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Original Study

On a fuzzy initial value problem

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Abstract

This work is on a fuzzy initial value problem with fuzzy coefficient. The problem is investigated via fuzzy Laplace transform method under generalized differentiability. The solutions of the problem are thus found. The problem is illustrated with an example. Using the generalized differentiability, solutions of a fuzzy initial value problem with fuzzy coefficient are investigated via fuzzy Laplace transform method. The graphics of the solutions are drawn. Conclusions are given.

Keywords: Fuzzy problem, fuzzy laplace transform method, generalized differentiability.
AMS 2020 codes: 03E72; 34A07; 44A10.

1 Introduction

Fuzzy number and fuzzy arithmetic were introduced by Zadeh in 1965 [1]. Fuzzy differential equations are an important topic in many fields. Thus, many researchers have studied fuzzy differential equation using different approach methods [2–11]. The first approach is the Hukuhara derivative [12, 13]. This approach has a drawback: the solution becomes fuzzier as time goes. Thus, the fuzzy solution behaves quite differently from the crisp solution. So, the generalized Hukuhara derivative was studied [7, 8, 11, 14–16]. The generalized Hukuhara derivative allows to resolve the above-mentioned shortcoming. The second approach is the extension principle [17]. The third approach is the differential inclusion one [18]. The Fuzzy Laplace transform (FLM) method is useful to solve fuzzy differential equations. In many articles, the solution of fuzzy differential equation has been studied by using FLM approach [19–24].

The rest of this paper is organized as follows: preliminaries and basic definitions used in this paper are given in Section 2. In Section 3, some applications and main results are reported. In Section 4, the conclusions of this paper are introduced.

2 Preliminaries

In this part, we present some important definitions and theorems which will be used in this paper.

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Definition 1. [25] A fuzzy number is a mapping u:$\mathbb{R} \to [0,1]$ satisfying the properties $\overline{\{x \in \mathbb{R} \mid u(x) > 0\}}$ is compact, u is normal, u is convex fuzzy set, u is upper semi-continuous on $\mathbb{R}$. Let's denote the set of all fuzzy numbers with $\mathbb{R}_F$.

Definition 2. [11] Let be $u \in \mathbb{R}_F$. The α-level set of u is $[u]^\alpha = [\underline{u}_\alpha, \overline{u}_\alpha] = \{x \in \mathbb{R} \mid u(x) \geq \alpha\}$, $0 < \alpha \leq 1$.

Definition 3. [11] The parametric form $[\underline{u}_\alpha, \overline{u}_\alpha]$ of u fuzzy number satisfies the following requirements:
1. The lower part $\underline{u}_\alpha$ is bounded non-decreasing left-continuous on $(0,1]$, right-continuous for $\alpha = 0$.
2. The upper part $\overline{u}_\alpha$ is bounded non-increasing left-continuous on $(0,1]$, right-continuous for $\alpha = 0$.
3. $\underline{u}_\alpha \leq \overline{u}_\alpha$, $0 \leq \alpha \leq 1$.

Definition 4. [25] If A is a symmetric triangular fuzzy number with support $[\underline{a}, \overline{a}]$, the $\alpha-$level set of A is $[A]^\alpha = \left[\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right]$.

Definition 5. [26] Let $u,\ v \in \mathbb{R}_F$. The generalized Hukuhara difference (gH-difference) between u and v is the set $w \in \mathbb{R}_F$ which $u \ominus_g v = w$ if and only if $u = v + w$ or $v = u + (-1)w$.

