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Original Study

New results for generalized Hurwitz-Lerch Zeta functions using Laplace transform

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Abstract

Fractional Kinetic equations (FKEs) including a wide variety of special functions are widely and successfully applied in describing and solving many important problems of physics and astrophysics. In this work, the solutions of the FKEs of the generalized Hurwitz-Lerch Zeta function using the Laplace transform are derived and examined.

Keywords: Hurwitz-Lerch Zeta function, generalized Hurwitz-Lerch Zeta function, fractional Kinetic equation, Mittag-Leffler function, Laplace transform.
AMS 2020 codes: 11M35; 44A20; 33E12.

1 Introduction

The Hurwitz-Lerch Zeta function is defined by Srivastava and Choi [1, 2]:

$$\Phi(z,r,\alpha)=\sum_{n=0}^{\infty}\frac{z^n}{(n+\alpha)^r}, \tag{1}$$

$$\left(\alpha\in\mathbb{C}\setminus\mathbb{Z}_0;\ r\in\mathbb{C}\text{ when }|z|<1;\ \Re(r)>1\text{ when }|z|=1\right).$$

In 1997, Goyal and Laddha [3] defined and studied an extension of (1) one variable in the following equation:

$$\Phi_{\varepsilon}^{\star}(z,r,\alpha)=\sum_{n=0}^{\infty}(\varepsilon)_n\frac{z^n}{(n+\alpha)^r\,n!}, \tag{2}$$

$$(\varepsilon\in\mathbb{C};\alpha\in\setminus\mathbb{Z}_0;|x|<1).$$

Afterwards, many researchers studied many different generalizations and extensions of the Hurwitz-Lerch Zeta function one variable. The readers who are interested can refer to these earlier publications for further researches and applications (Lin and Srivastava [4], Lin et al. [5], Choi et al. [6], Gupta et al. [7], Garg et al. [8], Srivastava et al. [9], Daman and Pathan [10], Srivastava et al. [11], and Srivastava [12]).

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Later on, in 2017, Choi and Parmar [13] introduced and examined a generalization of (1) two variables in the following equation:

$$\Phi_{\mathfrak{m},\mathfrak{n},\mathfrak{n}';\mathfrak{u}}(\mathfrak{v},\mathfrak{y},\mathfrak{r},\mathfrak{p}) = \sum_{\mathfrak{a},\mathfrak{b}=0}^{\infty} \frac{(\mathfrak{m})_{\mathfrak{a}+\mathfrak{b}}\,(\mathfrak{n})_{\mathfrak{a}}\,(\mathfrak{n}')_{\mathfrak{b}}}{(\mathfrak{u})_{\mathfrak{a}+\mathfrak{b}}\,\mathfrak{a}!\,\mathfrak{b}!}\,\frac{\mathfrak{v}^{\mathfrak{a}}\mathfrak{y}^{\mathfrak{b}}}{(\mathfrak{a}+\mathfrak{b}+\mathfrak{p})^{\mathfrak{r}}} \tag{3}$$

$$\left(\mathfrak{u},\mathfrak{p}\in\mathbb{C}\setminus\mathbb{Z}_0;\ \mathfrak{m},\mathfrak{n},\mathfrak{n}'\in\mathbb{C};\ \mathfrak{r},\mathfrak{v},\mathfrak{y}\in\mathbb{C}\text{ when }|\mathfrak{v}|<1\text{ and }|\mathfrak{y}|<1;\right.$$

$$\left.\Re(\mathfrak{r}-\mathfrak{u}-\mathfrak{m}-\mathfrak{n}-\mathfrak{n}')>1\text{ when }|\mathfrak{v}|=1\text{ and }|\mathfrak{y}|=1\right).$$

A large number of generalizations and extensions of the Hurwitz-Lerch Zeta function two variables have been studied by Choi et al. [14], and Srivastava et al. [15].

In 2011, Srivastava et al. [16] introduced and studied the following extension of the generalized Hurwitz-Lerch Zeta function:

$$\Phi_{\lambda,\mu;\omega}^{(\sigma,\rho,\kappa)}(z,s,a) = \sum_{n=0}^{\infty} \frac{(\lambda)_{\sigma n}\,(\mu)_{\rho n}}{(\omega)_{\kappa n}\,n!}\,\frac{z^n}{(n+a)^s}, \tag{4}$$

$$(\lambda,\mu\in\mathbb{C};\ a,\omega\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ \sigma,\rho,\kappa\in\mathbb{R}^+;\ \kappa-\sigma-\rho>-1\text{ when }s,z\in\mathbb{C};$$
$$\kappa-\sigma-\rho=-1\text{ and }s\in\mathbb{C}\text{ when }|z|<\delta^\star=\sigma^{-\sigma}\rho^{-\rho}\kappa^{\kappa};$$
$$\kappa-\sigma-\rho=-1\text{ and }\Re(s+\omega-\lambda-\mu)>1\text{ when }|z|=\delta^\star).$$

The main goal of this work is to demonstrate the generalized FKEs involving generalized Hurwitz-Lerch Zeta function (4). Here, we consider the Laplace transform methodology to get the results.

