end{array} \right] \\ &\quad \times \left[\begin{array}{c} z(\vartheta - \beta_1) \\ z(\vartheta - \beta_2) \\ \frac{1}{\beta_{21}} \int_{\vartheta-\beta_2}^{\vartheta-\beta_1} z(s) ds \end{array} \right] \end{aligned}$$

Since $g(D(r(\vartheta))z(\vartheta), y(\vartheta))$ satisfies sector condition (8), for any diagonal matrix $U > 0$, one can establish

$$\begin{cases} 0 \leq -2e^{\alpha\vartheta} \sum_{i=1}^{n_h} u_i g_i \left(d_{i\bar{p}}^T z(\vartheta), y(\vartheta) \right) \left[g_i \left(d_{i\bar{p}}^T z(\vartheta), y(\vartheta) \right) - k d_{i\bar{p}}^T z(\vartheta) \right] \\ 0 \leq -2e^{\alpha\vartheta} g^T \left(D(r(\vartheta))z(\vartheta), y(\vartheta) \right) U g \left(D(r(\vartheta))z(\vartheta), y(\vartheta) \right) + 2ke^{\alpha\vartheta} z^T(\vartheta) D(r(\vartheta))^T U g \left(D(r(\vartheta))z(\vartheta), y(\vartheta) \right) \end{cases}$$

From 16-26, it may be inferred that

$$\begin{aligned} \mathbb{E} \left\{ \mathcal{L} \mathcal{V}(z(\vartheta), \vartheta, r(\vartheta)) \right\} \leq & \mathbb{E} \left\{ e^{\alpha\vartheta} \xi^T(\vartheta) \left[\frac{\vartheta - \vartheta_k}{d_k} \Delta_1(d_k) \right. \right. \\ & + \frac{\vartheta_{k+1} - \vartheta}{d_k} \Delta_1(d_k) \\ & - z^T(\vartheta) E z(\vartheta) \\ & \left. \left. + w^T(\vartheta) F w(\vartheta) \right] \xi(\vartheta) \right\} \quad (26) \end{aligned}$$

with $\Delta_1(d_k)$ and $\Delta_2(d_k)$ are given below:

$$\begin{aligned} \Delta_1(d_k) = & \Xi_1 + d_k \Xi_2 \\ & + 2e^{-\alpha\beta_2} z_1^T L_2 L_3^{-1} L_2^T z_1 \\ & - 4e^{-\alpha\beta_2} z_1^T L_2 L_3^{-1} L_2^T z_8 \\ & + 2e^{-\alpha\beta_2} z_8^T L_2 L_3^{-1} L_2^T z_8, \end{aligned}$$

$$\begin{aligned} \Delta_2(d_k) = & \Xi_1 + d_k \Xi_3 \\ & + d_k \Pi_2^T M_1 \bar{U}_1^{-1} M_1^T \Pi_2 \\ & + \frac{1}{3} d_k e^{-\alpha d} \Pi_2^T M_2 \bar{U}_1^{-1} M_2^T \Pi_2 \\ & + 2e^{-\alpha\beta_2} z_1^T L_2 L_3^{-1} L_2^T z_1 \\ & - 4e^{-\alpha\beta_2} z_1^T L_2 L_3^{-1} L_2^T z_8 \\ & + 2e^{-\alpha\beta_2} z_8^T L_2 L_3^{-1} L_2^T z_8. \end{aligned}$$

In the meanwhile, if $R_0^{-1} \bar{\mathcal{K}}$ is the gain matrix, then $\Delta_1(d_k) < 0, \Delta_2(d_k) < 0$, depending on the Schur complement, may be deduced from 11, 12 and 27

$$\begin{aligned} \mathbb{E} \left\{ \mathcal{L} \mathcal{V}(z(\vartheta), \vartheta, r(\vartheta)) \right\} < & \mathbb{E} \left\{ -e^{\alpha\vartheta} z^T(\vartheta) E z(\vartheta) \right. \\ & \left. + e^{\alpha\vartheta} w^T(\vartheta) F w(\vartheta) \right\} \quad (27) \end{aligned}$$

respectively. Moreover, a simple computation using the Schur complement to the $\begin{bmatrix} P_{\bar{p}} & (\mathcal{K}_l - \tilde{G}_l)^T \\ \star & u_{0l}^2 \end{bmatrix}$ yields

$$(\mathcal{K}_l - \tilde{G}_l)^T (\mathcal{K}_l - \tilde{G}_l) \leq u_{0l}^2 P_{\bar{p}}.$$