Definition 6. [11] Let $f : [a,b] \to \mathbb{R}_F$ and $t_0 \in [a,b]$ .We say that f is (1)-differentiable at t_0, if there is an element $f'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small (near to 0), exist $f(t_0 + h) \ominus f(t_0)$, $f(t_0) \ominus f(t_0 - h)$ and the limits

$$\lim_{h\to 0} \frac{f(t_0+h) \ominus f(t_0)}{h} = \lim_{h\to 0} \frac{f(t_0) \ominus f(t_0-h)}{h} = f'(t_0),$$

and f is (2)-differentiable if for all $h > 0$ sufficiently small (near to 0), exist $f(t_0) \ominus f(t_0 + h)$, $f(t_0 - h) \ominus f(t_0)$ and the limits

$$\lim_{h\to 0} \frac{f(t_0) \ominus f(t_0+h)}{-h} = \lim_{h\to 0} \frac{f(t_0-h) \ominus f(t_0)}{-h} = f'(t_0).$$

Theorem 1. *[27] Let $f : [a,b] \to \mathbb{R}_F$ be fuzzy function, where $[f(t)]^\alpha = \left[\underline{f}_\alpha(t), \overline{f}_\alpha(t)\right]$, for each $\alpha \in [0,1]$.*

(i) If f is (1)-differentiable then $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable functions and $[f'(t)]^\alpha = \left[\underline{f}'_\alpha(t), \overline{f}'_\alpha(t)\right]$,

(ii) If f is (2)-differentiable then $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable functions and $[f'(t)]^\alpha = \left[\overline{f}'_\alpha(t), \underline{f}'_\alpha(t)\right]$.

Definition 7. [28] The fuzzy Laplace transform of fuzzy function f is

$$F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt = \left[\lim_{\rho\longrightarrow\infty} \int_0^\rho e^{-st} \underline{f}(t)\,dt\ , \lim_{\rho\longrightarrow\infty} \int_0^\rho e^{-st} \overline{f}(t)\,dt\right],$$

$$F(s,\alpha) = L\left([f(t)]^\alpha\right) = \left[L\left(\underline{f}_\alpha(t)\right), L\left(\overline{f}_\alpha(t)\right)\right].$$

$$L\left(\underline{f}_\alpha(t)\right) = \int_0^\infty e^{-st} \underline{f}_\alpha(t)\,dt = \lim_{\rho\longrightarrow\infty} \int_0^\rho e^{-st} \underline{f}_\alpha(t)\,dt\ ,$$

$$L\left(\overline{f}_\alpha(t)\right) = \int_0^\infty e^{-st} \overline{f}_\alpha(t)\,dt = \lim_{\rho\longrightarrow\infty} \int_0^\rho e^{-st} \overline{f}_\alpha(t)\,dt.$$

Theorem 2. *[19] Let $f'(t)$ be an integrable fuzzy function and $f(t)$ is primitive of $f'(t)$ on $(0,\infty]$.*
1. If f is (1)-differentiable, $L(f'(t)) = sL(f(t)) \ominus f(0)$.
2. If f is (2)-differentiable, $L(f'(t)) = (-f(0)) \ominus (-sL(f(t)))$.

Theorem 3. *[28] Let $f''(t)$ be an integrable fuzzy function and $f(t)$, $f'(t)$ are primitive of $f'(t)$, $f''(t)$ on $(0,\infty]$.*
1. If f and f' are (1)-differentiable, $L(f''(t)) = s^2 L(f(t)) \ominus sf(0) \ominus f'(0)$.
2. If f and f' are (2)-differentiable, $L(f''(t)) = s^2 L(f(t)) \ominus sf(0) - f'(0)$.
3. If f is (1)-differentiable and f' is (2)-differentiable, $L(f''(t)) = \ominus\left(-s^2\right) L(f(t)) - sf(0) - f'(0)$.
4. If f is (2)-differentiable and f' is (1)-differentiable, $L(f''(t)) = \ominus\left(-s^2\right) L(f(t)) - sf(0) \ominus f'(0)$.