The rest of the paper is organized as follows. The fractional Kinetic equations are given in Section 2. The numerical results and graphics are described in Section 3. The conclusion is introduced by Section 4.

2 Fractional Kinetic equations

In [17], experts designated the fractional differential equation for the rate of change of reaction. The destruction rate and the production rate are given by

$$\frac{d\mathfrak{z}}{d\mathfrak{y}} = -d(\mathfrak{z}_\mathfrak{y}) + p(\mathfrak{z}_\mathfrak{y}), \tag{5}$$

where $\mathfrak{z}=\mathfrak{z}(\mathfrak{y})$ the rate of the reaction, $d=d(\mathfrak{z})$ the rate of destruction, $p=p(\mathfrak{z})$ the rate of production and $\mathfrak{z}_\mathfrak{y}$ denotes the function defined by $\mathfrak{z}_\mathfrak{y}(\mathfrak{y}^\star)=\mathfrak{z}(\mathfrak{y}-\mathfrak{y}^\star), \mathfrak{y}^\star>0$.

The special case of equation (5) for spatial fluctuations and inhomogeneities in $\mathfrak{z}(\mathfrak{y})$ the quantities are neglected, that is the equation

$$\frac{d\mathfrak{z}}{d\mathfrak{y}} = -c_i\,\mathfrak{z}_i(\mathfrak{y}) \tag{6}$$

with the initial condition that $\mathfrak{z}_i(\mathfrak{y}=0)=\mathfrak{z}_0$ is the number of density of the species i at time $\mathfrak{y}=0$ and $c_i>0$. If we shift the index i and integrate the standard Kinetic equation (6), we have

$$\mathfrak{z}(\mathfrak{y}) - \mathfrak{z}_0 = -c\,{}_0\mathscr{D}_\mathfrak{y}^{-1}\,\mathfrak{z}(\mathfrak{y}) \tag{7}$$

where ${}_0\mathscr{D}_\mathfrak{y}^{-1}$ is the special case of the Riemann-Liouville integral operator ${}_0\mathscr{D}_\mathfrak{y}^{-\mathfrak{r}}$ given as Srivastava and Saxena [18],

$${}_0\mathscr{D}_\mathfrak{y}^{-\mathfrak{r}} f(\mathfrak{y}) = \frac{1}{\Gamma(\mathfrak{r})}\int_0^{\mathfrak{y}} (\mathfrak{y}-s)^{\mathfrak{r}-1} f(s)ds, \tag{8}$$

$$(\mathfrak{y} > 0, \Re(\mathfrak{x}) > 0).$$

The fractional generalisation of the standard Kinetic equation (7) is given by Haubold and Mathai [17] as follows:

$$\mathfrak{z}(\mathfrak{y}) - \mathfrak{z}_0 = -c^{\mathfrak{x}}\, {}_0\mathscr{D}_{\mathfrak{y}}^{-1}\mathfrak{z}(\mathfrak{y}) \tag{9}$$

and obtained the solution of (6) as follows:

$$\mathfrak{z}(\mathfrak{y}) = \mathfrak{z}_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\mathfrak{x}k+1)}(c\mathfrak{y})^{\mathfrak{x}k}. \tag{10}$$

Furthermore, Saxena and Kalla [19] considered the following fractional Kinetic equation:

$$\mathfrak{z}(\mathfrak{y}) - \mathfrak{z}_0 f(\mathfrak{y}) = -c^{\mathfrak{x}}\, {}_0\mathscr{D}_{\mathfrak{y}}^{-1}\mathfrak{z}(\mathfrak{y}) \ (\Re(\mathfrak{x}) > 0), \tag{11}$$

where $\mathfrak{z}(\mathfrak{y})$ denotes the number density of a given species at time $\mathfrak{y}$, $\mathfrak{z}_0 = \mathfrak{z}(0)$ is the number of density of that species at time $\mathfrak{y} = 0$, c is a constant and $f \in L(0,\infty)$.

By applying the Laplace transform (Erdélyi et al. [20], Srivastava and Karlsson [21], and Srivastava and Manocha [22]) to the equation (11), we obtain

$$\mathfrak{L}\{\mathfrak{z}(\mathfrak{y}); p\} = \mathfrak{z}_0 \frac{F(p)}{1 + c^{\mathfrak{x}} p^{-\mathfrak{x}}} = \mathfrak{z}_0 \left(\sum_{n=0}^{\infty} (-c^{\mathfrak{x}})^n p^{-\mathfrak{x}n} \right) F(p), \tag{12}$$

$$\left(n \in \mathfrak{z}_0, \left| \frac{c}{p} \right| < 1 \right).$$

The extension and generalisation of FKEs involving many fractional operators were found in (Saxena et al. [23], Saxena et al. [24], Chouhan and Sarwat [25], Chouhan et al. [26], Agarwal et al. [27], Agarwal et al. [28], Agarwal and Nisar [29], Baleanu et al. [30], Nisar et al. [31], Nisar [32], Şahin and Yağcı [33], Yağcı and Şahin [34], Akel et al. [35], and Hidan et al. [36]).