It follows from (27) that

$$\begin{aligned} & \mathcal{V}(z(\vartheta), \vartheta, r(\vartheta)) - \mathcal{V}(z(\vartheta_f), \vartheta_f, r(\vartheta_f)) \\ & + \mathcal{V}(z(\vartheta_f^-), \vartheta_f^-, r(\vartheta_f^-)) \\ & - \mathcal{V}(z(\vartheta_{f-1}), \vartheta_{f-1}, r(\vartheta_{f-1})) + \dots \\ & + \mathcal{V}(z(\vartheta_1^-), \vartheta_1^-, r(\vartheta_1^-)) - \mathcal{V}(z(0), 0, r(0)) \\ & \leq - \int_0^{\vartheta} e^{\alpha s} z^T(s) E z(s) ds + \int_0^{\vartheta} e^{\alpha s} w^T(s) F w(s) ds \quad (28) \end{aligned}$$

Owing to

$$\mathcal{V}(z(\vartheta_n^-), \vartheta_n^-, r(\vartheta_n^-)) - \mathcal{V}(z(\vartheta_n), \vartheta_n, r(\vartheta_n)) = 0, \quad \forall n = 1, 2, \dots, f, \text{ one has}$$

$$\begin{aligned} & \mathcal{V}(z(\vartheta), \vartheta, r(\vartheta)) - \mathcal{V}(z(0), 0, r(0)) \leq \\ & - \int_0^{\vartheta} e^{\alpha s} z^T(s) E z(s) ds + \int_0^{\vartheta} e^{\alpha s} w^T(s) F w(s) ds \quad (29) \end{aligned}$$

Equation (27) with Dynkin's formula yields the following result:

$$\begin{aligned} \mathbb{E} \left\{ \mathcal{V}(z(\vartheta), \vartheta, r(\vartheta)) \right\} \leq & \mathbb{E} \left\{ \mathcal{V}(z(0), 0, r(0)) \right\} \\ & - \int_0^{\vartheta} e^{\alpha s} z^T(s) E z(s) ds + \int_0^{\vartheta} e^{\alpha s} w^T(s) F w(s) ds \quad (30) \end{aligned}$$

On the other hand, it is noted that

$$\begin{aligned} \beth_1 = & \left(\frac{1 - e^{-\alpha\beta_1}}{\alpha} \right), \\ \beth_2 = & \left(\frac{1 - e^{-\alpha\beta_2}}{\alpha} \right), \\ \beth_3 = & \left(\frac{e^{-\alpha\beta_1} - e^{-\alpha\beta_2}}{\alpha} \right), \\ \beth_4 = & 2\beta_2 \left(\frac{e^{-\alpha\beta_2} + \alpha\beta_2 - 1}{\alpha^2} \right), \\ \beth_5 = & \beta_{21} \left(\frac{e^{\alpha\beta_2} - e^{\alpha\beta_1} - \alpha\beta_{21}}{\alpha^2} \right), \\ \beth_6 = & \beta_1 \left(\frac{e^{\alpha\beta_1} - \alpha\beta_1 - 1}{\alpha^2} \right), \\ \beth_7 = & \beta_{21} \left(\frac{e^{\alpha\beta_{21}} - \alpha\beta_{21} - 1}{\alpha^2} \right), \\ \beth_8 = & \beta_1^2 \left(\frac{2e^{\alpha\beta_1} - 2 - 2\alpha\beta_1 - \alpha^2\beta_1^2}{4\alpha^3} \right). \end{aligned}$$

