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Original Study

On a generalization of the Cahn-Hilliard type equation with logarithmic nonlinearities in island formation

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Abstract

In this paper, the main objective is to demonstrate the existence of solutions for an equation resembling the Cahn-Hilliard model, featuring a proliferation term and a logarithmic nonlinear term. This equation is conceptualized within the context of interactions in liquid-gas systems, particularly in the context of island formation. The primary challenge lies in the departure from the original Cahn-Hilliard equation, as we no longer maintain conservation of the spatial average mean of the order parameter. This departure introduces the complexity in establishing uniform estimates for the solutions of the approximated problems concerning the regularization parameter, as it may potentially result in finite-time blow-up.

Keywords: Cahn–Hilliard equation, existence, blow up, logarithmic nonlinear term, formation of islands.
AMS 2020 codes: 35K55; 35B45.

1 Introduction

The significance of the Cahn-Hilliard equation in material science cannot be overstated. This equation effectively captures the crucial qualitative aspects of two-phase systems, particularly in relation to phase separation processes. In the realm of material science, the resulting pattern formation is termed as the microstructure of the material, wielding a substantial influence on diverse material properties such as strength, hardness, and conductivity.

The broad applicability of the Cahn-Hilliard model across various evolutionary stages underscores its versatility. It serves as a robust model for early-stage systems, offering a qualitative description for intermediate times, and continues to be relevant for late-stage systems. Notably, the gradual evolution during late stages often occurs at such a slow pace that pattern formation essentially becomes frozen over the relevant time scales. Consequently, the observed practical behavior reflects the long-term dynamics of the system. For a more in-depth

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exploration, interested readers are directed to references such as [1–5].

Beyond its fundamental role in material science, the Cahn-Hilliard equation finds an application in modeling a diverse array of phenomena. This extends to areas such as population dynamics [6], bacterial films [7], thin films [8, 9], image processing [10, 11], and even celestial phenomena like the rings of Saturn [12].

In a related study [13], the authors explored a model put forth in [14]:

$$\frac{\partial \varsigma}{\partial t} + \Delta^2 \varsigma - \Delta f(\varsigma) + \eta(\varsigma) = 0, \tag{1}$$

where

$$\eta(s) = s(s-1) \tag{2}$$

and

$$f(s) = (s - \frac{1}{2})^3 - (s - \frac{1}{2}). \tag{3}$$

Furthermore, in [15] the author has analysed the model (1) with a regular nonlinear term (3), but with a general source term, which is given by

$$\eta(s) = \alpha s^2 + \beta s + \gamma, \tag{4}$$

where $\alpha > 0$ and $\beta, \gamma \in \mathbb{R}$. The authors in [13, 15] have proved that the solutions can blow up in finite time and exist globally under strong assumptions on the solutions and not only on the initial data. However, the author in [16] considered the model (1)-(2) but with logarithmic nonlinear terms f and proved the existence of a solution to the problem.

The Cahn-Hilliard equation incorporating a mass source term is expressed as

$$\frac{\partial \varsigma}{\partial t} + \Delta^2 \varsigma - \Delta f(\varsigma) + \eta(x, \varsigma) = 0, \tag{5}$$

where η represents the mass source term. This equation serves as a versatile model with applications in diverse biological contexts, notably in the growth of cancerous tumor and other biological entities. The choice of η determines specific behaviors, for instance, a linear function $\eta(x,s) = \alpha s,\ \alpha > 0$ yields the Cahn-Hilliard-Oono equation, capturing long-range interactions in phase separation (see [17]; see also [18] for the study of the limit dynamics when α approaches zero). A quadratic function $\eta(x,s) = \alpha s(s-1),\ \alpha > 0$ finds applications in biology [19–22], particularly in wound healing and tumor growth. Another relevant function, commonly employed in tumor growth scenarios, is $\eta(x,s) = \frac{\alpha}{2}(s+1) - \beta(1-s)^2(1+s)^2 + s(x,t)$, where α and β denote growth and death coefficients. Additionally, the function $\eta(x,\varsigma) = 1_{\Omega \setminus D}(x)\varsigma$ is associated with image inpainting applications, as documented in [23–27], and further explored with non-regular non-linear terms in [28].

In this paper, we explore the model (1) incorporating the nonlinear function η

$$\eta(s) = s^2 - p(1-s) \tag{6}$$

along with a logarithmic nonlinear term. The model is subject to Neumann boundary conditions. Under certain assumptions, we establish the existence of solutions for the problem. It is noteworthy that, as detailed in the article, the solutions in certain scenarios may experience a finite-time blow-up.

2 Mathematical problem

Let $\Omega \subset \mathbb{R}^n$, $n = 1,2$ or 3 be a bounded and regular domain with boundary Γ. We consider the following problem

$$\partial_t \varsigma + \Delta^2 \varsigma - \Delta f(\varsigma) + \eta(\varsigma) = 0, \tag{7}$$

$$\partial_\nu \varsigma = \partial_\nu \Delta \varsigma = 0, \text{ on } \Gamma, \tag{8}$$

$$\varsigma(0,x) = \varsigma_0(x), \text{ in } \Omega. \tag{9}$$

Assume that all constants are equal to one, ν is the outer unit normal vector to Γ, $f = F'$ as defined below and $\eta(s) = s^2 - p(1-s)$ where p is a strictly positive real number.
Note that

$$f' \geq -\lambda_1. \tag{10}$$

Let us write $F(s) = \frac{\lambda_1}{2}\left(1-s^2\right) + F_1(s)$ with $f_1 = F_1'$ and introduce for $N \in \mathbb{N}$ the approximated function $F_{1,N} \in C^4(\mathbb{R})$, which is defined by:

$$F_{1,N}^{(4)}(s) = \begin{cases} F_1^{(4)}\left(1-\dfrac{1}{N}\right); \text{ if } s \geq 1-\frac{1}{N}, \\ F_1^{(4)}(s); \qquad \text{if } |s| \leq 1-\frac{1}{N}, \\ F^{(4)}\left(-1+\frac{1}{N}\right); \text{ if } s \leq -1+\frac{1}{N}. \end{cases} \tag{11}$$

$$F_{1,N}^{(k)}(0) = F_1^{(k)}(0), \quad k = 0,1,2,3 \tag{12}$$

and $f_1(s) = f(s) + \lambda_1 s = \frac{\lambda_2}{2}\ln\left(\frac{1+s}{1-s}\right)$.
Hence,

$$F_{1,N}(s) = \begin{cases} \sum \dfrac{1}{k!} F_1^{(k)}\left(1-\dfrac{1}{N}\right)\left(s-1+\dfrac{1}{N}\right)^k; \quad \text{if } s \geq 1-\frac{1}{N}, \\ F_1(s); \qquad \text{if } |s| \leq 1-\frac{1}{N}, \\ \displaystyle\sum_{k=0}^{4} \frac{1}{k!} F_1^{(k)}\left(-1+\frac{1}{N}\right)\left(s+1-\frac{1}{N}\right)^k; \text{ if } s \leq -1+\frac{1}{N}. \end{cases} \tag{13}$$

Setting

$$F_N(s) = \frac{\lambda_1}{2}(1-s^2) + F_{1,N}(s),$$

$$f_{1,N} = F_{1,N}'$$

and

$$f_N = F_N',$$

there holds

$$f_{1,N}' \geq 0, \qquad f_N' \geq -\lambda_1, \tag{14}$$

$$F_N \geq -c_1, \qquad c_1 \geq 0, \tag{15}$$

$$f_N(s).s \geq c_2 F_N(s) + |f_{1,N}(s)| - c_3, \qquad c_2 > 0, c_3 \geq 0, s \in \mathbb{R}, \tag{16}$$

and

$$(f_N(s+m) - f_N(m))\, s \geq c_4\left(s^4 + m^2 s^2\right) - c_5, \qquad c_4 > 0, c_5 \geq 0, \text{ and } s,m \in \mathbb{R}. \tag{17}$$

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The constants $c_i, i = 1, \cdots, 5$ are independent of N for N large enough. More generally,

$$f_N(s+m)\,s \geq c_{6,m}\left(F_N(s+m) + |f_{1,N}(s+m)|\right) - c_{7,m},\ s \in \mathbb{R},\ m \in (-1,1), \tag{18}$$

where the constants $c_{6,m}$ and $c_{7,m}$ are independent of N for sufficiently large N and exhibit continuous and bounded dependence on m.
Finally, we arrive at