3 Applications

We investigate the solutions of the problem given as

$$u''(t) = [\lambda]^{\alpha} u(t), \;\; t > 0$$
$$u(0) = [\rho]^{\alpha}, \; u'(0) = [\nu]^{\alpha}$$

by the fuzzy Laplace transform and the generalized Hukuhara differentiability, where $[\lambda]^{\alpha} = \left[\underline{\lambda}_{\alpha}, \overline{\lambda}_{\alpha}\right]$, $\left(\underline{\lambda}_{\alpha} > 0, \;\; \overline{\lambda}_{\alpha} > 0\right)$, $[\rho]^{\alpha} = \left[\underline{\rho}_{\alpha}, \overline{\rho}_{\alpha}\right]$, $[\nu]^{\alpha} = [\underline{\nu}_{\alpha}, \overline{\nu}_{\alpha}]$ are symmetric triangular fuzzy numbers, $u(t)$ is positive fuzzy function, the fuzzy Laplace transform of fuzzy function $u(t)$ is $L(u(t)) = U(s)$ and (i,j)-solution means that u is (i)-differentiable, u' is (j)-differentiable, $i, j = 1, 2$.

1. (1,1)-solution: Let u, u' be (1)-differentiable. Then, the equation

$$s^2 U(s) \ominus su(0) \ominus u'(0) = [\lambda]^{\alpha} U(s)$$

is obtained by using FLT. From this, we have the equations

$$s^2 \underline{U}_{\alpha}(s) - s\underline{u}_{\alpha}(0) - \underline{u}'_{\alpha}(0) = \underline{\lambda}_{\alpha} \underline{U}_{\alpha}(s),$$
$$s^2 \overline{U}_{\alpha}(s) - s\overline{u}_{\alpha}(0) - \overline{u}'_{\alpha}(0) = \overline{\lambda}_{\alpha} \overline{U}_{\alpha}(s).$$

Using the initial conditions, we obtain

$$\underline{U}_{\alpha}(s) = \frac{s\underline{\rho}_{\alpha}}{s^2 - \underline{\lambda}_{\alpha}} + \frac{\underline{\nu}_{\alpha}}{s^2 - \underline{\lambda}_{\alpha}}, \; \overline{U}_{\alpha}(s) = \frac{s\overline{\rho}_{\alpha}}{s^2 - \overline{\lambda}_{\alpha}} + \frac{\overline{\nu}_{\alpha}}{s^2 - \overline{\lambda}_{\alpha}}.$$

Taking the inverse Laplace transform of these equations, $(1,1)$-solution is obtained as

$$\underline{u}_{\alpha}(t) = \underline{\rho}_{\alpha} \cosh\left(\sqrt{\underline{\lambda}_{\alpha}} t\right) + \frac{\underline{\nu}_{\alpha}}{\sqrt{\underline{\lambda}_{\alpha}}} \sinh\left(\sqrt{\underline{\lambda}_{\alpha}} t\right), \overline{u}_{\alpha}(t) = \overline{\rho}_{\alpha} \cosh\left(\sqrt{\overline{\lambda}_{\alpha}} t\right) + \frac{\overline{\nu}_{\alpha}}{\sqrt{\overline{\lambda}_{\alpha}}} \sinh\left(\sqrt{\overline{\lambda}_{\alpha}} t\right),$$
$$[u(t)]^{\alpha} = [\underline{u}_{\alpha}(t), \overline{u}_{\alpha}(t)].$$

2. (1,2)-solution: Let u be (1)-differentiable and u' be (2)-differentiable. Then, since

$$-u'(0) \ominus \left(-s^2 U(s)\right) - su(0) = [\lambda]^{\alpha} U(s),$$

we have the equations

$$-\overline{u}'_{\alpha}(0) + s^2 \overline{U}_{\alpha}(s) - s\overline{u}_{\alpha}(0) = \underline{\lambda}_{\alpha} \underline{U}_{\alpha}(s),$$
$$-\underline{u}'_{\alpha}(0) + s^2 \underline{U}_{\alpha}(s) - s\underline{u}_{\alpha}(0) = \overline{\lambda}_{\alpha} \overline{U}_{\alpha}(s).$$