2.1 Solution of generalized FKEs involving generalized Hurwitz-Lerch Zeta function

Here, we present the solution of the generalised FKEs by considering generalized Hurwitz-Lerch Zeta function (4).

Theorem 1. *If* $\mathfrak{d} > 0, \mathfrak{x} > 0; \lambda, \mu, \delta \in \mathbb{C}$, *and* $\mathfrak{d} \neq \delta$ *be such that* $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-; \sigma, \rho, \kappa \in \mathbb{R}^+$, *then the solution of the following fractional equation*

$$\mathfrak{z}(\mathfrak{y}) - \mathfrak{z}_0 \Phi_{\lambda,\mu;\omega}^{(\sigma,\rho,\kappa)}(\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}}, s, a) = -\delta^{\mathfrak{x}}\, {}_0\mathscr{D}_{\mathfrak{y}}^{-\mathfrak{x}} \tag{13}$$

is given by

$$\mathfrak{z}(\mathfrak{y}) = \mathfrak{z}_0 \sum_{n=0}^{\infty} \frac{(\lambda)_{\sigma n}(\mu)_{\rho n}\Gamma(\mathfrak{x}n+1)(\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}})^n}{(\omega)_{\kappa n}\, n!\,(n+a)^s}\, \mathscr{E}_{\mathfrak{x},\mathfrak{x}n+1}(-\delta^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}}), \tag{14}$$

where $\mathscr{E}_{\mathfrak{x},\mathfrak{x}n+1}(.)$ *is the Mittag-Leffler function (Mittag and Leffler [37]).*

Proof. The Laplace transform of the Riemann-Liouville fractional integral operator is given by Srivastava and Saxena [18], and Erdélyi et al. [20]:

$$\mathfrak{L}\{{}_0\mathscr{D}_{\mathfrak{y}}^{-\mathfrak{x}} f(\mathfrak{y}); s\} = s^{-\mathfrak{x}} F(s) \tag{15}$$

where $F(p)$ is given in Erdélyi et al. [20], Srivastava and Karlsson [21], and Srivastava and Manocha [22]. Now, applying the Laplace transform to both sides of (13), we obtain

$$\begin{aligned}
&\mathfrak{L}\{\mathfrak{z}(\mathfrak{y});p\} = \mathfrak{z}_0\,\mathfrak{L}\{\Phi^{(\sigma,\rho,\kappa)}_{\lambda,\mu;\omega}(\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}},s,a);p\} - \delta^{\mathfrak{x}}\,\mathfrak{L}\{{}_0\mathscr{D}^{-\mathfrak{x}}_{\mathfrak{y}}\mathfrak{z}(\mathfrak{y});p\}\\
&\mathfrak{z}(p) = \mathfrak{z}_0\left\{\int_0^{\infty} e^{-p\mathfrak{y}}\sum_{n=0}^{\infty}\frac{(\lambda)_{\sigma n}(\mu)_{\rho n}(\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}})^n}{(\omega)_{\kappa n}n!\,(n+a)^s}d\mathfrak{y}\right\} - \delta^{\mathfrak{x}}p^{-\mathfrak{x}}\mathfrak{z}(p)\\
&\mathfrak{z}(p)+\delta^{\mathfrak{x}}p^{-\mathfrak{x}}\mathfrak{z}(p)\\
&\quad = \mathfrak{z}_0\sum_{n=0}^{\infty}\frac{(\lambda)_{\sigma n}(\mu)_{\rho n}\mathfrak{d}^{\mathfrak{x}n}}{(\omega)_{\kappa n}n!\,(n+a)^s}\int_0^{\infty} e^{-p\mathfrak{y}}\mathfrak{y}^{\mathfrak{x}n}d\mathfrak{y}\\
&\quad = \mathfrak{z}_0\sum_{n=0}^{\infty}\frac{(\lambda)_{\sigma n}(\mu)_{\rho n}\mathfrak{d}^{\mathfrak{x}n}}{(\omega)_{\kappa n}n!\,(n+a)^s}\frac{\Gamma(\mathfrak{x}n+1)}{p^{\mathfrak{x}n+1}}\\
&\mathfrak{z}(p) = \mathfrak{z}_0\sum_{n=0}^{\infty}\frac{(\lambda)_{\sigma n}(\mu)_{\rho n}\Gamma(\mathfrak{x}n+1)\,\mathfrak{d}^{\mathfrak{x}n}}{(\omega)_{\kappa n}n!\,(n+a)^s}\,p^{-(\mathfrak{x}n+1)}\sum_{r=0}^{\infty}\left[-\left(\frac{p}{\delta}\right)^{-\mathfrak{x}}\right]^r.
\end{aligned}\tag{16}$$