Lemma 1.

$$|\eta(s+m) - \eta(m)| \leq c_8\left(s^2 + |ms|\right) + c_9 \tag{19}$$

and

$$|\eta(s+m) - \eta(m)|^2 \leq c_{10}\left(s^4 + s^2(m^2+1)\right) + c_{11}. \tag{20}$$

Proof.

$$\begin{aligned} |\eta(s+m) - \eta(m)| &= |s^2 + 2sm + ps| \\ &\leq s^2 + 2|ms| + p|s| \\ &\leq s^2 + 2|ms| + \frac{p^2}{2} + \frac{s^2}{2} \\ &\leq c_8\left(s^2 + |ms|\right) + c_9 \end{aligned}$$

and

$$\begin{aligned} |\eta(s+m) - \eta(m)|^2 &\leq \left(c\left(s^2 + |ms|\right) + c'\right)^2 \\ &\leq c\left(\left(s^2 + |ms|\right) + 1\right)^2 \\ &\leq c\left(s^4 + 2s^2|ms| + m^2s^2 + 2s^2 + 2|ms| + 1\right) \\ &\leq c\left(s^4 + 2s^2\left(\frac{m^2}{2} + \frac{s^2}{2}\right) + m^2s^2 + 2s^2 + 2\left(\frac{1}{2} + \frac{m^2s^2}{2}\right) + 1\right) \\ &\leq c_{10}\left(s^4 + m^2s^2 + s^2\right) + c_{11}. \end{aligned}$$

We now introduce the approximated problem

$$\partial_t \varsigma_N + \Delta^2 \varsigma_N - \Delta f_N(\varsigma_N) + \eta(\varsigma_N) = 0, \tag{21}$$

$$\partial_\nu \varsigma_N = \partial_\nu \Delta \varsigma_N = 0, \text{ on } \Gamma, \tag{22}$$

$$\varsigma_N(0,x) = \varsigma_0(x), \text{ in } \Omega. \tag{23}$$

From (17) and (19)-(20) it follows that we have the existence and uniqueness (depending on the regularity of ς_0) of the (at least) local in time solution ς_N of (21)-(23).

2.1 Notations

We denote by $(.)$ the usual L^2- scalar product with associated norm $||.||$. We also set $||.||_{-1} = ||(-\Delta)^{-1}||$, where $(-\Delta)^{-1}$ denotes the inverse of the minus Laplace operator associated with Neumann boundary conditions and acting on zero-mean functions.
More generally, we denote by $||.||_X$ the norm on the Banach space X.
We set $\langle . \rangle = \frac{1}{\text{Vol}(\Omega)} \int_\Omega .dx$, being understood that if $\zeta \in H^{-1}(\Omega) = H^1(\Omega)'$ then

$$\langle \zeta \rangle = \frac{1}{\text{Vol}(\Omega)} \langle \zeta, 1 \rangle_{H^{-1}(\Omega), H^1(\Omega)}.$$

We also set, whenever this makes sense

$$\overline{\zeta} = \zeta - \langle \zeta \rangle.$$

We note that:

$$\zeta \mapsto \left(||\overline{\zeta}||_{-1}^2 + \langle \zeta^2 \rangle \right)^{\frac{1}{2}},$$

$$\zeta \mapsto \left(||\overline{\zeta}||^2 + \langle \zeta^2 \rangle \right)^{\frac{1}{2}},$$

$$\zeta \mapsto \left(||\nabla \zeta||^2 + \langle \zeta^2 \rangle \right)^{\frac{1}{2}},$$

$$\zeta \mapsto \left(||\Delta \zeta||^2 + \langle \zeta^2 \rangle \right)^{\frac{1}{2}}$$

are all norms on $H^{-1}(\Omega), L^2(\Omega), H^1(\Omega)$ or $H^2(\Omega)$ that are equivalent to the usual norms on these spaces. Furthermore, $||.||_{-1}$ is a norm on $\{\zeta \in H^{-1}(\Omega), \langle \zeta \rangle = 0\}$ which is equivalent to the usual H^{-1}norm.
Note that the same letter c (and sometimes also c' or c'') in this paper stands for (generally positive) constants that are independent of N and may change from line to line. The same applies to constants such as c_δ, c'_δ and c''_δ which depend on a parameter δ.

3 Applications

In this part, we observe the existence and blow up properties of the results founded by using projected scheme.

3.1 Existence and blow up solutions

In this section, our objective is to establish estimates for u_N that are independent of N. These estimates can be rigorously justified based on the approximated problems. A crucial aspect involves obtaining a uniform (with respect to N) estimate for $f_N(\varsigma_N)$ in $L^2(\Omega \times (0,T))$, where $T > 0$ is independent of N. This is essential for passing to the limit in the nonlinear term and obtaining a solution to the singular initial problem. Notably, achieving this goal necessitates a uniform (with respect to N) strict separation property for $\langle \varsigma_N \rangle$ from the singular points -1 and 1. It is worth mentioning that such a strict separation property is straightforward for the original Cahn-Hilliard equation, given the conservation of the spatial average of the order parameter, provided that the same property holds for the initial datum.
We assume that $\varsigma_0 \in H^1(\Omega) \cap L^\infty(\Omega)$, with $|\varsigma_0(x)| < 1$ almost everywhere in Ω, and

$$|\langle \varsigma_0 \rangle| \leq 1 - 2\delta, \ \delta \in (0, \frac{1}{2}], \tag{24}$$

given. If we first integrate (21) over Ω, we find due to (22),

$$\frac{d\langle \varsigma_N \rangle}{dt} + \langle \eta(\varsigma_N) \rangle = 0. \tag{25}$$

In fact, we have that

$$\frac{d\varsigma_N}{dt} + \Delta^2 \varsigma_N - \Delta f_N(\varsigma_N) + \eta(\varsigma_N) = 0.$$

Noting that

$$\int_\Omega \Delta^2 \varsigma \, dx = -\int_\Omega \nabla \Delta \varsigma . \nabla 1 dx + \int_\Gamma \frac{\partial \Delta \varsigma}{\partial \nu} . \varsigma \, ds = 0.$$

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Furthermore,

$$\int_\Omega \Delta f_N(\varsigma_N)\,dx = -\int_\Omega \nabla f_N(\varsigma_N).\nabla 1 dx + \int_\Gamma \frac{\partial f_N(\varsigma_N)}{\partial \nu} 1\,ds = 0,$$

hence

$$\frac{d}{dt}\int_\Omega \varsigma_N dx + \int_\Omega \eta(\varsigma_N) dx = 0,$$

which yields

$$\frac{d\langle \varsigma_N\rangle}{dt} + \langle \eta(\varsigma_N)\rangle = 0.$$

On the other hand, setting $\varsigma_N = \langle \varsigma_N\rangle + \zeta_N, \langle \zeta_N\rangle = 0$ yields

$$\frac{d\langle \varsigma_N\rangle}{dt} + \langle \varsigma_N^2 - p(1-\varsigma_N)\rangle = 0$$

and

$$\frac{d\langle \varsigma_N\rangle}{dt} + \langle (\langle \varsigma_N\rangle + \zeta_N)^2 - p\,(1-\varsigma_N)\rangle = 0.$$

Therefore,

$$\frac{d\langle \varsigma_N\rangle}{dt} + \langle \langle \varsigma_N\rangle^2 + 2\langle \varsigma_N\rangle\langle \zeta_N\rangle + \zeta_N^2 - p\,(1-\varsigma_N)\rangle = 0,$$

so that

$$\frac{d\langle \varsigma_N\rangle}{dt} + \langle \varsigma_N\rangle^2 + 2\langle \varsigma_N\rangle\langle \zeta_N\rangle + \langle \zeta_N^2\rangle - p\,(1-\langle \varsigma_N\rangle) = 0,$$

it then follows

$$\frac{d\langle \varsigma_N\rangle}{dt} + \langle \varsigma_N\rangle^2 - p\,(1-\langle \varsigma_N\rangle) = -\langle \zeta_N^2\rangle.$$

Hence, we get

$$\frac{d\langle \varsigma_N\rangle}{dt} + \eta\langle \varsigma_N\rangle = -\langle \zeta_N^2\rangle. \tag{26}$$