According to (30) and Lemma 2, thus

$$\begin{aligned}
& \mathbb{E}\{\mathcal{V}(z(\vartheta), \vartheta, r(\vartheta))\} \\
& \leq \mathbb{E}\left\{\mathcal{V}(z(\vartheta), \vartheta, r(\vartheta)) + \int_0^{\vartheta} e^{\alpha s} w^T(s) w(s) ds\right\} \\
& \leq \left(\max_{1 \leq \bar{p} \leq n} \lambda_M(P_{\bar{p}}) \mathbb{E}\{z^T(0)z(0)\} + \beth_1 \lambda_{\max}(Q_3 + Q_6) \right. \\
& \quad + \beth_2 \lambda_{\max}(Q_4) + \beth_3 \lambda_{\max}(Q_5) + \beth_4 \lambda_{\max}(L_1) \\
& \quad + \beth_5 \lambda_{\max}(Q_7) + \beth_6 \lambda_{\max}(Q_8) + \beth_7 \lambda_{\max}(Q_9) \\
& \quad \left. + \beth_8 \lambda_{\max}(Q_{10}) \right) \\
& \quad \times \mathbb{E}\left\{\sup_{s \in [-\beta_2, 0]} \|\theta(s)\|_{\beta}^2\right\} \\
& \quad + \frac{(e^{\alpha \vartheta} - 1) \lambda_M(F)}{\alpha} \mathbb{E}\left\{\sup_{s \in [0, \vartheta]} \|w(s)\|^2\right\}. \quad (31)
\end{aligned}$$

Nevertheless, taking into account (11), (12) and Ξ_1 , one may infer that

$$\begin{aligned}
& \mathbb{E}\{\mathcal{V}(z(\vartheta), \vartheta, r(\vartheta))\} \\
& \geq \mathbb{E}\{e^{\alpha \vartheta} z^T(\vartheta) P_{\bar{p}} z(\vartheta) + \mathcal{V}_2(\vartheta, z(\vartheta)) \\
& \quad + \mathcal{V}_3(\vartheta, z(\vartheta)) + \mathcal{V}_4(\vartheta, z(\vartheta)) + \mathcal{V}_5(\vartheta, z(\vartheta))\} \\
& \geq \mathbb{E}\left\{e^{\alpha \vartheta} \frac{(\vartheta - \vartheta_k)}{d_k} \xi^T(\vartheta) \eta_1^T(\vartheta) \bar{P} \eta_1(\vartheta) \xi(\vartheta) \right. \\
& \quad \left. + e^{\alpha \vartheta} \frac{(\vartheta_{k+1} - \vartheta)}{d_k} \xi^T(\vartheta) \eta_1^T(\vartheta) \Xi_1(d_k) \eta_1(\vartheta) \xi(\vartheta) \right\} \\
& \geq \mathbb{E}\left\{e^{\alpha \vartheta} \frac{(\vartheta - \vartheta_k)}{d_k} \xi^T(\vartheta) \eta_1^T(\vartheta) \tilde{I} \eta_1(\vartheta) \xi(\vartheta) \right. \\
& \quad \left. + e^{\alpha \vartheta} \frac{(\vartheta_{k+1} - \vartheta)}{d_k} \xi^T(\vartheta) \eta_1^T(\vartheta) \tilde{I} \eta_1(\vartheta) \xi(\vartheta) \right\} \\
& = \mathbb{E}\{e^{\alpha \vartheta} \|z(\vartheta)\|^2\}. \quad (32)
\end{aligned}$$

Next, with reference to (31) and (32), one has

$$\begin{aligned}
& \mathbb{E}\{\|z(\vartheta)\|\} \leq \sqrt{\varepsilon_1} \mathbb{E}\left\{\sup_{s \in [-\beta_2, 0]} \|\theta(s)\|_{\beta}\right\} e^{\frac{-\alpha \vartheta}{2}} \\
& \quad + \sqrt{\frac{\lambda_M(F)}{\alpha}} \mathbb{E}\left\{\sup_{s \in [0, \vartheta]} \|w(s)\|\right\}. \quad (33)
\end{aligned}$$

where

$$\begin{aligned}
\varepsilon_1 = & \max_{1 \leq \bar{p} \leq n} \lambda_M(P_{\bar{p}}) + \beth_1 \lambda_{\max}(Q_3 + Q_6) \\
& + \beth_2 \lambda_{\max}(Q_4) + \beth_3 \lambda_{\max}(Q_5) + \beth_4 \lambda_{\max}(L_1) \\
& + \beth_5 \lambda_{\max}(Q_7) + \beth_6 \lambda_{\max}(Q_8) + \beth_7 \lambda_{\max}(Q_9) \\
& + \beth_8 \lambda_{\max}(Q_{10})
\end{aligned}$$

Mean square ISS is exhibited by the error system (10) according to Definition 1.