The inverse Laplace transform of (16) is given by Erdélyi et al. [20]

$$\mathfrak{L}^{-1}\{p^{-\mathfrak{x}};\mathfrak{y}\} = \frac{t^{\mathfrak{x}-1}}{\Gamma(\mathfrak{x})},\ (\Re(\mathfrak{x})>0),\tag{17}$$

we get

$$\begin{aligned}
L^{-1}\{\mathfrak{z}(p)\} &= \mathfrak{z}_0\sum_{n=0}^{\infty}\frac{(\lambda)_{\sigma n}(\mu)_{\rho n}\Gamma(\mathfrak{x}n+1)\,\mathfrak{d}^{\mathfrak{x}n}}{(\omega)_{\kappa n}n!\,(n+a)^s}\\
&\quad\times L^{-1}\left\{p^{-(\mathfrak{x}n+1)}\sum_{r=0}^{\infty}\left[-\left(\frac{p}{\delta}\right)^{-\mathfrak{x}}\right]^r\right\}\\
\mathfrak{z}(\mathfrak{y}) &= \sum_{n=0}^{\infty}\frac{(\lambda)_{\sigma n}(\mu)_{\rho n}\Gamma(\mathfrak{x}n+1)(\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}})^n}{(\omega)_{\kappa n}n!\,(n+a)^s}\sum_{r=0}^{\infty}(-1)^r\delta^{\mathfrak{x}r}\frac{\mathfrak{y}^{\mathfrak{x}r}}{\Gamma(\mathfrak{x}n+\mathfrak{x}r+1)}.
\end{aligned}\tag{18}$$

So, we can yield the required result (14).

Theorem 2. *If $\mathfrak{d}>0,\mathfrak{x}>0;\lambda,\mu\in\mathbb{C}$ be such that $a,\omega\in\mathbb{C}\setminus\mathbb{Z}_0^-;\sigma,\rho,\kappa\in\mathbb{R}^+$, then the solution of the following fractional equation*

$$\mathfrak{z}(\mathfrak{y}) - \mathfrak{z}_0\Phi^{(\sigma,\rho,\kappa)}_{\lambda,\mu;\omega}(\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}},s,a) = -\mathfrak{d}^{\mathfrak{x}}\,{}_0\mathscr{D}^{-\mathfrak{x}}_{\mathfrak{y}}\tag{19}$$

is given by

$$\mathfrak{z}(\mathfrak{y}) = \mathfrak{z}_0\sum_{n=0}^{\infty}\frac{(\lambda)_{\sigma n}(\mu)_{\rho n}\Gamma(\mathfrak{x}n+1)(\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}})^n}{(\omega)_{\kappa n}n!\,(n+a)^s}\mathscr{E}_{\mathfrak{x},\mathfrak{x}n+1}(-\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}}),\tag{20}$$

where $\mathscr{E}_{\mathfrak{x},\mathfrak{x}n+1}(.)$ is the Mittag-Leffler function (Mittag-Leffler [37]).

Proof. The proof of Theorem 2 is parallel to the proof of Theorem 1, thus the details are omitted.

Theorem 3. *If $\mathfrak{x}>0;\lambda,\mu,\delta\in\mathbb{C}$ be such that $a,\omega\in\mathbb{C}\setminus\mathbb{Z}_0^-;\sigma,\rho,\kappa\in\mathbb{R}^+$, then the solution of the following fractional equation*

$$\mathfrak{z}(\mathfrak{y}) - \mathfrak{z}_0\Phi^{(\sigma,\rho,\kappa)}_{\lambda,\mu;\omega}(\mathfrak{y},s,a) = -\delta^{\mathfrak{x}}\,{}_0\mathscr{D}^{-\mathfrak{x}}_{\mathfrak{y}}\tag{21}$$

is given by

$$\mathfrak{z}(\mathfrak{y}) = \mathfrak{z}_0\sum_{n=0}^{\infty}\frac{(\lambda)_{\sigma n}(\mu)_{\rho n}\Gamma(n+1)t^n}{(\omega)_{\kappa n}n!\,(n+a)^s}\mathscr{E}_{\mathfrak{x},n+1}(-\delta^{\mathfrak{x}}t^{\mathfrak{x}}),\tag{22}$$

where $\mathscr{E}_{\mathfrak{x},n+1}(.)$ is the Mittag-Leffler function (Mittag-Leffler [37]).

Proof. Theorem 3 can be easily acquired from Theorem 1, so the details are omitted.

2.2 Special cases

Choosing $\lambda = \sigma = 1$ in the equation (4), which is the generalized Hurwitz-Lerch Zeta function $\Phi_{\mu;\omega}^{\rho,\kappa}(z,s,a)$ introduced and studied by Lin and Srivastava [4].

Applying $\lambda = \sigma = 1$ in Theorem 1, Theorem 2, and Theorem 3 obtained the following forms:

Corollary 4. *If $\mathfrak{d} > 0, \mathfrak{x} > 0; \mu, \delta \in \mathbb{C}$, and $\mathfrak{d} \neq \delta$ be such that $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-; \rho, \kappa \in \mathbb{R}^+$, then the solution of the following fractional equation*

$$\mathfrak{z}(\mathfrak{y}) - \mathfrak{z}_0 \Phi_{\mu;\omega}^{(\rho,\kappa)}(\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}}, s, a) = -\delta^{\mathfrak{x}}{}_0\mathscr{D}_t^{-\mathfrak{x}} \tag{23}$$

is given by

$$\mathfrak{z}(\mathfrak{y}) = \mathfrak{z}_0 \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}\Gamma(\mathfrak{x}n+1)(\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}})^n}{(\omega)_{\kappa n}(n+a)^s} \mathscr{E}_{\mathfrak{x},\mathfrak{x}n+1}(-\delta^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}}). \tag{24}$$