Furthermore, ζ_N is a solution to:

$$\partial_t \zeta_N + \Delta^2 \zeta_N - \Delta f_N(\varsigma_N) + \overline{\eta(\varsigma_N)} = 0, \tag{27}$$

$$\partial_\nu \zeta_N = \partial_\nu \Delta \zeta_N = 0, \text{ on } \Gamma, \tag{28}$$

$$\zeta_N|_{t=0} = \zeta_0(x), \text{ in } \Omega. \tag{29}$$

Since (21) gives,

$$\begin{aligned}
\frac{\partial\,(\zeta_N + \langle \varsigma_N\rangle)}{\partial t} + \Delta^2 \varsigma_N - \Delta f_N(\varsigma_N) + \eta(\varsigma_N) &= 0\\
\frac{\partial \zeta_N}{\partial t} + \frac{\partial \langle \varsigma_N\rangle}{\partial t} + \Delta^2 \varsigma_N - \Delta f_N(\varsigma_N) + \eta(\varsigma_N) &= 0\\
\frac{\partial \zeta_N}{\partial t} - \langle g(\varsigma_N)\rangle + \Delta^2 \zeta_N - \Delta f_N(\varsigma_N) + \eta(\varsigma_N) &= 0\\
\frac{\partial \zeta_N}{\partial t} + \Delta^2 \zeta_N - \Delta f_N(\varsigma_N) + \overline{\eta(\varsigma_N)} &= 0,
\end{aligned}$$

$$\frac{\partial \varsigma_N}{\partial \nu} = \frac{\partial \zeta_N}{\partial \nu} \text{ and } \frac{\partial \Delta \varsigma_n}{\partial \nu} = \frac{\partial \Delta \zeta_n}{\partial \nu}.$$

These equations can be written equivalently, if we multiply by $-(\Delta)^{-1}$, as:

$$(-\Delta)^{-1}\partial_t\zeta_N - \Delta\zeta_N + \overline{f_N(\varsigma_N)} + (-\Delta)^{-1}\overline{\eta(\varsigma_N)} = 0, \text{ in } \Omega\times(0,T), \tag{30}$$

$$\partial_\nu\zeta_N = 0, \text{ on } \Gamma, \tag{31}$$

$$\zeta_N|_{t=0} = \zeta_0, \text{ in } \Omega. \tag{32}$$

Multiplying (30) by ζ_N and integrating over Ω and by parts, we have:

$$\frac{1}{2}\frac{d}{dt}||\zeta_N||^2_{-1} + \|\nabla\zeta_N\|^2 + [\overline{f_N(\varsigma_N)},\zeta_N] + [\eta(\varsigma_N),(-\Delta^{-1})^{-1}\zeta_N] = 0. \tag{33}$$

Since

$$\begin{aligned}\int_\Omega (-\Delta)^{-1}\frac{\partial\zeta_N}{\partial t}\zeta_N\,dx &= \frac{1}{2}\frac{d}{dt}\int_\Omega(-\Delta)^{-1}\zeta_N^2\,dx\\ &= \frac{1}{2}\frac{d}{dt}||\zeta_N||^2_{-1}\end{aligned}$$

and

$$\begin{aligned}\int_\Omega -\Delta\zeta_N\,\zeta_N\,dx &= \int_\Omega\nabla\zeta_N\cdot\nabla\zeta_N\,dx - \int_\Gamma\frac{\partial\zeta_N}{\partial\nu}\zeta_N\,ds\\ &= ||\nabla\zeta_N||^2.\end{aligned}$$

Note that

$$[\overline{f_N(\varsigma_N)},\zeta_N] = [f_N(\varsigma_N) - f_N(\langle\varsigma_N\rangle),\zeta_N]$$

and taking $s=\zeta_N$, $m=\langle\varsigma_N\rangle$ in (17), we get

$$[\overline{f_N(\varsigma_N)},\zeta_N] \geq c_4\int_\Omega\left(\zeta_N^4 + \langle\varsigma_N\rangle^2\zeta_N^2\right)dx - c. \tag{34}$$

Indeed, from (17), we have

$$\begin{aligned}(f_N(s+m)-f_N(m))\,s &\geq c_4\left(s^4+m^2s^2\right) - c_5\\ \left(f_N(\varsigma_N) - f_N\langle\varsigma_N\rangle\right)\zeta_N &\geq c_4\left(\zeta_N^4 + \langle\varsigma_N\rangle^2\zeta_N^2\right) - c_5,\end{aligned}$$

so

$$\begin{aligned}[f_N(\varsigma_N) - f_N\langle\varsigma_N\rangle,\zeta_N] &= \int_\Omega\left(f_N(\varsigma_N) - f_N\langle\varsigma_N\rangle\right)\zeta_N\,dx\\ &\geq c_4\int_\Omega\left(\zeta_N^4 + \langle\varsigma_N\rangle^2\zeta_N^2\right)dx - c_5.\end{aligned}$$

Furthermore,

$$\begin{aligned}|[\overline{\eta(\varsigma_N)},(-\Delta^{-1})\zeta_N]| &= |[\eta(\varsigma_N) - \eta(\langle\varsigma_N\rangle),(-\Delta^{-1})\zeta_N]]|\\ &\leq c||\zeta_N||\,||\eta(\varsigma_N) - g(\langle\varsigma_N\rangle)||,\end{aligned}$$

by Cauchy Schwartz and using $||\zeta_N||_{-1}\leq c||\zeta_N||$ which results after reapplying Lemma (1) with $s=\zeta_N$ and $m=\langle\varsigma_N\rangle$ and Young's inequality:

$$\begin{aligned}|[\overline{\eta(\varsigma_N)},(-\Delta^{-1})\zeta_N]| &\leq \frac{c_4}{4}\int_\Omega\left(\zeta_N^4 + \langle\varsigma_N\rangle^2\zeta_N^2\right)dx + c||\zeta_N||^2\\ &\leq \frac{c_4}{4}\int_\Omega\left(\zeta_N^4 + \langle\varsigma_N\rangle^2\zeta_N^2\right)dx + c.\end{aligned} \tag{35}$$

Indeed, we have that

$$\begin{aligned}|[\overline{\eta(\varsigma_N)},(-\Delta^{-1})\zeta_N]| &\leq c||\zeta_N||\,||\eta(\varsigma_N)-\eta(\langle\varsigma_N\rangle)||\\ &\leq c\left(\frac{||\zeta_N||^2}{2}+\frac{||\eta(\varsigma_N)-\eta(\langle\varsigma_N\rangle)||^2}{2}\right)\\ &\leq \frac{c}{2}\left(||\zeta_N||^2+c_9\int_\Omega\left(\zeta_N^4+\langle\varsigma_N\rangle^2\zeta_N^2+\zeta_N^2\right)dx+c_{10}\right)\\ &\leq \frac{c_4}{2}\int_\Omega\left(\zeta_N^4+\langle\varsigma_N\rangle^2\zeta_N^2\right)dx+c||\zeta_N||^2+c'\\ &\leq \frac{c_4}{2}\int_\Omega\left(\zeta_N^4+\langle\varsigma_N\rangle^2\zeta_N^2\right)dx+c.\end{aligned}$$

It follows from (33)-(35), that

$$\frac{d}{dt}||\zeta_N||_{-1}^2+c\left(||\zeta_N||_{H^1(\Omega)}^2+\int_\Omega\left(\zeta_N^4+\langle\varsigma_N\rangle^2\zeta_N^2\right)dx\right)\leq c',\ c>0. \tag{36}$$

Since

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\zeta_N||_{-1}^2+||\nabla\zeta_N||^2 &= -[\overline{f_N(\varsigma_N)},\zeta_N]-[\overline{\eta(\varsigma_N)},-\Delta^{-1}\zeta_N]\\ &\leq -c_4\int_\Omega\left(\zeta_N^4+\langle\varsigma_N\rangle^2\zeta_N^2\right)dx+c+\frac{c_4}{2}\int_\Omega\left(\zeta_N^4+\langle\varsigma_N\rangle^2\zeta_N^2\right)dx+c.\end{aligned}$$

Next, multiplying (30) by $-\Delta\zeta_N$ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}||\zeta_N||^2+||\Delta\zeta_N||^2+[\overline{f_N(\varsigma_N)},-\Delta\zeta_N]+[\overline{\eta(\varsigma_N)},\zeta_N]=0. \tag{37}$$