Remark 3: ISS is a crucial notion in control theory that assesses a dynamical system's resilience in the face of external disturbances. Introduced by Sontag in the realm of nonlinear control systems, ISS is particularly significant for systems subject to bounded inputs. MJ systems are dynamic systems capable of switching between different modes based on a Markov process, with each mode exhibiting distinct dynamics. This mode-switching introduces complexity in analysis and control, necessitating strategies that account for both the current state and potential future modes. Lur'e systems are characterized by nonlinear feedback, often represented by state-dependent nonlinear functions, which further complicate ISS synchronization due to their inherent nonlinearity. Utilizing LKF, especially in the form of LKFs, is instrumental in establishing ISS for MJ systems with Lur'e characteristics. These functionals can effectively incorporate both the nonlinear feedback and the stochastic nature of the system, offering a comprehensive criterion for ISS. The integration of ISS, MJ systems and Lur'e systems forms a robust framework for the analysis and control of complex dynamical systems. This combination not only addresses the challenges posed by nonlinearities and stochastic behaviors but also facilitates the development of effective control strategies that ensure ISS synchronization and resilience across a wide range of applications.

Remark 4: By including the functional expression $\int_{\vartheta - \beta(\vartheta)}^{\vartheta} e^{\alpha s} z^T(s) Q_4 z(s) ds$ and introducing a corresponding matrix block in the matrices $\Delta_2(d_k)$ and $\Delta_1(d_k)$. This addition leads to the creation of a matrix block on the diagonal of the principal matrices, specifically, $-(1 - \mu)e^{-\alpha \beta_2} z_9^T Q_4 z_9$. When the magnitude of the scalar μ is smaller than 1, it can be determined analytically that $-(1 - \mu)e^{-\alpha \beta_2} z_9^T Q_4 z_9 < 0$. This further enhances the negativity definiteness of these two matrices, making them even more negative.

Remark 5: The representation of the error system (10) may be expressed as follows when $r(\vartheta) = 1$ is

used to reduce the following system:

$$\begin{aligned} \dot{z}(\vartheta) = & Az(\vartheta) + Bz(\vartheta - \beta(\vartheta)) \\ & + Wg(Dz(\vartheta), y(\vartheta)) - \mathcal{K}z(\vartheta_k) \\ & - Gw(\vartheta) + \wp(\mathcal{K}z(\vartheta_k)). \end{aligned} \quad (34)$$

$$\Delta_1(d_k) = \begin{bmatrix} \bar{\Xi}_1 + d_k \Xi_2 & \sqrt{2}e^{-\alpha\beta_2}(e_1^T L_2 - e_8^T L_2) \\ \star & -L_3 \end{bmatrix} < 0 \quad (35)$$

$$\Delta_2(d_k) = \begin{bmatrix} \bar{\Xi}_1 + d_k \Xi_3 & \sqrt{d_k}e^{-\alpha d} \Pi_2^T M_1 \\ \star & -\bar{U}_1 \\ \star & \star \\ \star & \star \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{d_k}e^{-\alpha d} \Pi_2^T M_2 & \sqrt{2}e^{-\alpha\beta_2}(e_1^T L_2 - e_8^T L_2) \\ 0 & 0 \\ -3\bar{U}_1 & 0 \\ \star & -L_3 \end{bmatrix} < 0 \quad (36)$$

$$\begin{aligned} \bar{\Xi}_4 = & \bar{P} + d_k W - \hat{I} > 0, \\ \bar{P} = & \begin{bmatrix} P & \check{O}_{n \times n} \\ \star & \check{O}_{n \times n} \end{bmatrix}, \\ \hat{I} = & \begin{bmatrix} \hat{I}_n & \check{O}_{n \times n} \\ \star & \check{O}_{n \times n} \end{bmatrix}, \\ & \begin{bmatrix} P & (\mathcal{K}_1 - \tilde{G}_1)^T \\ \star & u_{01}^2 \end{bmatrix} \geq 0 \end{aligned}$$