Using the initial conditions and making the necessary operations, we obtain the equations

$$\underline{U}_{\alpha}(s) = \frac{\overline{\lambda}_{\alpha}\overline{\nu}_{\alpha}}{s^4 - \underline{\lambda}_{\alpha}\overline{\lambda}_{\alpha}} + \frac{s\overline{\lambda}_{\alpha}\overline{\rho}_{\alpha}}{s^4 - \underline{\lambda}_{\alpha}\overline{\lambda}_{\alpha}} + \frac{s^2\underline{\nu}_{\alpha}}{s^4 - \underline{\lambda}_{\alpha}\overline{\lambda}_{\alpha}} + \frac{s^3\underline{\rho}_{\alpha}}{s^4 - \underline{\lambda}_{\alpha}\overline{\lambda}_{\alpha}},$$
$$\overline{U}_{\alpha}(s) = \frac{\underline{\lambda}_{\alpha}\underline{\nu}_{\alpha}}{s^4 - \underline{\lambda}_{\alpha}\overline{\lambda}_{\alpha}} + \frac{s\underline{\lambda}_{\alpha}\underline{\rho}_{\alpha}}{s^4 - \underline{\lambda}_{\alpha}\overline{\lambda}_{\alpha}} + \frac{s^2\overline{\nu}_{\alpha}}{s^4 - \underline{\lambda}_{\alpha}\overline{\lambda}_{\alpha}} + \frac{s^3\overline{\rho}_{\alpha}}{s^4 - \underline{\lambda}_{\alpha}\overline{\lambda}_{\alpha}}.$$

Then, $(1,2)$-solution is obtained as

$$\underline{u}_{\alpha}(t) = \left(\frac{\overline{\lambda}_{\alpha}\overline{\nu}_{\alpha}}{2\sqrt[4]{\left(\underline{\lambda}_{\alpha}\overline{\lambda}_{\alpha}\right)^3}} + \frac{\underline{\nu}_{\alpha}}{2\sqrt[4]{\left(\underline{\lambda}_{\alpha}\overline{\lambda}_{\alpha}\right)}}\right)\left(\sin\left(\sqrt[4]{\left(\underline{\lambda}_{\alpha}\overline{\lambda}_{\alpha}\right)}t\right) + \sinh\left(\sqrt[4]{\left(\underline{\lambda}_{\alpha}\overline{\lambda}_{\alpha}\right)}t\right)\right)$$

$$+\left(\frac{\overline{\lambda}_\alpha\overline{\rho}_\alpha}{2\sqrt{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)}}+\frac{\underline{\rho}_\alpha}{2}\right)\left(\cos\left(\sqrt[4]{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)}t\right)+\cosh\left(\sqrt[4]{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)}t\right)\right),$$

$$\overline{u}_\alpha(t)=\left(\frac{\underline{\lambda}_\alpha\underline{\nu}_\alpha}{2\sqrt[4]{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)^3}}+\frac{\overline{\nu}_\alpha}{2\sqrt[4]{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)}}\right)\left(\sin\left(\sqrt[4]{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)}t\right)+\sinh\left(\sqrt[4]{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)}t\right)\right)$$

$$+\left(\frac{\underline{\lambda}_\alpha\underline{\rho}_\alpha}{2\sqrt{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)}}+\frac{\overline{\rho}_\alpha}{2}\right)\left(\cos\left(\sqrt[4]{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)}t\right)+\cosh\left(\sqrt[4]{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)}t\right)\right),$$

$$[u(t)]^\alpha=[\underline{u}_\alpha(t),\overline{u}_\alpha(t)].$$

3. (2,1)-solution: Let u be (2)-differentiable and u' be (1)-differentiable. From the equation

$$\ominus\left(-s^2U(s)\right)-su(0)\ominus u'(0)=[\lambda]^\alpha U(s),$$

we have the equations

$$s^2\overline{U}_\alpha(s)-\underline{\lambda}_\alpha\underline{U}_\alpha(s)=s\overline{\rho}_\alpha+\underline{\nu}_\alpha, s^2\underline{U}_\alpha(s)-\overline{\lambda}_\alpha\overline{U}_\alpha(s)=s\underline{\rho}_\alpha+\overline{\nu}_\alpha.$$