In fact, we have

$$\begin{aligned}\int_\Omega -\Delta^{-1}\frac{\partial\zeta_N}{\partial t}(-\Delta\zeta_N)\,dx &= \int_\Omega\frac{\partial\zeta_N}{\partial t}\zeta_N\,dx\\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega\zeta_N^2\,dx\\ &= \frac{1}{2}\frac{d}{dt}||\zeta_N||^2.\end{aligned}$$

Furthermore, owing to (14)

$$\begin{aligned}[\overline{f_N(\varsigma_N)},-\Delta\zeta_N] &= [f_N(\varsigma_N),-\Delta\zeta_N]\\ &= -\int_\Omega f_N(\varsigma_N)\Delta\zeta_N\,dx\\ &= \int_\Omega\nabla f_N(\varsigma_N).\nabla\zeta_N\,dx\\ &= \int_\Omega f'(\varsigma_N)\nabla\varsigma_N\nabla\zeta_N\,dx\\ &= [f'(\varsigma_N)\nabla\varsigma_N,\nabla\zeta_N]\\ &\geq -\lambda_1||\nabla\zeta_N||^2\end{aligned} \tag{38}$$

and owing once more to Lemma (1), we have

$$\begin{aligned}|[\overline{\eta(\varsigma_N)},\zeta_N]| &= |[\eta(\varsigma_N)-\eta(\langle\varsigma_N\rangle),\zeta_N]|\\ &\leq \tfrac{1}{2}||\Delta\zeta_N||^2+c\int_\Omega\left(\zeta_N^4+\langle\varsigma_N\rangle^2\zeta_N^2\right)dx+c'.\end{aligned} \tag{39}$$

Indeed,

$$\begin{aligned}|[\eta(\varsigma_N)-\eta(\langle\varsigma_N\rangle),\zeta_N]| &\le ||\eta(\varsigma_N)-\eta(\langle\varsigma_N\rangle)||\,||\zeta_N|| \\ &\le \frac{||\zeta_N||^2}{2}+\frac{||\eta(\varsigma_N)-\eta(\langle\varsigma_N\rangle)||^2}{2} \\ &\le \frac{1}{2}||\Delta\zeta_N||^2+c\int_\Omega\left(\zeta_N^4+\langle\varsigma_N\rangle^2\zeta_N^2\right)dx+c'.\end{aligned}$$

From (37)-(39) it follows that

$$\frac{d}{dt}||\zeta_N||^2+||\Delta\zeta_N||^2\le c\left(||\zeta_N||^2_{H^1(\Omega)}+\int_\Omega\left(\zeta_N^4+\langle\varsigma_N\rangle^2\zeta_N^2\right)dx\right)+c'. \tag{40}$$

By (37), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\zeta_N||^2+||\Delta\zeta_N||^2 &= [\overline{f_N(\varsigma_N)},-\Delta\zeta_N]-[\overline{\eta(\varsigma_N)},\zeta_N] \\ &\le \lambda_1||\nabla\zeta_N||^2+\frac{1}{2}||\Delta\zeta_N||^2+c\int_\Omega\left(\zeta_N^4+\langle\varsigma_N\rangle^2\zeta_N^2\right)dx+c',\end{aligned}$$

so,

$$\frac{d}{dt}||\zeta_N||^2+||\Delta\zeta_N||^2\le c\left(||\nabla\zeta_N||^2+\int_\Omega\left(\zeta_N^4+\langle\varsigma_N\rangle^2\zeta_N^2\right)dx\right)+c'.$$

Finally, summing (36) and γ_1 times (40), where $\gamma_1>0$ is small enough and independent of N, we find

$$\frac{d}{dt}\left(||\zeta_N||^2_{-1}+\gamma_1||\zeta_N||^2\right)+c\left(||\zeta_N||^2_{H^2(\Omega)}+\int_\Omega\left(\zeta_N^4+\langle\varsigma_N\rangle^2\zeta_N^2\right)dx\right)\le c',\quad c>0. \tag{41}$$

Indeed, we have (36)+γ_1(40) gives

$$\frac{d}{dt}\left(||\zeta_N||^2+\gamma_1||\zeta_N||^2\right)+c||\zeta_N||_{H^1(\Omega)}+\gamma_1||\Delta\zeta_N||^2+c\int_\Omega\left(\zeta_N^4+\langle\varsigma_N\rangle^2\zeta_N^2\right)dx\le c',$$

then

$$\frac{d}{dt}\left(||\zeta_N||^2+\gamma_1||\zeta_N||^2\right)+c\left(||\zeta_N||_{H^2(\Omega)}+\int_\Omega\left(\zeta_N^4+\langle\varsigma_N\rangle^2\zeta_N^2\right)dx\right)\le c'.$$

We now come back to (25)-(26). Noting that $g(s)\ge ps-p$, we have

$$\begin{aligned}\frac{d\langle\varsigma_N\rangle}{dt}+\eta(\langle\varsigma_N\rangle) &= -\langle\zeta_N^2\rangle\le 0 \\ \frac{d\langle\varsigma_N\rangle}{dt} &\le -\eta(\langle\varsigma_N\rangle)\le -p\langle\varsigma_N\rangle+p \\ \frac{d\langle\varsigma_N\rangle}{dt}+p\langle\varsigma_N\rangle &\le p.\end{aligned}$$

Consider the ODE:

$$\frac{d\langle\varsigma_N\rangle}{dt}+p\langle\varsigma_N\rangle=p,$$

then

$$\frac{d\langle\varsigma_N\rangle}{dt}=-p\left(\langle\varsigma_N\rangle-1\right),$$

hence

$$\frac{d\langle\varsigma_N\rangle}{(\langle\varsigma_N\rangle-1)}=-p\,dt,$$

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we infer

$$\ln|\langle \varsigma_N\rangle - 1| = -pt + k,$$

so

$$\langle \varsigma_N\rangle = ce^{-pt} + 1,$$

but at $t = 0$, $\langle \varsigma_N\rangle = \langle \varsigma_0\rangle$, hence $c = \langle \varsigma_0\rangle - 1$.

Finally,

$$\langle \varsigma_N(t)\rangle \leq (\langle \varsigma_0\rangle - 1)\, e^{-pt} + 1, \tag{42}$$

as long as it exists. In particular,

$$\langle \varsigma_N(t)\rangle < 1. \tag{43}$$

Note that it follows from (41) that:

$$\frac{d}{dt}\left(||\zeta_N||^2_{-1} + \gamma_1||\zeta_N||^2\right) + c\left(||\zeta_N||^2_{-1} + \gamma_1||\zeta_N||^2\right) \leq c',\ c > 0.$$

Indeed, we have that

$$\begin{aligned} c\left(||\zeta_N||^2_{H^2(\Omega)} + \int_\Omega \left(\zeta_N^4 + \langle \varsigma_N\rangle^2 \zeta_N^2\right) dx\right) &\geq c||\zeta_N||^2_{H(\Omega)} - \frac{1}{2}||\Delta\zeta_N||^2 - c' \\ &\geq c||\Delta\zeta_N||^2 - c' \\ &\geq c||\zeta_N||^2 - c' \\ &\geq \frac{c}{2}||\zeta_N||^2 + \frac{c}{2}||\zeta_N||^2 - c' \\ &\geq c\left(||\zeta_N||^2_{-1} + \gamma_1||\zeta_N||^2\right) - c', \end{aligned}$$

but by (41)

$$\begin{aligned} &\frac{d}{dt}\left(||\zeta_N||^2_{-1} + \gamma_1||\zeta_N||^2\right) + c\left(||\zeta_N||^2_{-1} + \gamma_1||\zeta_N||^2\right) \\ &\leq \frac{d}{dt}\left(||\zeta_N||^2_{-1} + \gamma_1||\zeta_N||^2\right) + c\left(||\zeta_N||^2_{H^2(\Omega)} + \int_\Omega \left(\zeta_N^4 + \langle \varsigma_N\rangle^2 \zeta_N^2\right) dx\right) + c' \\ &\leq 2c'. \end{aligned}$$