$$\bar{\Xi}_1 =$$

$$\begin{aligned} & z_1^T \alpha P z_1 + z_1^T P z_2 - z_1^T \text{sym} \left\{ \frac{X_1}{2} \right\} z_1 - z_1^T (X_2 - X_1) z_3 \\ & - z_1^T X_3 z_4 - z_3^T \text{sym} \left\{ \frac{X_1}{2} - X_2 \right\} z_3 - z_3^T X_4 z_4 \\ & - z_4^T \text{sym} \left\{ \frac{X_5}{2} \right\} z_4 - 2z_5^T N z_5 + z_3^T \tilde{G}^T N z_5 \\ & + 2\beta_2^2 z_2^T L_1 z_2 + e^{-\alpha d} \{ \Pi_2^T M_1 \Pi_3 + \Pi_2^T M_2 \Pi_4 \} + \\ & + z_1^T Q_3 z_1 - e^{-\alpha\beta_1} z_7^T Q_3 z_7 + z_1^T Q_4 z_1 \\ & - (1 - \mu) e^{-\alpha\beta_2} z_9^T Q_4 z_9 + e^{-\alpha\beta_1} z_7^T Q_5 z_7 \\ & - e^{-\alpha\beta_2} z_8^T Q_5 z_8 + z_2^T Q_6 z_2 - e^{-\alpha\beta_1} z_6^T Q_6 z_6 \\ & + \wp_2 z_1^T Q_7 z_1 - z_{11}^T Q_7 z_{11} + \wp_1 z_1^T Q_8 z_1 - z_{10}^T Q_8 z_{10} \\ & + \wp_4 z_2^T Q_{10} z_2 - \beta_1^2 z_1^T Q_{10} z_1 + \beta_1 z_1^T Q_{10} z_{10} \\ & - z_{10}^T Q_{10} z_{10} + \wp_3 z_6^T Q_9 z_6 - 4z_7^T Q_9 z_7 - 2z_7^T Q_9 z_8 \end{aligned}$$

$$\begin{aligned} & + \frac{6}{\beta_{21}} z_7^T Q_9 z_{11} - 4z_8^T Q_9 z_8 + \frac{6}{\beta_{21}} z_8^T Q_9 z_{11} \\ & - \frac{12}{\beta_{21}^2} z_{11}^T Q_9 z_{11} - 2e^{-\alpha\beta_2} z_1^T L_1 z_1 + 2\beta_2^2 e^{-\alpha\beta_2} z_1^T L_3 z_1 \\ & + 4\beta_2 e^{-\alpha\beta_2} z_1^T L_2 z_1 + 2e^{-\alpha\beta_2} z_1^T L_1 z_8 \\ & - 2\beta_2 e^{-\alpha\beta_2} z_1^T L_2 z_8 - 2e^{-\alpha\beta_2} z_8^T L_1 z_8 \\ & - z_1^T \text{sym} \left(\frac{Y}{2} \right) z_1 - z_1^T (-Y + Y_1) z_3 \\ & - z_3^T \text{sym} \left(\frac{Y}{2} - Y_1 \right) z_3 - z_1^T R_0^T z_2 + 2z_1^T R_0^T A z_1 \\ & + z_1^T R_0^T B z_9 + z_1^T R_0^T W z_{12} - z_1^T \mathcal{R} z_3 - z_1^T R_0^T G z_{13} \\ & + z_1^T R_0^T z_5 - 2z_2^T R_0^T z_2 + z_2^T R_0^T A z_1 + z_2^T R_0^T B z_9 \\ & + z_2^T R_0^T W z_{12} - z_2^T \mathcal{R} z_3 - z_2^T R_0^T G z_{13} + z_2^T R_0^T z_5 \\ & - 2z_{12}^T U z_{12} + k z_1^T D U z_{12} + z_1^T E z_1 - z_{13}^T F z_{13} \end{aligned}$$

and the other notations are the same as those in Theorem 3.1. Then, under controller (8) with gain $\mathcal{K} = R_0^{-1} \mathcal{R}$, the ISS synchronization of error system (34) can be achieved.

Proof: The proof of this corollary is similar to that of Theorem 3.1.