Corollary 5. *If $\mathfrak{d} > 0, \mathfrak{x} > 0; \mu \in \mathbb{C}$ be such that $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-; \rho, \kappa \in \mathbb{R}^+$, then the solution of the following fractional equation*

$$\mathfrak{z}(\mathfrak{y}) - \mathfrak{z}_0 \Phi_{\mu;\omega}^{(\rho,\kappa)}(\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}}, s, a) = -\mathfrak{d}^{\mathfrak{x}}{}_0\mathscr{D}_{\mathfrak{y}}^{-\mathfrak{x}} \tag{25}$$

is given by

$$\mathfrak{z}(\mathfrak{y}) = \mathfrak{z}_0 \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}\Gamma(\mathfrak{x}n+1)(\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}})^n}{(\omega)_{\kappa n}(n+a)^s} \mathscr{E}_{\mathfrak{x},\mathfrak{x}n+1}(-\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}}). \tag{26}$$

Corollary 6. *If $\mu, \delta \in \mathbb{C}$ be such that $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-; \rho, \kappa \in \mathbb{R}^+$, then the solution of the following fractional equation*

$$\mathfrak{z}(\mathfrak{y}) - \mathfrak{z}_0 \Phi_{\mu;\omega}^{(\rho,\kappa)}(\mathfrak{y}, s, a) = -\delta^{\mathfrak{x}}{}_0\mathscr{D}_{\mathfrak{y}}^{-\mathfrak{x}} \tag{27}$$

is given by

$$\mathfrak{z}(\mathfrak{y}) = \mathfrak{z}_0 \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}\Gamma(n+1)\mathfrak{y}^n}{(\omega)_{\kappa n}(n+a)^s} \mathscr{E}_{\mathfrak{x},n+1}(-\delta^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}}). \tag{28}$$

Setting $\sigma = \rho = \kappa = 1$ in the equation (4), which is the generalized Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\omega}(z,s,a)$ introduced and studied by Garg et al. [8].

Applying $\sigma = \rho = \kappa = 1$ in Theorem 1, Theorem 2, and Theorem 3 obtained the following forms:

Corollary 7. *If $\mathfrak{d} > 0, \mathfrak{x} > 0; \lambda, \mu, \delta \in \mathbb{C}$, and $\mathfrak{d} \neq \delta$ be such that $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$, then the solution of the following fractional equation*

$$\mathfrak{z}(\mathfrak{y}) - \mathfrak{z}_0 \Phi_{\lambda,\mu;\omega}(\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}}, s, a) = -\delta^{\mathfrak{x}}{}_0\mathscr{D}_{\mathfrak{y}}^{-\mathfrak{x}} \tag{29}$$

is given by

$$\mathfrak{z}(\mathfrak{y}) = \mathfrak{z}_0 \sum_{n=0}^{\infty} \frac{(\lambda)_n(\mu)_n\Gamma(\mathfrak{x}n+1)(\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}})^n}{(\omega)_n n!(n+a)^s} \mathscr{E}_{\mathfrak{x},\mathfrak{x}n+1}(-\delta^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}}). \tag{30}$$

Corollary 8. *If $\mathfrak{d} > 0, \mathfrak{x} > 0; \lambda, \mu \in \mathbb{C}$ be such that $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$, then the solution of the following fractional equation*

$$\mathfrak{z}(\mathfrak{y}) - \mathfrak{z}_0 \Phi_{\lambda,\mu;\omega}(\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}}, s, a) = -\mathfrak{d}^{\mathfrak{x}}{}_0\mathscr{D}_{\mathfrak{y}}^{-\mathfrak{x}} \tag{31}$$

is given by

$$\mathfrak{z}(\mathfrak{y}) = \mathfrak{z}_0 \sum_{n=0}^{\infty} \frac{(\lambda)_n(\mu)_n\Gamma(\mathfrak{x}n+1)(\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}})^n}{(\omega)_n n!(n+a)^s} \mathscr{E}_{\mathfrak{x},\mathfrak{x}n+1}(-\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}}). \tag{32}$$

Corollary 9. *If* $\lambda, \mu, \delta \in \mathbb{C}$ *be such that* $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$, *then the solution of the following fractional equation*

$$\mathfrak{z}(\mathfrak{y}) - \mathfrak{z}_0 \Phi_{\lambda,\mu;\omega}(\mathfrak{y}, s, a) = -\delta^{\mathfrak{x}}\, {}_0\mathscr{D}_{\mathfrak{y}}^{-\mathfrak{x}} \tag{33}$$

is given by

$$\mathfrak{z}(\mathfrak{y}) = \mathfrak{z}_0 \sum_{n=0}^{\infty} \frac{(\lambda)_n (\mu)_n \Gamma(n+1) t^n}{(\omega)_n n! (n+a)^s} \mathscr{E}_{\mathfrak{x},n+1}(-\delta^{\mathfrak{x}} t^{\mathfrak{x}}). \tag{34}$$

Upon taking $\sigma = \rho = \kappa = 1$ and $\lambda = \omega$ in the equation (4), which is the generalized Hurwitz-Lerch Zeta function $\Phi_{\mu}^{\star}(z, s, a)$ introduced and studied by Goyal and Laddha [3].