Which yields

$$||\zeta_N(t)||^2 \leq ce^{-c't}||\zeta_0||^2 + c',\ c'' > 0. \tag{44}$$

In fact,

$$\frac{d}{dt}\left(||\zeta_N||^2_{-1} + \gamma_1||\zeta_N||^2\right) + c\left(||\zeta_N||^2_{-1} + \gamma_1||\zeta_N||^2\right) \leq c',\ c > 0,$$

then by Gronwall lemma

$$\begin{aligned} ||\zeta_N||^2_{-1} + \gamma_1||\zeta_N||^2 &\leq \left(||\zeta_0||^2_{-1} + \gamma_1||\zeta_0||^2\right) + c'' \\ \gamma_1||\zeta_N||^2 &\leq e^{-ct}\left(||\zeta_0||^2_{-1} + \gamma_1||\zeta_0||^2\right) + c'' \\ ||\zeta_N||^2 &\leq \frac{1}{\gamma_1}e^{-ct}\left(||\zeta_0||^2_{-1} + \gamma_1||\zeta_0||^2\right) + \frac{c''}{\gamma_1}, \end{aligned}$$

but $||\zeta_0||_{-1}^2 \leq q||\zeta_0||^2$, then

$$||\zeta_N||^2 \leq \tfrac{1}{\gamma_1} e^{-ct}\left(q||\zeta_0||^2 + \gamma_1||\zeta_0||^2\right) + \frac{c''}{\gamma_1}$$
$$\leq ce^{-c't}||\zeta_0||^2 + c''.$$

As long as it exists, in particular

$$\langle \zeta_N^2 \rangle = k||\zeta_N^2(t)|| \leq c||\zeta_0||^2 + c', \text{ since } e^{-c't} \leq 1,$$

and $\zeta_0 = \varsigma_0 - \langle \varsigma_0 \rangle$ and $|\langle \varsigma_0 \rangle| \leq 1 - 2\delta$, so

$$\langle \zeta_N^2(t) \rangle \leq c(\varsigma_0, \delta), \tag{45}$$

as long as it exists.
Let $y_\pm$ be the solution of the Ricatti ODE's

$$y_+' + \eta(y_+) = 0,\ y_+(0) = \langle \varsigma_0 \rangle \tag{46}$$

$$y_-' + \eta(y_-) = -c(\varsigma_0, \delta),\ y_-(0) = \langle \varsigma_0 \rangle, \tag{47}$$

where $c(\varsigma_0, \delta)$ is the constant in (45). Then it follows from the comparison principle that, as long as this makes sense

$$y_-(t) \leq \langle \varsigma_N(t) \rangle \leq y_+(t). \tag{48}$$

Indeed, we have that

$$\frac{d\langle \varsigma_N \rangle}{dt} + g(\langle \varsigma_N \rangle) = -\langle \zeta_N^2 \rangle \leq 0,$$

so, $\langle \varsigma_N(t) \rangle \leq y_+(t)$, also

$$\langle \zeta_N^2 \rangle \leq c(\varsigma_0, \delta),$$

hence

$$\frac{d\langle \varsigma_N \rangle}{dt} + \eta(\langle \varsigma_N \rangle) \geq -c(\varsigma_0, \delta),$$

so $\langle \varsigma_N(t) \rangle \geq y_-(t)$.
In particular, it follows that (at least) a local in times solution exists on some $[0, T]$, where $T > 0$ is independent of N. Note also that

$$y_+(t) = \frac{y_2 - cy_1 e^{\frac{t}{c_0}}}{1 - ce^{\frac{t}{c_0}}},$$

with

$$y_1 = \frac{p + \sqrt{p^2 + 4p}}{2},$$
$$y_2 = \frac{p - \sqrt{p^2 + 4p}}{2}$$

and

$$c = \frac{y_2 - \langle \varsigma_0 \rangle}{y_1 - \langle \varsigma_0 \rangle}.$$

$y_+(t)$ is the solution of the ODE: $y' + \eta(y) = 0$, so $y' + y^2 - p(1-y) = 0$, so

$$\frac{dy}{dt} = p + py - y^2$$
$$\int \frac{dy}{p + py - y^2} = \int dt.$$

We have

$$p+py-y^2=0:$$

$$\Delta=p^2-4(p)(-1)=p^2+4p,$$

hence the roots of the quadratic equation are

$$y_1=\frac{-p-\sqrt{p^2+4p}}{-2}=y_1=\frac{p+\sqrt{p^2+4p}}{2},$$

$$y_2=\frac{-p+\sqrt{p^2+4p}}{-2}=y_2=\frac{p-\sqrt{p^2+4p}}{2},$$

so

$$\int\frac{-dy}{(y-y_1)(y-y_2)}=t+c,$$

$$\frac{1}{(y-y_1)(y-y_2)}=\frac{A}{y-y_1}+\frac{B}{y-y_2}.$$

Where

$$A=\lim_{y\to y_1}\frac{1}{y-y_2}=\frac{1}{y_1-y_2}=\frac{1}{\sqrt{p^2+4p}}=c_0,$$

$$B=\lim_{y\to y_2}\frac{1}{y-y_1}=\frac{1}{y_2-y_1}=\frac{-1}{\sqrt{p^2+4p}}=-c_0.$$

We infer

$$\int\left(\frac{-c_0}{y-y_1}+\frac{c_0}{y-y_2}\right)dt=t+c,$$

$$c_0\left(\ln|y-y_2|-\ln|y-y_1|\right)=t+\ln c,$$

$$c_0\ln|\frac{y-y_2}{y-y_1}|=t+\ln c,$$

$$\ln|\frac{y-y_2}{y-y_1}|^{c_0}=t+\ln c,$$

$$\left(\frac{y-y_2}{y-y_1}\right)^{c_0}=ce^t,$$

$$\frac{y-y_2}{y-y_1}=ce^{\frac{t}{c_0}},$$

$$y-y_2=ce^{\frac{t}{c_0}}(y-y_1),$$

$$y\left(1-ce^{\frac{t}{c_0}}\right)=y_2-cy_1e^{\frac{t}{c_0}},$$

$$y=\frac{y_2-cy_1e^{\frac{t}{c_0}}}{1-ce^{\frac{t}{c_0}}}.$$

But $y|_{t=0}=\langle\varsigma_0\rangle$, so

$$\langle\varsigma_0\rangle=\frac{y_2-cy_1}{1-c},$$

$$y_2-cy_1=\langle\varsigma_0\rangle(1-c),$$

$$y_2-\langle\varsigma_0\rangle=c\left(y_1-\langle\varsigma_0\rangle\right),$$

$$c = \frac{y_2 - \langle \varsigma_0 \rangle}{y_1 - \langle \varsigma_0 \rangle}.$$

We assume from now on that $t \in [0, T]$, where T is as above, we again multiply (30) by ζ_N and we have

$$\frac{1}{2}\frac{d}{dt}||\zeta_N||^2_{-1} + ||\nabla \zeta_N||^2 + [f_N(\varsigma_N), \zeta_N] \leq c \int_\Omega \left(\zeta_N^4 + \langle \varsigma_N \rangle^2 \zeta_N^2\right) dx + c',$$

since by (33)

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||\zeta_N||^2_{-1} + ||\nabla \zeta_N||^2 + [\overline{f_N(\varsigma_N)}, \zeta_N] &= -[\overline{\eta(\varsigma_N)}, -\Delta^{-1}\zeta_N] \\ &\leq c \int_\Omega \left(\zeta_N^4 + \langle \varsigma_N \rangle^2 \zeta_N^2\right) dx + c'. \end{aligned} \tag{49}$$

Which yields, employing (18) with $s = \zeta_N$ and $m = \langle \varsigma_N \rangle$,

$$\frac{d}{dt}||\zeta_N||^2_{-1} + c_\delta \left(||f_N(\varsigma_N)||_{L^1(\Omega)} + \int_\Omega F_N(\varsigma_N) dx\right) \leq c' \int_\Omega \left(\zeta_N^4 + \langle \varsigma_N \rangle^2 \zeta_N^2\right) dx + c''_\delta,\ c_{\delta>0}. \tag{50}$$