From this, making the necessary operations, $(2,1)$-solution is obtained as

$$\underline{u}_\alpha(t)=\left(\frac{\overline{\lambda}_\alpha\underline{\nu}_\alpha}{2\sqrt[4]{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)^3}}+\frac{\overline{\nu}_\alpha}{2\sqrt[4]{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)}}\right)\left(\sin\left(\sqrt[4]{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)}t\right)+\sinh\left(\sqrt[4]{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)}t\right)\right)$$

$$+\left(\frac{\overline{\lambda}_\alpha\overline{\rho}_\alpha}{2\sqrt{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)}}+\frac{\underline{\rho}_\alpha}{2}\right)\left(\cos\left(\sqrt[4]{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)}t\right)+\cosh\left(\sqrt[4]{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)}t\right)\right),$$

$$\overline{u}_\alpha(t)=\left(\frac{\underline{\lambda}_\alpha\overline{\nu}_\alpha}{2\sqrt[4]{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)^3}}+\frac{\underline{\nu}_\alpha}{2\sqrt[4]{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)}}\right)\left(\sin\left(\sqrt[4]{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)}t\right)+\sinh\left(\sqrt[4]{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)}t\right)\right)$$

$$+\left(\frac{\underline{\lambda}_\alpha\underline{\rho}_\alpha}{2\sqrt{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)}}+\frac{\overline{\rho}_\alpha}{2}\right)\left(\cos\left(\sqrt[4]{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)}t\right)+\cosh\left(\sqrt[4]{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)}t\right)\right),$$

$$[u(t)]^\alpha=[\underline{u}_\alpha(t),\overline{u}_\alpha(t)].$$

4. (2,2)-solution: Let u and u' be (2)-differentiable. Since

$$s^2U(s)\ominus su(0)-u'(0)=[\lambda]^\alpha U(s),$$

$(2,2)$-solution is obtained as

$$\underline{u}_\alpha(t)=\underline{\rho}_\alpha\cosh\left(\sqrt{\underline{\lambda}_\alpha}t\right)+\frac{\overline{v}_\alpha}{\sqrt{\underline{\lambda}_\alpha}}\sinh\left(\sqrt{\underline{\lambda}_\alpha}t\right),\overline{u}_\alpha(t)=\overline{\rho}_\alpha\cosh\left(\sqrt{\overline{\lambda}_\alpha}t\right)+\frac{\underline{v}_\alpha}{\sqrt{\overline{\lambda}_\alpha}}\sinh\left(\sqrt{\overline{\lambda}_\alpha}t\right),$$

$$[u(t)]^\alpha=[\underline{u}_\alpha(t),\overline{u}_\alpha(t)].$$

These solutions must be valid fuzzy functions.

Example 3.1. Consider the solutions of the fuzzy problem

$$u''(t)=[1]^\alpha u(t),$$

$$u(0)=[1]^\alpha,\ \ u'(0)=[2]^\alpha,$$

where $[1]^\alpha=[\alpha,2-\alpha]$, $[2]^\alpha=[1+\alpha,3-\alpha]$.

$(1,1)$-solution is

$$\underline{u}_\alpha(t)=\alpha\cosh\left(\sqrt{\alpha}t\right)+\frac{1+\alpha}{\sqrt{\alpha}}\sinh\left(\sqrt{\alpha}t\right),$$

$$\overline{u}_\alpha(t)=(2-\alpha)\cosh\left(\sqrt{(2-\alpha)}t\right)+\frac{3-\alpha}{\sqrt{(2-\alpha)}}\sinh\left(\sqrt{(2-\alpha)}t\right),$$

$$[u(t)]^\alpha=[\underline{u}_\alpha(t),\overline{u}_\alpha(t)],$$

$(1,2)$-solution is

$$\underline{u}_\alpha(t)=\left(\frac{(2-\alpha)(3-\alpha)}{2\sqrt[4]{(\alpha(2-\alpha))^3}}+\frac{1+\alpha}{2\sqrt[4]{(\alpha(2-\alpha))}}\right)\left(\sin\left(\sqrt[4]{(\alpha(2-\alpha))}t\right)+\sinh\left(\sqrt[4]{(\alpha(2-\alpha))}t\right)\right)$$