Applying $\sigma = \rho = \kappa = 1$ and $\lambda = \omega$ in Theorem 1, Theorem 2, and Theorem 3 obtained the following forms:

Corollary 10. *If* $\mathfrak{d} > 0, \mathfrak{x} > 0; \mu, \delta \in \mathbb{C}$, *and* $\mathfrak{d} \neq \delta$ *be such that* $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, *then the solution of the following fractional equation*

$$\mathfrak{z}(\mathfrak{y}) - \mathfrak{z}_0 \Phi_{\mu}^{\star}(\mathfrak{d}^{\mathfrak{x}} \mathfrak{y}^{\mathfrak{x}}, s, a) = -\delta^{\mathfrak{x}}\, {}_0\mathscr{D}_{\mathfrak{y}}^{-\mathfrak{x}} \tag{35}$$

is given by

$$\mathfrak{z}(\mathfrak{y}) = \mathfrak{z}_0 \sum_{n=0}^{\infty} \frac{(\mu)_n \Gamma(\mathfrak{x} n+1) (\mathfrak{d}^{\mathfrak{x}} \mathfrak{y}^{\mathfrak{x}})^n}{n! (n+a)^s} \mathscr{E}_{\mathfrak{x},\mathfrak{x}n+1}(-\delta^{\mathfrak{x}} \mathfrak{y}^{\mathfrak{x}}). \tag{36}$$

Corollary 11. *If* $\mathfrak{d} > 0, \mathfrak{x} > 0; \mu \in \mathbb{C}$ *be such that* $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, *then the solution of the following fractional equation*

$$\mathfrak{z}(\mathfrak{y}) - \mathfrak{z}_0 \Phi_{\mu}^{\star}(\mathfrak{d}^{\mathfrak{x}} \mathfrak{y}^{\mathfrak{x}}, s, a) = -\mathfrak{d}^{\mathfrak{x}}\, {}_0\mathscr{D}_{\mathfrak{y}}^{-\mathfrak{x}} \tag{37}$$

is given by

$$\mathfrak{z}(\mathfrak{y}) = \mathfrak{z}_0 \sum_{n=0}^{\infty} \frac{(\mu)_n \Gamma(\mathfrak{x} n+1) (\mathfrak{d}^{\mathfrak{x}} \mathfrak{y}^{\mathfrak{x}})^n}{n! (n+a)^s} \mathscr{E}_{\mathfrak{x},\mathfrak{x}n+1}(-\mathfrak{d}^{\mathfrak{x}} \mathfrak{y}^{\mathfrak{x}}). \tag{38}$$

Corollary 12. *If* $\lambda, \mu, \delta \in \mathbb{C}$ *be such that* $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, *then the solution of the following fractional equation*

$$\mathfrak{z}(\mathfrak{y}) - \mathfrak{z}_0 \Phi_{\mu}^{\star}(\mathfrak{y}, s, a) = -\delta^{\mathfrak{x}}\, {}_0\mathscr{D}_{\mathfrak{y}}^{-\mathfrak{x}} \tag{39}$$

is given by

$$\mathfrak{z}(\mathfrak{y}) = \mathfrak{z}_0 \sum_{n=0}^{\infty} \frac{(\mu)_n \Gamma(n+1) t^n}{n! (n+a)^s} \mathscr{E}_{\mathfrak{x},n+1}(-\delta^{\mathfrak{x}} t^{\mathfrak{x}}). \tag{40}$$

Upon taking $\sigma = \rho = \mu = 1$ and $z = \frac{z}{\lambda}$. Then, the limit case of (4) when $\lambda \to \infty$, would yield the Mittag-Leffler type function $\mathscr{E}_{\kappa,\omega}^{(a)}(s; t)$ studied by Barnes [38], that is,

$$\mathscr{E}_{\kappa,\omega}^{(a)}(s; z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s \Gamma(\omega + \kappa n)}, \tag{41}$$

$$(a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-; \Re(\kappa) > 0; s, z \in \mathbb{C}).$$

Applying $\sigma = \rho = \mu = 1$ and $z = \frac{z}{\lambda}$. Then, the limit case of (4) when $\lambda \to \infty$ in Theorem 1, Theorem 2, and Theorem 3 obtained the following forms:

Corollary 13. *If* $\mathfrak{d} > 0, \mathfrak{x} > 0; \kappa, \delta \in \mathbb{C}$, *and* $\mathfrak{d} \neq \delta$ *be such that* $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$, *then the solution of the following fractional equation*

$$\mathfrak{z}(\mathfrak{y}) - \mathfrak{z}_0 \mathscr{E}_{\kappa,\omega}^{(a)}(s; \mathfrak{d}^{\mathfrak{x}} \mathfrak{y}^{\mathfrak{x}}) = -\delta^{\mathfrak{x}}\, {}_0\mathscr{D}_{\mathfrak{y}}^{-\mathfrak{x}} \tag{42}$$

is given by

$$\mathfrak{z}(\mathfrak{y}) = \mathfrak{z}_0 \sum_{n=0}^{\infty} \frac{\Gamma(\mathfrak{x} n+1) (\mathfrak{d}^{\mathfrak{x}} \mathfrak{y}^{\mathfrak{x}})^n}{(n+a)^s \Gamma(\omega + \kappa n)} \mathscr{E}_{\mathfrak{x},\mathfrak{x}n+1}(-\delta^{\mathfrak{x}} \mathfrak{y}^{\mathfrak{x}}). \tag{43}$$