In fact, we know from (18) that

$$\begin{aligned} f_N(\varsigma_N)\zeta_N &\geq c_\delta \left(F_N(\varsigma_N) + |f_{1,N}(\varsigma_N)|\right) - c'_\delta \\ \int_\Omega f_N(\varsigma_N)\zeta_N dx &\geq c_\delta \left(\int_\Omega F_N(\varsigma_N) dx + \int_\Omega |f_{1,N}(\varsigma_N)| dx\right) - c'_\delta, \\ \frac{1}{2}\frac{d}{dt}||\zeta_N||^2_{-1} + c_\delta &\left(||f_{1,1}(\varsigma_N)||_{L^1(\Omega)} + \int_\Omega F_N(\varsigma_N) dx\right) - c'_\delta \\ &\leq c \int_\Omega \left(\zeta_N^4 + \langle \varsigma_N \rangle^2 \zeta_N^2\right) dx + c', \end{aligned}$$

but $f_1(\varsigma_N) = f_N(\varsigma_N) + \lambda \varsigma_N$. We then infer that

$$\frac{d}{dt}||\zeta_N||^2_{-1} + c_\delta \left(||f_N(\varsigma_N)||_{L^1(\Omega)} + \int_\Omega F_N(\varsigma_N) dx\right) \leq c' \int_\Omega \left(\zeta_N^4 + \langle \varsigma_N \rangle^2 \zeta_N^2\right) dx + c''_\delta,\ c_{\delta>0}.$$

Summing (41) and $\gamma_2 \times$ (50), where $\gamma_2 > 0$ is small enough and independent of N and δ, we obtain a differential inequality of the form

$$\begin{aligned} \frac{dE_{1,N}}{dt} + c_\delta \left(E_{1,N} + ||\zeta_N||^2_{H^2(\Omega)}\right) + \int_\Omega \left(\zeta_N^4 + \langle \varsigma_N \rangle^2 \zeta_N^2\right) dx + ||f_N(\varsigma_N)||_{L^1(\Omega)} \\ + \int_\Omega F_N(\varsigma_N) dx \leq c'_\delta,\ c_\delta > 0, \end{aligned} \tag{51}$$

where $E_{1,N} = (1 + \gamma_2)||\zeta_N||^2_{-1} + \gamma_1 ||\zeta_N||^2$.
In fact, (41)+γ_2(50) give

$$\begin{aligned} \frac{d}{dt}\left(||\zeta_N||^2_{-1} + \gamma_1 ||\zeta_N||^2\right) + c &\left(||\zeta_N||^2_{H^2(\Omega)} + \int_\Omega \left(\zeta_N^4 + \langle \varsigma_N \rangle^2 \zeta_N^2\right) dx\right) \\ + \gamma_2 \frac{d}{dt}||\zeta_N||^2_{-1} + \gamma_2 c_\delta &\left(||f_N(\varsigma_N)||_{L^1(\Omega)} + \int_\Omega F_N(\varsigma_N) dx\right) \\ &\leq c' + \gamma_2 c' \int_\Omega \left(\zeta_N^4 + \langle \varsigma_N \rangle^2 \zeta_N^2\right) dx + \gamma_2 c''_\delta,\ c_\delta > 0. \end{aligned}$$

Hence,

$$\frac{dE_{1,N}}{dt} + c_\delta \left(||\zeta_N||^2_{H^2(\Omega)} + \int_\Omega \left(\zeta_N^4 + \langle \varsigma_N \rangle^2 \zeta_N^2\right) dx + ||f_N(\varsigma_N)||_{L^1(\Omega)} + \int_\Omega F_N(\varsigma_N) dx\right)$$

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$$\leq c'_\delta, \qquad c_\delta > 0,$$

but

$$\begin{aligned} E_{1,N} &= (1+\gamma_2)||\zeta_N||^2_{-1} + \gamma_1||\zeta_N||^2 \\ &\leq (1+\gamma_2)k||\zeta_N||^2_{H^2(\Omega)} + \gamma_1 k'||\zeta_N||^2_{H^2(\Omega)}, \end{aligned}$$

so,

$$E_{1,N} \leq k||\zeta_N||^2_{H^2(\Omega)},$$

then

$$\frac{dE_{1,N}}{dt} + c_\delta \left(E_{1,N} + ||\zeta_N||^2_{H^2(\Omega)} + \int_\Omega \left(\zeta_N^4 + \langle \varsigma_N \rangle^2 \zeta_N^2\right) dx \right) + ||f_N(\varsigma_N)||_{L^1(\Omega)}$$

$$+ \int_\Omega F_N(\varsigma_N) dx \leq c'_\delta,\ c_\delta > 0.$$

If we note that $|\langle \varsigma_N \rangle \leq 1$, we see that

$$\frac{d\langle \varsigma_N \rangle^2}{dt} = 2\langle \varsigma_N \rangle \left(-\langle \zeta_N^2 \rangle - \langle \varsigma_N \rangle^2 + p(1 - \langle \varsigma_N \rangle) \right) \leq c||\zeta_N||^2,$$

which results in the following

$$\frac{d\langle \varsigma_N \rangle^2}{dt} + \langle \varsigma_N \rangle^2 \leq c||\zeta_N||^2 + c', \tag{52}$$

since

$$\begin{aligned} \frac{d\langle \varsigma_N \rangle^2}{dt} &= 2\langle \varsigma_N \rangle \frac{d}{dt}\langle \varsigma_N \rangle \\ &= 2\langle \varsigma_N \rangle \left(-g(\langle \varsigma_N \rangle) - \langle \zeta_N^2 \rangle \right) \\ &= 2\langle \varsigma_N \rangle \left(-\langle \varsigma_N \rangle^2 + p(1 - \langle \varsigma_N \rangle) - \langle \zeta_N^2 \rangle \right) \\ &= -2\langle \varsigma_N \rangle^3 + 2p\langle \varsigma_N \rangle - 2p\langle \varsigma_N \rangle^2 - 2\langle \varsigma_N \rangle \langle \zeta_N \rangle^2 \\ &\leq c||\zeta_N||^2 + c', \end{aligned}$$

so

$$\begin{aligned} \frac{d\langle \varsigma_N \rangle^2}{dt} + \langle \varsigma_N \rangle^2 &\leq c||\zeta_N||^2 + c' + 1 \\ &\leq c||\zeta_N||^2 + c''. \end{aligned}$$

Summing (51) and $\gamma_3 \times$ (52), where $\gamma_3 > 0$ (independent of N) is small enough, we get

$$\frac{dE_{2,N}}{dt} + c_\delta \left(E_{2,N} + ||\varsigma_N||^2_{H^2(\Omega)} + \int_\Omega \left(\zeta_N^4 + \langle \varsigma_N \rangle^2 \zeta_N^2\right) dx \right) + ||f_N(\varsigma_N)||_{L^1(\Omega)} \tag{53}$$

$$+ \int_\Omega F_N(\varsigma_N) dx \leq c'_\delta,\ c_\delta > 0,$$

where

$$E_{2,N} = E_{1,N} + \gamma_3 \langle \varsigma_N \rangle^2,$$

since (51) $+\gamma_3$ (52) give

$$\frac{d}{dt}E_{1,N} + c_\delta \left(E_{1,N} + ||\zeta_N||^2_{H^2(\Omega)} + \int_\Omega \left(\zeta_N^4 + \langle \varsigma_N \rangle^2 \zeta_N^2\right) dx + ||f_N(\varsigma_N)||_{L^1(\Omega)} + \int_\Omega F_N(\varsigma_N) dx \right)$$

$$+\gamma_3 \frac{d}{dt}\langle \varsigma_N\rangle^2 + \gamma^3 \langle \varsigma_N\rangle^2$$
$$\leq c'_\delta + c\gamma_3 ||\zeta_N||^2 + \gamma_3 c',$$

but

$$||\zeta_N||^2 = \int_\Omega \zeta_N^2 dx \leq c \int_\Omega \zeta_N^4 dx,$$

so we get (53).
$E_{2,N}$ satisfies

$$c||\varsigma_N||^2 \leq E_{2,N} \leq c'||\varsigma_N||^2,\ c, c' > 0.$$

Indeed, we have

$$E_{2,N} = E_{1,N} + \gamma_3 \langle \varsigma_N\rangle^2$$
$$= (1+\gamma_2)||\zeta_N||^2_{-1} + \gamma_1 ||\zeta_N||^2 + \gamma_3 \langle \varsigma_N\rangle^2,$$

but $||\zeta_N||_{-1} \sim ||\zeta_N||_{L^2(\Omega)}$, since $\langle \zeta_N\rangle = 0$, so $\exists c_1, c_2 > 0$, such that

$$c_1 ||\zeta_N||^2_{L^2(\Omega)} \leq ||\zeta_N||^2_{-1} \leq c_2 ||\zeta_N||^2_{L^2(\Omega)}.$$

$$\gamma_3 \langle \varsigma_N\rangle^2 + (1+\gamma_2) c_1 ||\zeta_N||^2_{L^2(\Omega)} + \gamma_1 ||\zeta_N||^2 \leq E_{2,N}$$
$$\leq (1+\gamma_2) c_2 ||\zeta_N||^2_{L^2(\Omega)} + \gamma_1 ||\zeta_N||^2 + \gamma_3 \langle \varsigma_N\rangle^2$$
$$c_3 \left(||\zeta_N||^2_{L^2(\Omega)} + \langle \varsigma_N\rangle^2 \right) \leq E_{2,N} \leq c_4 \left(||\zeta_N||^2_{L^2(\Omega)} + \langle \varsigma_N\rangle^2 \right),$$

then

$$c||\varsigma_N||^2 \leq E_{2,N} \leq c'||\varsigma_N||^2.$$

In the next step, we multiply (30) by $\frac{d\varsigma_N}{dt}$, then integrate over Ω and have

$$||\frac{d\zeta_N}{dt}||^2_{-1} + \frac{1}{2}\frac{d}{dt}||\nabla \varsigma_N||^2 + [f_N(\varsigma_N), \frac{d\varsigma_N}{dt}] + [\varsigma_N^2 - p(1-\varsigma_N), (-\Delta)^{-1}\frac{d\varsigma_N}{dt}] = 0. \tag{54}$$