$$+\left(\frac{(2-\alpha)^2}{2\sqrt{(\alpha(2-\alpha))}}+\frac{\alpha}{2}\right)\left(\cos\left(\sqrt[4]{(\alpha(2-\alpha))}t\right)+\cosh\left(\sqrt[4]{(\alpha(2-\alpha))}t\right)\right),$$

$$\overline{u}_\alpha(t)=\left(\frac{\alpha(1+\alpha)}{2\sqrt[4]{(\alpha(2-\alpha))^3}}+\frac{3-\alpha}{2\sqrt[4]{(\alpha(2-\alpha))}}\right)\left(\sin\left(\sqrt[4]{(\alpha(2-\alpha))}t\right)+\sinh\left(\sqrt[4]{(\alpha(2-\alpha))}t\right)\right)$$

$$+\left(\frac{\alpha^2}{2\sqrt{(\alpha(2-\alpha))}}+\frac{2-\alpha}{2}\right)\left(\cos\left(\sqrt[4]{(\alpha(2-\alpha))}t\right)+\cosh\left(\sqrt[4]{(\alpha(2-\alpha))}t\right)\right),$$

$$[u(t)]^\alpha=[\underline{u}_\alpha(t),\overline{u}_\alpha(t)],$$

$(2,1)$-solution is

$$\underline{u}_\alpha(t)=\left(\frac{(2-\alpha)(1+\alpha)}{2\sqrt[4]{(\alpha(2-\alpha))^3}}+\frac{(3-\alpha)}{2\sqrt[4]{(\alpha(2-\alpha))}}\right)\left(\sin\left(\sqrt[4]{(\alpha(2-\alpha))}t\right)+\sinh\left(\sqrt[4]{(\alpha(2-\alpha))}t\right)\right)$$

$$+\left(\frac{(2-\alpha)^2}{2\sqrt{(\alpha(2-\alpha))}}+\frac{\alpha}{2}\right)\left(\cos\left(\sqrt[4]{(\alpha(2-\alpha))}t\right)+\cosh\left(\sqrt[4]{(\alpha(2-\alpha))}t\right)\right),$$

$$\overline{u}_\alpha(t)=\left(\frac{\alpha(3-\alpha)}{2\sqrt[4]{(\alpha(2-\alpha))^3}}+\frac{(1+\alpha)}{2\sqrt[4]{(\alpha(2-\alpha))}}\right)\left(\sin\left(\sqrt[4]{(\alpha(2-\alpha))}t\right)+\sinh\left(\sqrt[4]{(\alpha(2-\alpha))}t\right)\right)$$

$$+\left(\frac{\alpha^2}{2\sqrt{(\alpha(2-\alpha))}}+\frac{2-\alpha}{2}\right)\left(\cos\left(\sqrt[4]{(\alpha(2-\alpha))}t\right)+\cosh\left(\sqrt[4]{(\alpha(2-\alpha))}t\right)\right),$$