Corollary 14. *If $\mathfrak{d} > 0, \mathfrak{x} > 0$; $\kappa \in \mathbb{C}$ be such that $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$, then the solution of the following fractional equation*

$$\mathfrak{z}(\mathfrak{y}) - \mathfrak{z}_0 \mathscr{E}_{\kappa,\omega}^{(a)}(s; \mathfrak{d}^{\mathfrak{x}} \mathfrak{y}^{\mathfrak{x}}) = -\mathfrak{d}^{\mathfrak{x}}\, {}_0\mathscr{D}_{\mathfrak{y}}^{-\mathfrak{x}} \tag{44}$$

is given by

$$\mathfrak{z}(\mathfrak{y}) = \mathfrak{z}_0 \sum_{n=0}^{\infty} \frac{\Gamma(\mathfrak{x}n+1)(\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}})^n}{(n+a)^s \Gamma(\omega+\kappa n)} \mathscr{E}_{\mathfrak{x},\mathfrak{x}n+1}(-\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}}). \tag{45}$$

Corollary 15. *If $\kappa, \delta \in \mathbb{C}$ be such that $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$, then the solution of the following fractional equation*

$$\mathfrak{z}(\mathfrak{y}) - \mathfrak{z}_0 \mathscr{E}_{\kappa,\omega}^{(a)}(s; \mathfrak{y}) = -\delta^{\mathfrak{x}}\, {}_0\mathscr{D}_{\mathfrak{y}}^{-\mathfrak{x}} \tag{46}$$

is given by

$$\mathfrak{z}(\mathfrak{y}) = \mathfrak{z}_0 \sum_{n=0}^{\infty} \frac{\Gamma(n+1)t^n}{(n+a)^s \Gamma(\omega+\kappa n)} \mathscr{E}_{\mathfrak{x},n+1}(-\delta^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}}). \tag{47}$$

Finally, upon setting $\lambda, \mu, \omega, \sigma, \rho, \kappa = 1$ in the equation (4) gives the equation (1) [1, 2].
Choosing $\lambda, \mu, \omega, \sigma, \rho, \kappa = 1$ in Theorem 1, Theorem 2, and Theorem 3 obtained the following forms:

Corollary 16. *If $\mathfrak{d} > 0$; $\delta, \mathfrak{x} \in \mathbb{C}, a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, and $\mathfrak{d} \neq \delta$, then the solution of the following fractional equation*

$$\mathfrak{z}(\mathfrak{y}) - \mathfrak{z}_0 \Phi(\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}}, s, a) = -\delta^{\mathfrak{x}}\, {}_0\mathscr{D}_{\mathfrak{y}}^{-\mathfrak{x}} \tag{48}$$

is given by

$$\mathfrak{z}(\mathfrak{y}) = \mathfrak{z}_0 \sum_{n=0}^{\infty} \frac{\Gamma(\mathfrak{x}n+1)(\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}})^n}{(n+a)^s} \mathscr{E}_{\mathfrak{x},\mathfrak{x}n+1}(-\delta^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}}). \tag{49}$$

Corollary 17. *If $\mathfrak{d} > 0$; $\mathfrak{x} \in \mathbb{C}, a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, then the solution of the following fractional equation*

$$\mathfrak{z}(\mathfrak{y}) - \mathfrak{z}_0 \Phi(\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}}, s, a) = -\mathfrak{d}^{\mathfrak{x}}\, {}_0\mathscr{D}_{\mathfrak{y}}^{-\mathfrak{x}} \tag{50}$$

is given by

$$\mathfrak{z}(\mathfrak{y}) = \mathfrak{z}_0 \sum_{n=0}^{\infty} \frac{\Gamma(\mathfrak{x}n+1)(\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}})^n}{(n+a)^s} \mathscr{E}_{\mathfrak{x},\mathfrak{x}n+1}(-\mathfrak{d}^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}}). \tag{51}$$

Corollary 18. *If $\delta \in \mathbb{C}, a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, then the solution of the following fractional equation*

$$\mathfrak{z}(\mathfrak{y}) - \mathfrak{z}_0 \Phi(\mathfrak{y}, s, a) = -\delta^{\mathfrak{x}}\, {}_0\mathscr{D}_{\mathfrak{y}}^{-\mathfrak{x}} \tag{52}$$

is given by

$$\mathfrak{z}(\mathfrak{y}) = \mathfrak{z}_0 \sum_{n=0}^{\infty} \frac{\Gamma(n+1)t^n}{(n+a)^s} \mathscr{E}_{\mathfrak{x},n+1}(-\delta^{\mathfrak{x}} t^{\mathfrak{x}}). \tag{53}$$

3 Numerical result and graphics

This section obtains the general solutions of the fractional Kinetic equation involving the generalized Hurwitz-Lerch Zeta function using Laplace transform.