Indeed, (30) results in

$$(-\Delta)^{-1}\frac{d\varsigma_N}{dt} - \Delta\zeta_N + \overline{f_N(\varsigma_N)} + (-\Delta)^{-1}\overline{\eta(\varsigma_N)} = 0,$$

but

$$\int_\Omega -\Delta\zeta_N \frac{d\zeta_N}{dt} dx =$$
$$= \frac{1}{2}\frac{d}{dt}\left(\int_\Omega \nabla\zeta_N \cdot \nabla\zeta_N\, dx - \int_\Gamma \zeta_N \frac{\partial \zeta_N}{\partial \nu} ds \right)$$
$$= \frac{1}{2}\frac{d}{dt}||\nabla\zeta_N||^2.$$

Furthermore

$$[f_N(\varsigma_N), \frac{\partial\zeta_N}{\partial t}] = [f_N(\varsigma_N), \frac{\partial\varsigma_N}{\partial t}] - [f_N(\varsigma_N), \frac{\partial\langle\varsigma_N\rangle}{\partial t}]$$
$$= \frac{d}{dt}\int_\Omega F_N(\varsigma_N) dx + [f_N(\varsigma_N), \langle\zeta_N\rangle^2 + \langle\varsigma_N\rangle^2 - p(1-\langle\varsigma_N\rangle)]$$
$$= \frac{d}{dt}\int_\Omega F_N(\varsigma_N) dx + \mathrm{Vol}(\Omega)\langle f_N(\varsigma_N)\rangle[\langle\zeta_N\rangle^2 + \langle\varsigma_N\rangle^2 - p(1-\langle\varsigma_N\rangle)]$$

$$\leq ||f_N(\varsigma_N)||_{L^1(\Omega)} \left(||\zeta_N||^2_{L^2(\Omega)} + 1 \right). \tag{55}$$

We also note that

$$\begin{aligned} |[\varsigma_N^2 - p(1-\varsigma_N), (-\Delta)^{-1}\frac{\partial \zeta_N}{\partial t}]| &= [\varsigma_N^2 - \langle \varsigma_N^2 \rangle + p(\varsigma_N - \langle \varsigma_N \rangle), (-\Delta)^{-1}\tfrac{\partial \zeta_N}{dt}] \\ &\leq \frac{1}{2}||\frac{\partial \zeta_N}{\partial t}||^2_{-1} + c\int_\Omega \left(\zeta_N^4 + \langle \varsigma_N \rangle^2 \zeta_N^2\right) dx + c'. \end{aligned} \tag{56}$$

It follows from (54)-(56) that

$$\frac{d}{dt}\left(||\nabla \zeta_N||^2 + 2\int_\Omega F_N(\varsigma_N) dx \right) + ||\frac{\partial \zeta_N}{\partial t}||^2_{-1} \tag{57}$$

$$\leq c\left(||f_N(\varsigma_N)||_{L^1(\Omega)} \left(||\zeta_N||^2 + 1 \right) + \int_\Omega \left(\zeta_N^4 + \langle \varsigma_N \rangle^2 \zeta_N^2\right) dx \right).$$

By (54)

$$||\frac{\partial \zeta_N}{\partial t}||^2_{-1} + \frac{1}{2}\frac{d}{dt}||\nabla \zeta_N||^2 = -[f_N(\varsigma_N), \frac{\partial \zeta_N}{\partial t}] - [\varsigma_N^2 - p(1-\varsigma_N), (-\Delta)^{-1}\frac{\partial \varsigma_N}{\partial t}]$$

$$\leq -\frac{d}{dt}\int_\Omega F_N(\varsigma_N) dx + c||f_N(\varsigma_N)||_{L^1(\Omega)} \left(||\zeta_N||^2 + 1 \right) + \frac{1}{2}||\frac{\partial \zeta_N}{\partial t}||^2_{-1} + c\int_\Omega \left(\zeta_N^4 + \langle \varsigma_N \rangle^2 \zeta_N^2\right) dx,$$

then

$$\frac{1}{2}||\frac{\partial \zeta_N}{\partial t}||^2_{-1} + \frac{1}{2}\frac{d}{dt}||\nabla \zeta_N||^2 + \frac{d}{dt}\int_\Omega F_N(\varsigma_N) dx$$

$$\leq c\left(||f_N(\varsigma_N)||_{L^1(\Omega)} \left(||\zeta_N||^2 + 1 \right) + \int_\Omega \left(\zeta_N^4 + \langle \varsigma_N \rangle^2 \zeta_N^2\right) dx \right),$$

and next we multiply by 2 and get (57). It follows from (53) that ς_N is bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$, and $\int_\Omega \left(\zeta_N^4 + \langle \varsigma_N \rangle^2 \zeta_N^2\right) dx$ is bounded in $L^1(0,T)$ which implies that ς_N is bounded in $L^4((0,T)\times\Omega)$ and $f_N(\varsigma_N)$ is bounded in $L^1((0,T)\times\Omega)$ independently of N. It therefore follows from (57) that ς_N is also bounded in $L^\infty(0,T;H^1(\Omega))$ and $\frac{\partial \varsigma_N}{\partial t}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$ independently of N (note that F_N is bounded on $[-1,1]$, independently of N).

We finally multiply (30) by $\overline{f_N(\varsigma_N)}$ and integrate over Ω and have

$$||\overline{f_N(\varsigma_N)}||^2 - [\Delta \zeta_N, \overline{f_N(\varsigma_N)}] + [(-\Delta)^{-1}\frac{\partial \zeta_N}{\partial t}, \overline{f_N(\varsigma_N)}] + [(-\Delta)^{-1}\overline{\eta(\varsigma_N)}, \overline{f_N(\varsigma_N)}] = 0,$$

noting that

$$\begin{aligned} [\Delta \zeta_N, \overline{f_N(\varsigma_N)}] &= [\Delta \zeta_N, f_N(\varsigma_N)] \\ &= -[f_N'(\varsigma_N)\nabla \varsigma_N, \nabla \varsigma_N] \\ &\leq \lambda_1 ||\nabla \varsigma_N||^2. \end{aligned}$$

Indeed, we have that

$$\begin{aligned}[\Delta\zeta_N, \overline{f_N(\varsigma_N)}] &= [\Delta\zeta_N, f_N(\varsigma_N) - \langle f_N(\varsigma_N)\rangle] \\ &= [\Delta\zeta_N, f_N(\varsigma_N)] \\ &= \int_\Omega \Delta\varsigma_N f_N(\varsigma_N)\, dx \\ &= -\int_\Omega \nabla\varsigma_N \cdot \nabla f_N(\varsigma_N) dx \\ &= -\int_\Omega \nabla\varsigma_N \cdot \nabla(\varsigma_N) f_N'(\varsigma_N) dx \\ &\le \lambda_1 ||\nabla\varsigma_N||^2.\end{aligned}$$

Proceeding as above, we get the inequality

$$||\overline{f_N(\varsigma_N)}||^2 \le c\left(||\varsigma_N||^2_{H^1(\Omega)} + \int_\Omega \left(\zeta_N^4 + \langle\varsigma_N\rangle^2\zeta_N^2\right) dx + ||\frac{\partial\zeta_N}{\partial t}||^2_{-1}\right). \tag{58}$$