$$[u(t)]^{\alpha}=[\underline{u}_{\alpha}(t),\overline{u}_{\alpha}(t)],$$

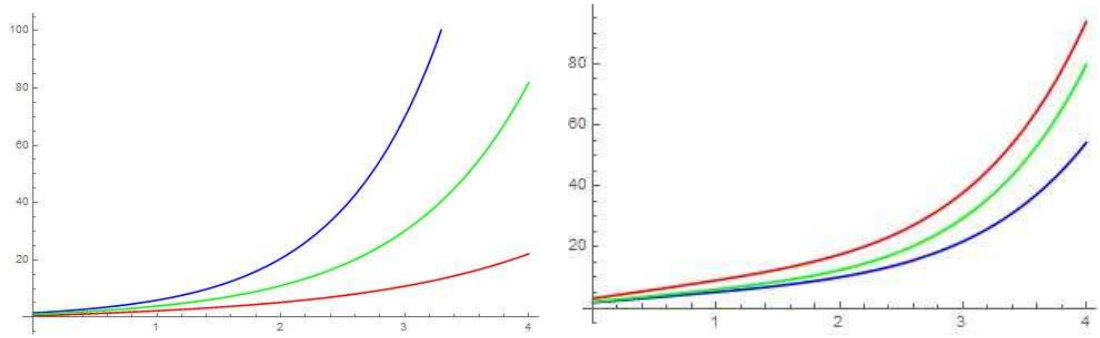
and $(2,2)$-solution is

$$\underline{u}_{\alpha}(t)=\alpha\cosh\left(\sqrt{\alpha}t\right)+\frac{3-\alpha}{\sqrt{\alpha}}\sinh\left(\sqrt{\alpha}t\right),$$

$$\overline{u}_{\alpha}(t)=(2-\alpha)\cosh\left(\sqrt{(2-\alpha)}t\right)+\frac{1+\alpha}{\sqrt{(2-\alpha)}}\sinh\left(\sqrt{(2-\alpha)}t\right),$$

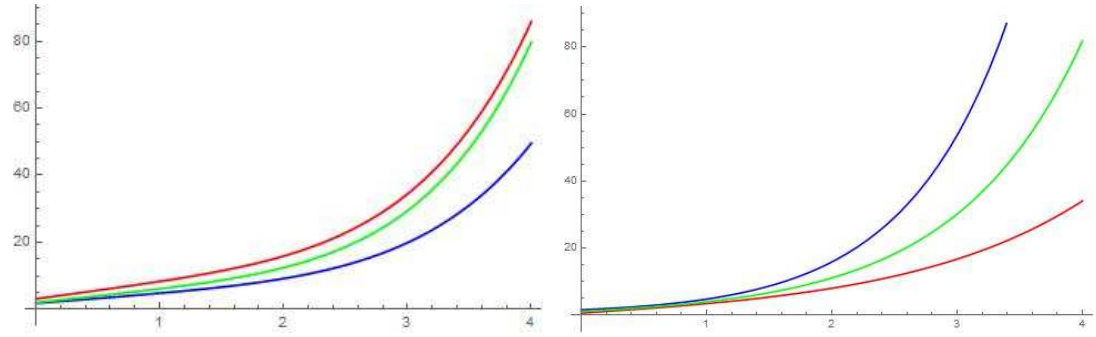
$$[u(t)]^{\alpha}=[\underline{u}_{\alpha}(t),\overline{u}_{\alpha}(t)].$$

From Definition 3, according to Figure (1a) and Figure (2b), $(1,1)$ and $(2,2)$ solutions are valid fuzzy functions. But, according to Figure (1b) and Figure (2a), $(1,2)$ and $(2,1)$ solutions are not valid fuzzy functions.



(a) (b)

Fig. 1 a) Graphic of (1,1)-solution for $\alpha=0.5$. b) Graphic of (1,2)-solution for $\alpha=0.5$.



(a) (b)

Fig. 2 a) Graphic of (2,1)-solution for $\alpha=0.5$. b)Graphic of (2,2)-solution for $\alpha=0.5$.

In these figures, blue is used to symbolize $\rightarrow\overline{y}_{\alpha}(t)$, red is for $\rightarrow\underline{y}_{\alpha}(t)$ and green is used to explain $\rightarrow\overline{y}_{1}(t)=\underline{y}_{1}(t)$.

4 Conclusion

In this work, we studied a fuzzy initial problem with fuzzy coefficient. We used the generalized differentiability and the fuzzy Laplace transform. We solved an application on the problem and drew the figures of the solutions. Finally, it is shown that the solutions are valid fuzzy functions or not.

5 Declarations

5.1 Conflict of interest:

The authors hereby declare that there is no conflict of interests regarding the publication of this paper.

5.2 Funding:

Not applicable.

5.3 Author's contribution:

H.G.Ç.-Methodology, Writing-Original Draft, Validation, Conceptualization, Formal Analysis, Investigation, Writing-Review and Editing. All authors read and approved the final submitted version of this manuscript.

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5.5 Data availability statement:

All data that support the findings of this study are included within the article.

5.6 Using of AI tools:

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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