Remark 6: Under the conditions outlined in Theorem 3.1, $w(\rho) = 0$ it is obvious from the proof above that the system (10) exhibits exponential stability with a decay rate ε of the order of under the conditions outlined in the preceding proof.

4 Numerical example

This section provides numerical example to illustrate the effectiveness and practical applicability of the proposed method.

Example 1: The dynamical network may be thought a delayed Chua's circuit system with two modes ($\bar{p} = 1, 2$), which is explained as follows:

$$\begin{cases} \dot{x}_1(\rho) = a_{\bar{p}}(x_2(\rho) - m_{\bar{p}}x_1(\rho) + \hat{g}(x_1(\rho))) - b_{\bar{p}}x_1(\rho - \beta(\rho)), \\ \dot{x}_2(\rho) = x_1(\rho) - x_2(\rho) + x_3(\rho) - b_{\bar{p}}x_1(\rho - \beta(\rho)), \\ \dot{x}_3(\rho) = -b_{\bar{p}}x_2(\rho) + c_{\bar{p}}(2x_1(\rho - \beta(\rho)) - x_3(\rho - \beta(\rho))). \end{cases} \quad (37)$$

with the nonlinear function

$$\hat{g}(x_1(\rho)) = \frac{1}{2}(m_{\bar{p}} - m_0)(|x_1(\rho) + 1| - |x_1(\rho) - 1|)$$

and the parameters $m_0 = \frac{-1}{7}$, $m_1 = \frac{1}{7}$, $m_2 = \frac{2}{7}$, $a_1 = a_2 = 9$, $b_1 = 14.52$, $b_2 = 14.28$, $c_1 = 0.1$, $c_2 = 0.09$. It can be found that the corresponding system parameters of Lur'e form are as follows:

$$\begin{aligned} A_{\bar{p}} &= \begin{bmatrix} -a_{\bar{p}}m_{\bar{p}} & a_{\bar{p}} & 0 \\ 1 & -1 & 1 \\ 0 & -b_{\bar{p}} & 0 \end{bmatrix}, \\ B_{\bar{p}} &= \begin{bmatrix} -c_{\bar{p}} & 0 & 0 \\ -c_{\bar{p}} & 0 & 0 \\ 2c_{\bar{p}} & 0 & -c_{\bar{p}} \end{bmatrix}, \\ W_{\bar{p}} &= \begin{bmatrix} -a_{\bar{p}}(m_0 - m_{\bar{p}}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ D_{\bar{p}} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{p} = 1, 2. \end{aligned}$$

with the nonlinear $\sigma(D_{\bar{p}}x(\rho)) = \text{col}\{\frac{1}{2}(|x_1(\rho) + 1| - |x_1(\rho) - 1|), 0, 0\}$ belonging to the sector $[0, 1]$. Let $C_{\bar{p}} = \text{diag}\{1, 1, 1\}$ and $G_{\bar{p}} = \text{diag}\{0.1, 0.1, 0.1\}$. The external disturbance is defined uniformly across both modes as $w(\rho) = 0.2 \sin(10\rho)$. Let $r(\rho)$ be a right-continuous Markov process with known transition rate matrix:

$$\Pi = \begin{bmatrix} -0.2 & 0.2 \\ 0.4 & -0.4 \end{bmatrix}$$

By selecting $\alpha = 0.3$ and solving the LMIs (11)-(12), for a specific sampling period of $d = 0.256$, the corresponding controller gain matrix can be calculated as follows:

$$\mathcal{K} = \begin{bmatrix} -2.1015 & 0.1214 & -0.3221 \\ -1.1421 & -0.0122 & 0.0015 \\ 4.1331 & 0.1212 & -0.3224 \end{bmatrix}.$$