Application Let $\lambda, \mu, \omega, \rho, \kappa, \sigma, a, s = 1$, $\mathfrak{d}, \delta = 4$, and $\mathfrak{z}_0 = 3$ in the (13). Then, we have the following equation:

$$\mathfrak{z}(\mathfrak{y}) - 3\Phi_{1,1;1}^{(1,1,1)}(4^{0,4}\mathfrak{y}^{\mathfrak{x}}, 1, 1) = -4^{\mathfrak{x}}\, {}_0\mathscr{D}_{\mathfrak{y}}^{-\mathfrak{x}}. \tag{54}$$

Applying the Laplace transform to the both sides of (54), we get

$$\begin{aligned}\mathfrak{L}\{\mathfrak{z}(\mathfrak{y});p\} &= 3\,\mathfrak{L}\{\Phi_{1,1;1}^{(1,1,1)}(4^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}},1,1);p\} - 4^{\mathfrak{x}}\,\mathfrak{L}\{{}_0\mathscr{D}_{\mathfrak{y}}^{-\mathfrak{x}}\,\mathfrak{z}(\mathfrak{y});p\}\\ \mathfrak{z}(p) &= 3\sum_{n=0}^{\infty}\frac{(1)_n(1)_n\Gamma(\mathfrak{x}n+1)4^{\mathfrak{x}n}}{(1)_n(n+1)!}\,p^{-(\mathfrak{x}n+1)}\sum_{r=0}^{\infty}\left[-\left(\frac{p}{4}\right)^{-\mathfrak{x}}\right]^r.\end{aligned} \tag{55}$$

Then, using the inverse Laplace transform of (55), we obtain

$$\begin{aligned}L^{-1}\{\mathfrak{z}(p)\} &= 3\sum_{n=0}^{\infty}\frac{(1)_n(1)_n\Gamma(\mathfrak{x}n+1)4^{\mathfrak{x}n}}{(1)_n(n+1)!}\\ &\quad\times L^{-1}\left\{p^{-(\mathfrak{x}n+1)}\sum_{r=0}^{\infty}\left[-\left(\frac{p}{4}\right)^{-\mathfrak{x}}\right]^r\right\}\\ \mathfrak{z}(\mathfrak{y}) &= \sum_{n=0}^{\infty}\frac{(1)_n(1)_n\Gamma(\mathfrak{x}n+1)(4^{\mathfrak{x}}\mathfrak{y}^{\mathfrak{x}})^n}{(1)_n(n+1)!}\sum_{r=0}^{\infty}(-1)^r\,1^{\mathfrak{x}r}\,\frac{\mathfrak{y}^{\mathfrak{x}r}}{\Gamma(\mathfrak{x}n+\mathfrak{x}r+1)}.\end{aligned} \tag{56}$$

In Figure (1), we plotted the 2D graphs of solution yielded by (56), for the different values of $\mathfrak{x}$ using Mathematica.

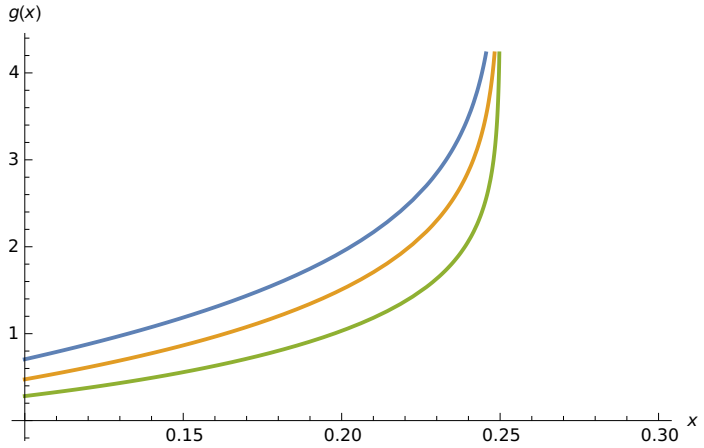


Fig. 1 The graphs of (56) for the values $\mathfrak{x}=0.4$ (blue), $\mathfrak{x}=0.5$ (orange) and $\mathfrak{x}=0.6$ (green).

4 Conclusions

The fractional Kinetic equation involving the generalized Hurwitz-Lerch Zeta function was studied using the Laplace transform. The results obtained in this study have remarkable significance as the solution of the equations are general and many new and known solutions of FKEs can be reproduced involving various types of special functions.

5 Declarations

5.1 Conflict of interest:

The authors hereby declare that there is no conflict of interests regarding the publication of this paper.

5.2 Funding:

Not applicable.

5.3 Author's contribution:

O.Y.-Writing Original Draft; R.Ş.-Supervision. K.S.N.- Writing Review Editing, Methodology. All authors read and approved the final submitted version of this manuscript.

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5.5 Data availability statement:

All data that support the findings of this study are included within the article.

5.6 Using of AI tools:

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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