Indeed, we have that

$$\begin{aligned}|[-\Delta^{-1}\frac{\partial\zeta_N}{\partial t}, \overline{f_N(\varsigma_N)}]| &\le ||-\Delta^{-1}\tfrac{\partial\zeta_N}{\partial t}||\,||\overline{f_N(\varsigma_N)}|| \\ &\le \frac{\varepsilon}{2}||\frac{\partial\zeta_N}{\partial t}||^2_{-1} + \frac{1}{2\varepsilon}||\overline{f_N(\varsigma_N)}||^2 - |[-\Delta^{-1}\overline{\eta(\varsigma_N)}, \overline{f_N(\varsigma_N)}]| \\ &\le ||-\Delta^{-1}\overline{\eta(\varsigma_N)}||\,||\overline{f_N(\varsigma_N)}|| \\ &\le \frac{\varepsilon}{2}||-\Delta^{-1}\overline{\eta(\varsigma_N)}||^2 + \frac{1}{2\varepsilon}||\overline{f_N(\varsigma_N)}||^2 \\ &\le c||\overline{\eta(\varsigma_N)}||^2 + ||\overline{f_N(\varsigma_N)}||^2 \\ &\le c\int_\Omega \left(\zeta_N^4 + \langle\varsigma_N\rangle^2\zeta_N^2\right) dx + c' + \frac{1}{2\varepsilon}||\overline{f_N(\varsigma_N)}||^2.\end{aligned}$$

Therefore

$$||\overline{f_N(\varsigma_N)}||^2 = [\Delta\zeta_N, \overline{f_N(\varsigma_N)}] - [(-\Delta)^{-1}\frac{\partial\zeta_N}{\partial t}, \overline{f_N(\varsigma_N)}]$$

$$\le \lambda_1||\nabla\varsigma_N||^2 + \frac{\varepsilon}{2}||\frac{\partial\zeta_N}{\partial t}||^2_{-1} + \frac{1}{2\varepsilon}||\overline{f_N(\varsigma_N)}||^2 + c\int_\Omega \left(\zeta_N^4 + \langle\varsigma_N\rangle^2\zeta_N^2\right) dx + c' + \frac{1}{2\varepsilon}||\overline{f_N(\varsigma_N)}||^2,$$

and we get (58).

This results in a uniform (with respect to N) estimate for $\overline{f_N(\varsigma_N)}$ in $L^2((0,T)\times\Omega)$. From (18) it follows that

$$|\langle f_N(\varsigma_N)\rangle| \le c_\delta||\zeta_N||\,||\overline{f_N(\varsigma_N)}|| + c'_\delta, \tag{59}$$

we find a uniform (with respect to N) estimate for $f_N(\varsigma_N)$ in $L^2((0,T)\times\Omega)$.

Besides, owing to (18) with $\langle\varsigma_N\rangle = m$ and $\zeta_N = s$,

$$f_N(\varsigma_N) - \varsigma_N \ge c_6\left(F_N(\varsigma_N) + |f_{1,N}(\varsigma_N)|\right) - c_7$$

$$|f_{1,N}(\varsigma_N)| \le \frac{1}{c_6} f_N(\varsigma_N)\,\zeta_N - F_N(\varsigma_N) + \frac{c_7}{c_6}$$

$$\begin{aligned}|\langle f_{1,N}(\varsigma_N)\rangle| &\le \langle|f_{1,N}(\varsigma_N)|\rangle \\ &\le \frac{1}{c_6}\frac{1}{\mathrm{Vol}(\Omega)}\int_\Omega f_N(\varsigma_N)\zeta_N dx - \frac{1}{\mathrm{Vol}(\Omega)}\int_\Omega F_N(\varsigma_N) dx + \frac{c_7}{c_6},\end{aligned}$$

but $F_N(\varsigma_N) \leq cf(\varsigma_N)\varsigma_N + |f_{1,N}(\varsigma_N)|$ and $f_N(\varsigma_N) = f_{1,N}(\varsigma_N) - \lambda_1\varsigma_N$.

$$\begin{aligned}|\langle f_N(\varsigma_N)\rangle| &\leq |\langle f_{1,N}(\varsigma_N)\rangle| + \lambda_1|\langle\varsigma_N\rangle| \\ &\leq |\langle f_{1,N}(\varsigma_N)\rangle| + c||\varsigma_N||_{L^2(\Omega)} \\ &\leq c||\varsigma_N||_{L^2(\Omega)} + ||\overline{f_N(\varsigma_N)}||.\end{aligned}$$

3.2 Existence of solutions

We have the following theorems.

Theorem 2. *We assume that $\varsigma_0 \in H^1(\Omega)$ is such that $|\langle\varsigma_0\rangle| < 1$ and $-1 < \varsigma_0(x) < 1$ a.e., $x \in \Omega$, then there exists $T = T(\varsigma_0) > 0$ and a solution of (7)-(9) on $[0,T]$ such that $\varsigma \in C([0,T];H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap L^4((0,T)\times\Omega)$ and $\frac{\partial\varsigma}{\partial t} \in L^2(0,T;H^{-1}(\Omega))$.*
Furthermore, $-1 < \varsigma(x,t) < 1$ a.e. $(x,t) \in \Omega\times(0,T)$.

The proof of this theorem is standard due to the uniform estimates obtained in the previous section.

Theorem 3. *Assume that the same assumptions apply as in Theorem 2, then the solution ς is global in time.*

Proof. Consider $[0,T^*)$, such that $T^* > 0$ is the maximal time interval in which the solution ς is given in Theorem 2 exists, so that

$$|\varsigma(x,t)| \leq 1 \text{ a.e. } (x,t) \in \Omega\times[0,T^*).$$

Furthermore, $\langle\varsigma\rangle$ satisfies

$$\frac{d\langle\varsigma\rangle}{dt} + \langle\varsigma^2 - p(1-\varsigma)\rangle = 0.$$

Using the fact that

$$\frac{d\langle\varsigma\rangle}{dt} + p\langle\varsigma\rangle = -\langle\varsigma^2 - 1\rangle,$$

which yields

$$\langle\varsigma(t)\rangle = e^{-pt}\langle\varsigma_0\rangle - e^{-pt}\int_0^t e^{ps}\langle\varsigma^2 - 1\rangle dx.$$

Noting that

$$|\langle\varsigma^2 - 1\rangle| \leq 2,$$

this yields

$$|\langle\varsigma(t)\rangle| \leq e^{-pt}\langle\varsigma_0\rangle + 1 - e^{-pt}, \qquad t \in [0,T^*).$$

In particular, it follows from the last inequality that

$$|\varsigma(x,t)| \leq 1, \qquad t \in [0,T^*).$$

4 Conclusions

In this study, we examined a variation of the Cahn-Hilliard equation featuring a logarithmic nonlinear term and a proliferation term given by $\eta(s) = s^2 - p(1-s)$. The model, subject to Neumann boundary conditions, captures interactions between liquid and gas, with p denoting the gas pressure. Specifically, the model finds application in understanding the formation of islands. We successfully demonstrated the existence of a solution

to the problem. Notably, our challenge stemmed from the singularities in the nonlinear terms. Constructing approximated problems posed difficulty, as we could not rule out the possibility of solutions to these approximated problems experiencing blowup in finite time. Consequently, deriving uniform estimates for the approximated problems in the presence of singularities became a more intricate task.

It is noteworthy that our future work will delve into the exploration of the Cahn-Hilliard equation incorporating a fidelity term of the form $\lambda_0 \xi_{\Omega\setminus D}(x)(u-h)$. Here, λ_0 stands as a suitably large constant, and D represents the inpainting model. The results indicate that inpainting in this scenario is faster and more efficient compared to a model with a regular polynomial nonlinear term.

5 Declarations

5.1 Conflict of interest:

Not applicable.

5.2 Funding:

Not applicable.

5.3 Author's contribution:

H.F.-Data Curation, Conceptualization, Design, Formal Analysis, Project Administration. M.B.-Data Curation, Conceptualization, Design. H.A.-Writing - Original Draft, Investigation. Y.A.-Writing - Original Draft, Resources. All authors reviewed the results and approved the final version of the manuscript.

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5.5 Data availability statement:

All data that support the findings of this study are included within the article.

5.6 Using of AI tools:

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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