The network simulations begin with specific initial conditions of master and slave are taken as $x(0) = (0.3, 0.1, -0.4)$ and $y(0) = (0.6, -0.5, 0.2)$. It is important to highlight that a longer sampling period leads to several consequences. Firstly, it reduces the use of communication channels, resulting in fewer behaviors. Secondly, it decreases the amount of signal transmission. Consider differentiable delay function $\beta(\rho) = \mu_2 + \mu_1 \sin(\rho)$, which means that $\beta_1 = \mu_2 - \mu_1$ and $\beta_2 = \mu_2 + \mu_1$. Set $\alpha = 0.01$, then, either Theorem 3.1 or Corollary 3.2 can be used to determine the maximum allowable upper bound d . Table 1 shows the maximum

allowable upper bounds d for different μ_1 and μ_2 , where the first entry and the second entry of the ordered pairs are obtained by using Theorem 3.1 and Corollary 3.2, respectively. It can be seen that the maximum allowable d decreases as μ_1 and μ_2 increase. Note also that the first entry is always smaller than the second entry, which indicates that the condition in Theorem 3.1 is more conservative than that in Corollary 3.2. The chaotic attractors of master system described in (1) are illustrated in Figure 1, while those of the slave system defined in (3) are depicted in Figure 2. Figure 3 displays the sampling instants and sampling intervals. The dynamic state trajectories of the error system in (9) are shown in Figures 4 and 5, representing the responses without and with disturbance, respectively. Figure 6 captures the evolution of the Markov jump signal.

Remark 7: To achieve synchronization of Lur'e systems, various control strategies have been employed, including adaptive control [13] and SDC [14]. A sampled-data method of tracking the trajectory of a system is striking since it is simple to implement and has the capability to track the trajectories of a system smoothly and accurately during normal operation, which is distinctive from these other approaches [14]. It is quite evident that this control method is the best option when it comes to drive systems that are influenced by structural and parameter MJ systems.

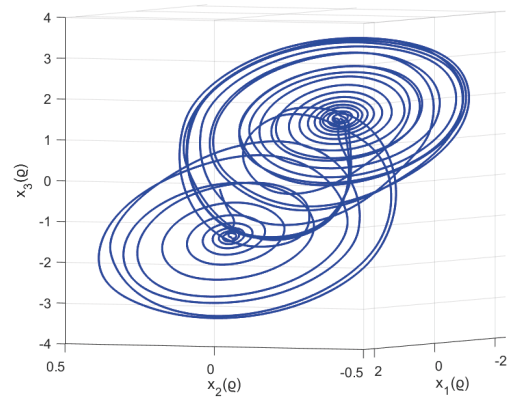


Figure 1. Chaotic attractors of master system $x(\rho)$ in Example 1

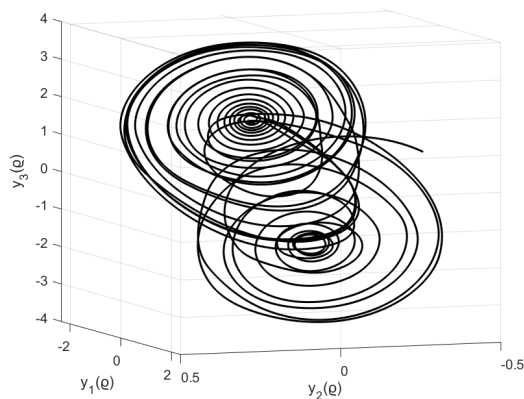


Figure 2. Chaotic attractors of master system $y(\rho)$ in Example 1

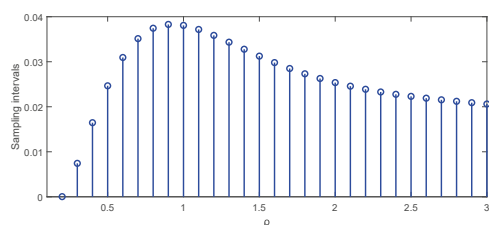


Figure 3. The sampling instants and sampling intervals in Example 1

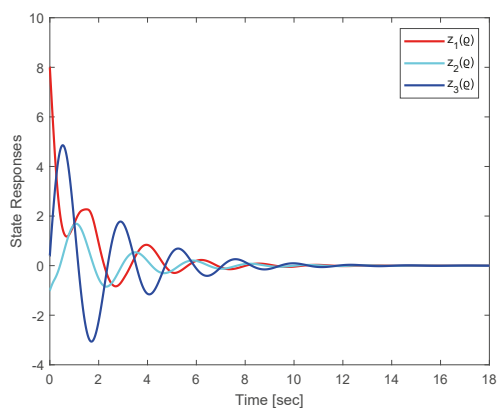


Figure 4. The state response for error system without disturbance in Example 1

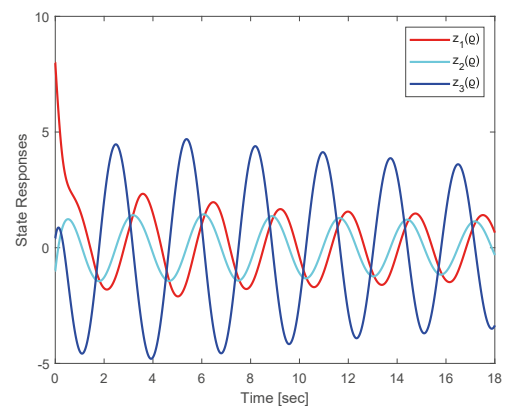


Figure 5. The state response for error system with disturbance in Example 1

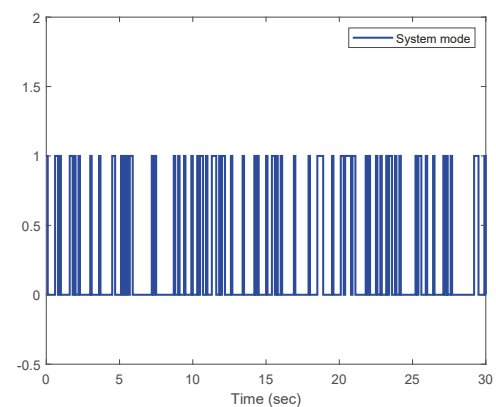


Figure 6. Markov jump signal in Example 1

5 Conclusion

This study presents a robust approach to achieving exponential synchronization in actuator-saturated SDC for MJ Lur'e networks through the application of ISS. By directly addressing the issues given by actuator saturation and external disturbances, the suggested technique offers great promise for improving system performance and ISS synchronization. Specifically, the LKFs used in this study have been engineered to incorporate information about the state and mode of the system over the sampling period, which helps to reduce the amount of conservatism in the subsequent analysis of the system. As a result, we have developed a set of sufficient conditions that allows us to design SDCs, which can guarantee mean-square ISS of the error system, as they are presented in Theorems 3.1 and Corollary 3.2. In this paper, we propose a novel

approach by combining an MFMBII with a flexible inequality involving an additional free weighting matrix in order to provide better synchronization criteria than are available in previous methods. It has been demonstrated that these criteria can be solved effectively through numerical optimization techniques that are well established and are formulated as LMIs. The findings significantly contribute to the expanding body of knowledge in control theory and pave the way for future research and practical applications across various fields. Subsequent work may focus on extending these concepts to more complex networks and developing advanced control algorithms. In the context of smart grids, this approach can enhance the ISS synchronization and performance of distributed energy resources and loads. In healthcare systems, it can improve the coordination of robotic surgical systems or teleoperated medical devices, where precise synchronization is vital for ensuring patient safety and treatment efficacy. Overall, the potential applications of this synchronization method highlight its importance in addressing contemporary challenges in control systems across diverse industries. The potential applications of this research span a wide range of areas, including distributed power grids, biological neural networks, industrial process and communication networks, where the robust and reliable control of complex, interconnected systems is of critical importance. As a future direction, advanced learning-based techniques such as recurrent neural network (RNN)-based disturbance observers (e.g., gated RBF and PINN models) may be explored to estimate hidden disturbances or delay variations, which can then be integrated with the existing saturated controller. Furthermore, federated learning and multi-agent reinforcement learning approaches also offer promising avenues for enhancing system performance under complex conditions [23].

6 Declaration

Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Conflict of Interest

This work does not have any conflicts of interest.

Funding Statement

This work was supported by the National Natural Science Foundation of China under Grant No. 62103103, the Natural Science Foundation of Jiangsu Province of China under Grant No. BK20210223, the Open Research Project of State Key Laboratory of Industrial Control Technology, China under Grant No. ICT2025B37, the Open Research Project of the Ministry of Education Key Laboratory of NSLSCS under Grant No. 202502, and the Jiangsu Provincial Scientific Research Center of Applied Mathematics under Grant No. BK20233002.

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